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An application of fuzzy set theory for supply chain coordination*

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Abstract. One of the important issues in supply chain management is the coordination among different members of the supply chain. In this paper, we have studied a two-stage supply chain coordination problem under uncertain cost and demand information. The aim of the paper is to design a coordination mechanism through quantity discount policy under asymmetric information environment that allows the system to perform as closely as that of under complete information. Fuzzy set theory is applied to estimate the uncertainty associated with the input parameters and triangular membership function has been used to analyze the model. Finally, the model is illustrated with a suitable numerical example.

Keywords: supply chain, coordination, quantity discount, asymmetric information, fuzzy sets

1 Introduction

In the last two decades, Supply Chain Management (SCM) has received maximum attention from both academia and industry. People from both areas have shown keen interest on the subject realizing its potential to improve performance of business operations at a reduced cost and delivery time. A single vendor and a single buyer together constitute a simple two-stage supply chain and form the basic building block of any complex supply chain. Simchi Levi et al. (2000)[34] have mentioned that, supply chain management is a set of approaches utilized to efficiently integrate suppliers, manufactures, warehouses, and stores, so that merchandise is produced and distributed at the right quantities, to the right locations, and to the right time, in order to minimize system-wide costs while satisfying service level requirements. Present business environment is compelling producers, distributors, and raw materials supplier to coordinate their business operations to have efficient and effective supply chains. However, conflicting objectives of different independent members lead to inefficiency in the supply chain. Supply chain management involves the coordination of independently managed business organizations who seek to maximize their individual profits, and one of the major issues of supply chain management is to develop suitable mechanisms to coordinate different activities that are controlled by different members of the chain.

Coordination between the buyer and the seller in a de-centralized supply chain setting under complete information has been extensively studied in the literature. Most of the authors have considered quantity discount as the coordination mechanism and Munson and Rosenblatt (1998)[39] have mentioned that quantity discount is one of the most popular and effective mechanism of coordination in business for quite a long time. The quantity discount policy was initiated with the focus to entice the buyers to procure at a larger quantity and thereby reducing the supplier's total operating cost.

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A wide-ranging literature is available on supply chain coordination based on quantity discount policy. A detailed review on the subject has been provided by Goyal and Gupta (1989)[22], Benton and Park (1996) [11], Munson and Rosenblatt (1998)[39] and recently by Sarmah et al. (2006)[45]. The literature on supply chain coordination through quantity discounts under complete information can be broadly divided into the following three categories (Munson and Rosenblatt, 1998)[39]. They are (i) buyer's perspective models (ii) seller's perspective models and (iii) joint buyer-seller perspective models.

In the class of buyer's perspective models (Abad, 1988; Chakravarty 1984, 1985, 1986; Bassok and Anupindi, 1997)[1, 6, 10, 12, 13], the authors have derived the economic order quantity of the buyer for the different pricing schemes offered by the seller under certain assumptions and constraints. One of the common assumptions here is that the quantity discount schedule is already available and the buyers have to optimize their own requirement in that situation. Further, it is assumed that the sellers have complete information on buyer's cost structures.

On the other hand, in the seller's perspective models, Banerjee (1986)[9], Goyal (1987, 1987)[20, 21], Hwang and Kim (1986)[25], Kim and Hwang (1988,)[26], Lee and Rosenblatt (1986)[30], Monahan (1984)[38], Parlar and Wang (1994)[42], Rosenblatt and Lee (1985)[44], Weng and Wong (1993) have studied the problem from the seller's point of view. Here, the authors have derived quantity discount pricing schemes that induce the buyers to change their order quantities from that of undiscounted price. The pricing scheme is profitable to the seller as long as the discount offered to the buyer(s) is less than the seller's cost savings.

A considerable amount of research has been carried out on the joint buyer-seller models by authors such as Abad (1994, 2003)[7, 8], Banerjee (1986)[9], Chakravarty and Martin (1988)[14], Kohli and Park (1989)[28], Weng (1995)[51], Munson and Rosenblatt (2001)[40], Yang and Wee (2001)[52], Viswanathan and Wang (2003)[49], Qin, Tang and Guo (2007)[43]. These models derived quantity discount strategies to maximize system profit in such a way that no member ends up with a lesser profit as compared to the earlier non-coordinated approach. During the development of the models, the authors have different modeling assumptions and for example, Li and Liu (2006)[35], Shin and Benton (2006)[46] and Zhou and Li (2007)[5] have considered a stochastic annual demand while the other authors have assumed mostly deterministic demand which is either price-sensitive or constant.

However, one common assumption taken in all these models is that, the seller has complete information on the cost structures (including ordering cost and holding cost) of the buyer which is unlikely the case in many real situations. Most of the times for the fear of loosing competitive advantage, independent members of supply chain do not reveal their complete information. Therefore in many cases information available in a supply chain is either incomplete or asymmetric and, under such circumstances, how quantity discount policy can be used to make a meaningful coordination between different supply-chain members is an important area of study.

From different literatures, it is found that in all such cases where the exact parameter value is not available, it is only subjectively judged or assumed approximately e.g. the ordering cost of the buyer may be around 'h', etc. In such cases, to analyze the model with uncertainty or vagueness, application of fuzzy set theory is highly effective and recently it has got wide applications. The concept of fuzzy sets deals with possibility distribution unlike probability distribution in stochastic processes. In fuzzy sets, the boundary is not precise and hence the members can belong to different sets with different membership value between 0 and 1. A brief overview on fuzzy sets is included in appendix.

Ever since Zadeh (1965)[54] developed the concept of fuzzy sets to deal with uncertainty, an extensive research had followed along the path to enrich the field. Recently, the theory of fuzzy sets and fuzzy logic has found wide applications in operations management also and Guiffrida and Nagi (1998)[23] have carried out a detailed review. As a part of operations management, inventory control and supply chain management have also seen an exhaustive applications of fuzzy sets. A brief review of inventory models based on fuzzy sets is discussed below.

Park (1987)[41] developed an EOQ model with uncertain inventory cost under arithmetic operations of extension principle and the author used trapezoidal fuzzy numbers to represent the inventory costs. Later, Yao and Lee (1996)[53] considered shortages while developing their EOQ model with fuzzy order quantity. Vujosevic M et al. (1996)[50] developed different methods to obtain EOQ when the inventory holding cost and ordering costs are fuzzy in nature. Chen et al. (1996)[16] analyzed a fuzzy inventory model with backorder option. Hsieh (2002) [24] has introduced two fuzzy inventory models with fuzzy parameters and derived the optimal production quantity by using graded mean integration representation method and extended Lagrangean method. The author had shown that a crisp model is a specific case of the fuzzy model. Gen et al. (1997)[19], Lee and Yao (1998, 1999)[31, 32], Lin and Yao (2000)[36], Yao and Chang (2000)[15], Yao and Su (2000)[4], discussed different inventory problems that consider inventory with backorder, inventory without backorder and production inventory in the fuzzy sense. Yao and Chiang (2003)[3] considered an inventory model without backorder where total demand and holding costs were assumed to be fuzzy in nature and the authors had used different methods to derive total cost.

Mahata et al. (2005)[37] have investigated the joint economic lot size (JELS) model for both purchaser and vendor in fuzzy sense. The authors have extended Banerjee's (1986)[9] model with the assumption that the order quantity for the purchaser is a fuzzy variable while the rest of the parameters are deterministic. However, the authors have not derived the amount of coordination benefit and the mechanism to share the benefit among the different partners.

From the literature review, it is found that there has been significant development of inventory models under uncertainty using the concept of fuzzy sets. But the issue of developing quantity discount policy for a buyer-vendor integrated system under uncertainty has not yet been addressed in the literature adequately except Lam and Wong (1996)[29], Corbett and Groote (2000)[17] and Eric Sucky (2006)[48].

Lam and Wong (1996)[29] have extended Dolan's (1978)[18] deterministic model in a fuzzy sense. The authors have applied fuzzy mathematical programming to solve a joint economic lot size problem to determine the number of price breaks, quantity discount and order quantity at each price break, to achieve the optimal joint cost. An efficient algorithm has been developed to solve the above problem simultaneously from the perspectives of the seller as well as the buyer and finally, the system profit has been divided between the buyer and the seller, based on the same level of satisfaction. Here satisfaction has been measured by a linear membership function derived on the basis of 'best' and 'worst' situations and the model considers uncertainty only at the time of splitting the system profit, and the rest of the model is quite deterministic with constant demand, holding cost and ordering/ set up cost.

Eric Sucky (2006)[48] has addressed the two-stage supply chain coordination problem under asymmetric information with an assumption that the buyers could have only two sets of predefined cost structures (all sets are deterministic) and the seller does not know in which particular cluster does a particular buyer belongs. The author has developed a pricing scheme to accommodate the buyers in any of the two pre-defined clusters. However, the limitation of the model is that, it is not flexible enough to address a wide range of cost structures that a buyer can 'possibly' have. The buyers' cost structures can have any range or any distribution. Therefore, sometimes plotting a stochastic distribution for a particular parameter becomes impossible if there is no adequate historical data available therein.

Thus, this work is motivated to bridge the gap in the literature by proposing a two stage supply chain coordination model through quantity discount policy where the seller does not have complete information on the cost structure of the buyer. The model considers the single buyer and the single vendor which has been extended to fuzzy sense and to solve the model, and the technique developed by Vujosevic M et al. (1996)[50] has been adopted. This work is an extension of the authors' earlier work (Sinha, and Sarmah, 2007)[47].

The rest of the paper is organized in the following way: we develop the mathematical model in section 2, section 3 deals with the solution methodology; a numerical example has been carried out in section 4 while conclusion of the paper is included in section 5. A brief overview on fuzzy set theory is included further in appendix.

2 Development of the mathematical model

While developing the model, it is assumed that the buyer is a retailer who purchases a single product from a certain vendor and sells it in the market. Henceforth, the words, 'buyer' and 'retailer' would be used interchangeably in the subsequent sections. The seller is assumed to be either a manufacturer who manufactures the items for the retailer/buyer or is another vendor who procures the same from another source.

It is further assumed that the vendor, having more market power than that of the buyer, takes the initiative to design a Pareto-optimal quantity discount policy; hence, the entire model is developed in the perspective of the vendor only. While developing the model, we have further assumed that the product has not achieved maturity in the market and hence its annual market demand is not known precisely to the vendor. In this model, we have relaxed the earlier modeling assumption of complete information and have taken a more realistic situation where the ordering cost and holding cost of the buyer is not known to the vendor and thus the vendor estimates the ordering cost and holding cost of the buyer, and the annual market demand of the product by subjective judgment.

In this paper, the terms 'asymmetric information', 'incomplete information' and 'imprecise information' are also used interchangeably to convey the similar meaning. Further, in the development of the model, it is assumed that the vendor uses lot-for-lot policy and hence does not keep any inventory (Chakravarty and Martin, 1988)[14].

Notation

A_b Ordering cost of the buyer in crisp (actual)
$\tilde{A}_b$ Ordering cost of the buyer in fuzzy (estimated)
A_s Set up cost of the seller
D Annual demand
$\tilde{D}$ Annual demand (estimated)
h_b Inventory holding cost per unit per year of the buyer in crisp (actual)
$\tilde{h}_b$ Inventory holding cost per unit per year of the buyer in fuzzy (estimated)
P_o Per unit production cost of the vendor
P Un-discounted per unit sell price of the seller
p Discounted per unit sell price of the seller
Q Joint economic lot size (JELS) of the system
Q^* Economic order quantity (EOQ) of the buyer
S Per unit sell-price of the retailer

2.1 Non-coordinated policy

The profit equation of the retailer - as it appears to the vendor is given as follows,

$$B_{nc} = (S-P)\tilde{D} - \left[\frac{\tilde{A}_b\tilde{D}}{Q} + \frac{\tilde{h}_b Q}{2}\right] \tag{1}$$

With imprecise information about the annual demand, order cost and holding cost of the buyer, the vendor can estimate the expected economic order quantity of the buyer which minimizes 1 as,

$$Q^* = \sqrt{\frac{2\tilde{A}_b\tilde{D}}{\tilde{h}_b}} \tag{2}$$

Thus

$$(B_{nc})_{\min} = B^*_{nc} = (S-P)\tilde{D} - \left[\frac{\tilde{A}_b\tilde{D}}{Q^*} + \frac{\tilde{h}_b Q^*}{2}\right] \tag{3}$$

The procedure to derive Q^* is shown in section 3. When no discount is offered and the buyer's economic order quantity (Q^*) is accepted, the profit function of the seller is as follows,

$$S^*_{nc} = \left[(P-P_0)\tilde{D} - \frac{A_s\tilde{D}}{Q^*})\right] \tag{4}$$

2.2 Coordinated Policy

If the seller offers a discount and the discounted per-unit price is p, then for a particular lot size Q,

$$B_c = (S-p)\tilde{D} - - \left[\frac{\tilde{A}_b\tilde{D}}{Q} + \frac{\tilde{h_b}Q}{2}\right] \tag{5}$$

And, the profit function of the seller is as follows,

$$S_c = (p-P_0)\tilde{D} - \frac{A_s\tilde{D}}{Q} \tag{6}$$

Benefit of the buyer if quantity discount is implemented,

$$b = (B - B^*_{nc}) = (P-p)\tilde{D} + \tilde{A}_b\tilde{D}\left(\frac{1}{Q^*} - \frac{1}{Q}\right) + \frac{\tilde{h_b}}{2}(Q^* - Q) \tag{7}$$

Benefit of the seller if quantity discount is implemented,

$$s = (S_c - S^*_{nc}) = A_s\tilde{D}\left(\frac{1}{Q^*} - \frac{1}{Q}\right) - (P-p)\tilde{D} \tag{8}$$

The objective is to maximize total coordination benefit for both the buyer and seller,

$$\max. \qquad J = (b+c) = \left[(\tilde{A}_b + A_s)\tilde{D}\left(\frac{1}{Q^*} - \frac{1}{Q}\right) + \frac{\tilde{h_b}}{2}(Q^* - Q)\right] \tag{9}$$

It is assumed that the vendor offers the buyer a certain level of coordination benefit such that,

$$b = \gamma s \tag{10}$$

where, $\gamma \geq 0$ and is a pre-defined parameter to split the coordination benefit between buyer and seller.

From (7), (8) and (10),

$$p = P - \left[\frac{(\gamma A_s - \tilde{A}_b)}{(\gamma+1)}\left(\frac{1}{Q^*} - \frac{1}{Q}\right) - \frac{\tilde{h_b}(Q^* - Q)}{2\tilde{d}(\gamma+1)}\right] \tag{11}$$

Once optimal Q is derived, one can derive the total coordination benefit (J) and the seller's discounted per-unit selling price (p) - corresponding to a particular value of γ. The procedures are described in section 3.

3 Solution methodology

From (9), the optimal Q is derived as follows,

$$Q = \sqrt{\frac{2\tilde{D}(\tilde{A}_b + A_s)}{\tilde{h_b}}} \tag{12}$$

However, $\tilde{A}_b$, $\tilde{h_b}$ and D are not known precisely and let $\tilde{A}_b$, $\tilde{h_b}$ and D, and be defined by triangular fuzzy numbers such that $\tilde{A}_b = [a_1, a_2, a_3]$, $\tilde{h_b} = [h_{b1}, h_{b2}, h_{b3}]$ and $\tilde{D} = [d_1, d_2, d_3]$ where, $(a_1 < a_2 < a_3)$, $(h_{b1} < h_{b2} < h_{b3})$ and $(d_1 < d_2 < d_3)$ - based on subjective judgment. We apply arithmetic operators on fuzzy quantities and then de-fuzzify the same to convert them to crisp output.

The membership functions of $\mu_{\tilde{A}_b}(\tilde{A}_b)$, $\mu_{\tilde{h_b}}(\tilde{h_b})$ and $\mu_{\tilde{D}}(\tilde{D})$ are defined as follows

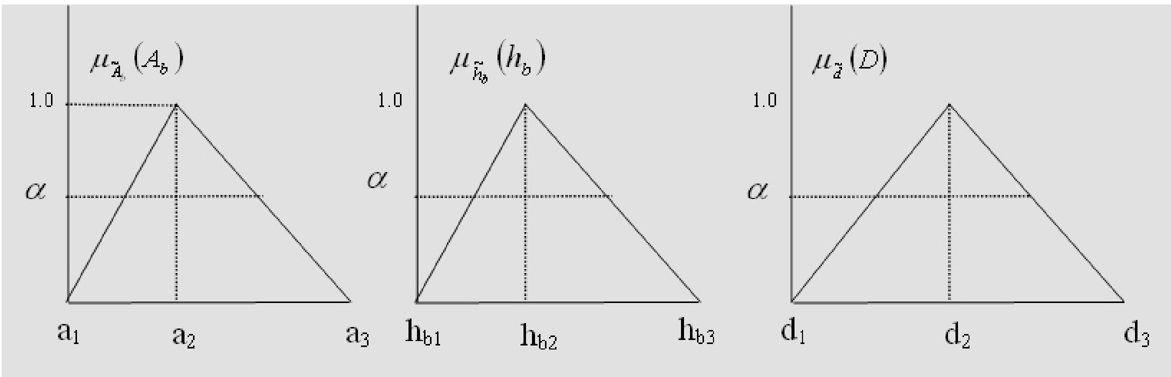

Fig. 1. Membership functions for $\tilde{a_b}$, $\tilde{h_b}$ and $\tilde{d}$

$$\mu_{\tilde{A}_b}(\tilde{A}_b) = \begin{cases} 0, & \text{if} \quad A_b < a_1 \\ \frac{A_b - a_1}{a_2 - a_1} & \text{if} \quad a_1 \le A_b < a_2 \\ \frac{a_3 - A_b}{a_3 - a_2} & \text{if} \quad a_2 \le A_b < a_3 \\ 0, & \text{if} \quad A_b \ge a_3 \end{cases} \quad \text{(A)} \qquad \mu_{\tilde{h_b}}(\tilde{h_b}) = \begin{cases} 0, & \text{if} \quad h_b < h_{b1} \\ \frac{h_b - h_{b1}}{h_{b2} - h_{b1}} & \text{if} \quad h_{b1} \le h_b < h_{b2} \\ \frac{h_{b3} - h_b}{h_{b3} - h_{b2}} & \text{if} \quad h_{b2} \le h_b < h_{b3} \\ 0, & \text{if} \quad h_b \ge h_{b3} \end{cases} \quad \text{(B)}$$

$$\mu_{\tilde{D}}(\tilde{D}) = \begin{cases} 0, & \text{if} \quad D < d_1 \\ \frac{D - d_1}{D - d_1} & \text{if} \quad d_1 \le D < d_2 \\ \frac{d_3 - D}{d_3 - d_2} & \text{if} \quad d_2 \le D < d_3 \\ 0, & \text{if} \quad D \ge d_3 \end{cases} \qquad \text{(C)}$$

One can see the method described in appendix. Using the concept of 'α-cut' method, from (A), (B) and (C),

$$\begin{aligned} {}^{\alpha}A_b &= [\alpha(a_2 - a_1) + a_1,\ a_3 - \alpha(a_3 - a_2)] \quad \text{for} \quad \alpha \in [0,1], \\ {}^{\alpha}h_b &= [\alpha(h_{b2} - h_{b1}) + h_{b1},\ h_{b3} - \alpha(h_{b3} - h_{b2})] \quad \text{for} \quad \alpha \in [0,1], \\ {}^{\alpha}D &= [\alpha(d_2 - ad_1) + d_1,\ d_3 - \alpha(d_3 - d_2)] \quad \text{for} \quad \alpha \in [0,1], \end{aligned}$$

3.1 Derivation of Q^*

It is derived that, $Q^* = \sqrt{\frac{2\tilde{A}_b\tilde{D}}{\tilde{h_b}}}$. From inequality

$$\left\{ \frac{a_3 - \alpha(a_3 - a_2)}{\alpha(h_{b2} - h_{b1}) + h_{b1}} \right\} \ge \left\{ \frac{\alpha(a_2 - a_1) + a_1}{h_{b3} - \alpha(h_{b3} - h_{b2})} \right\} \quad \text{for} \quad \alpha \in [0,1]$$

Thus,

$$^{\alpha}\left\{ \frac{\tilde{A}_b}{\tilde{h_b}} \right\} = \left[\left\{ \frac{\alpha(a_2 - a_1) + a_1}{h_{b3} - \alpha(h_{b3} - h_{b2})} \right\}, \left\{ \frac{a_3 - \alpha(a_3 - a_2)}{\alpha(h_{b2} - h_{b1}) + h_{b1}} \right\} \right]$$

The value of $^{\alpha}\left[(Q^*)^2\right] = {}^{\alpha}\left[\frac{2\tilde{A}_b\tilde{D}}{\tilde{h_b}}\right]$ is derived as follows,

$$^{\alpha}\left[(Q^*)^2\right] = \left[2\{\alpha(d_2 - d_1) + d_1\} \left\{ \frac{\alpha(a_2 - a_1) + a_1}{h_{b3} - \alpha(h_{b3} - h_{b2})} \right\},\ 2\{d_3 - \alpha(d_3 - d_2)\} \left\{ \frac{a_3 - \alpha(a_3 - a_2)}{\alpha(h_{b2} - h_{b1}) + h_{b1}} \right\} \right] \tag{13}$$

We put $\alpha = 0$ and $\alpha = 1$ in (13) and obtain an approximate triangular fuzzy number for Q^* as below (K. H. Lee, 2005)[33].

$$(Q^*) = \left[\sqrt{\frac{2d_1a_1}{h_{b3}}}, \sqrt{\frac{2d_2a_2}{h_{b2}}}, \sqrt{\frac{2d_3a_3}{h_{b1}}}\right] = [q_1, q_2, q_3], \tag{14}$$

where $q_1 = \sqrt{\frac{2d_1a_1}{h_{b3}}},\ q_2 = \sqrt{\frac{2d_2a_2}{h_{b2}}},\ q_3 = \sqrt{\frac{2d_3a_3}{h_{b1}}}$.

Thus, the membership function for $\mu_{\tilde{Q}^*}(\tilde{Q^*})$ is given as,

$$\mu_{\tilde{Q}^*}(Q^*) = \begin{cases} 0, & \text{if} \quad Q^* < q_1 \\ \frac{Q^*-q_1}{Q^*-q_1} & \text{if} \quad q_1 \le Q^* < q_2 \\ \frac{q_3-Q^*}{q_3-q_2} & \text{if} \quad q_2 \le Q^* < q_3 \\ 0, & \text{if} \quad Q^* \ge q_3 \end{cases} \tag{D}$$

Q^* is de-fuzzified to a crisp value by the 'centre of gravity' method. The de-fuzzified Q^* is found as,

$$Q^* = defuzz(\tilde{Q^*}) = \frac{\int_R Q^* \mu_{\tilde{Q}^*}(Q^*)dQ^*}{\int_R \mu_{\tilde{Q}^*}(Q^*)dQ^*} \tag{15}$$

3.2 Derivation of optimal *Q*

Since, A_s is a crisp element, it may be represented as: ${}^{\alpha}A_s = [A_s, A_s]$ for $\alpha \in [0,1]$,

Using arithmetic operators,

$${}^{\alpha}(\tilde{A}_b + A_s) = [\{\alpha(a_2 - a_1) + a_1 + A_s\},\ \{a_3 + A_s - \alpha(a_3 - a_2)\}] \quad \text{for} \quad \alpha \in [0,1]. \tag{16}$$

From inequality,

$$\left\{\frac{\{(a_3 + A_s) - \alpha(a_3 - a_2)\}}{\alpha(h_{b2} - h_{b1}) + h_{b1}}\right\} \ge \left\{\frac{\alpha(a_2 - a_1) + a_1 + A_s}{h_{b3} - \alpha(h_{b3} - h_{b2})}\right\} \quad \text{for} \quad \alpha \in [0,1]. \tag{17}$$

Thus,

$${}^{\alpha}\left(\frac{\tilde{A}_b + A_s}{\tilde{h_b}}\right) = \left[\left\{\frac{\alpha(a_2 - a_1) + a_1 + A_s}{h_{b3} - \alpha(h_{b3} - h_{b2})}\right\}, \left\{\frac{(a_3 + A_s) - \alpha(a_3 - a_2)}{\alpha(h_{b2} - h_{b1}) + h_{b1}}\right\}\right] \tag{18}$$

And, the value of ${}^{\alpha}\left(Q^2\right) = {}^{\alpha}\left\{2\tilde{D}\left(\frac{\tilde{A}_b + A_s}{\tilde{h_b}}\right)\right\}$ is derived as follows,

$$= \left[2\alpha\{(d_2 - d_1) + d_1\}\left\{\frac{\alpha(a_2 - a_1) + a_1 + A_s}{h_{b3} - \alpha(h_{b3} - h_{b2})}\right\},\ 2\{d_3 - \alpha(d_3 - d_2)\}\left\{\frac{(a_3 + A_s) - \alpha(a_3 - a_2)}{\alpha(h_{b2} - h_{b1}) + h_{b1}}\right\}\right] \tag{19}$$

We put $\alpha = 0$ and $\alpha = 1$ in (18) and obtain an approximate triangular fuzzy number for (K. H. Lee, 2005)[33].

Thus,

$$(Q) = \left[\sqrt{\frac{2d_1(a_1 + A_s)}{h_{b3}}}, \sqrt{\frac{2d_2(a_2 + A_s)}{h_{b2}}}, \sqrt{\frac{2d_3(a_3 + A_s)}{h_{b1}}}\right] = [r_1, r_2, r_3] \tag{20}$$

where $r_1 = \sqrt{\frac{2d_1(a_1+A_s)}{h_{b3}}},\ r_2 = \sqrt{\frac{2d_2(a_2+A_s)}{h_{b2}}},\ r_3 = \sqrt{\frac{2d_3(a_3+A_s)}{h_{b1}}}$.

The membership function for $\mu_{\tilde{Q}}(Q)$ is derived as,

$$\mu_{\tilde{Q}}(Q) = \begin{cases} 0, & \text{if} \quad Q < r_1 \\ \frac{Q-r_1}{Q-r_1} & \text{if} \quad r_1 \le Q < r_2 \\ \frac{r_3-Q}{r_3-r_2} & \text{if} \quad r_2 \le Q < r_3 \\ 0, & \text{if} \quad Q \ge r_3 \end{cases} \tag{E}$$

Similarly, Q is de-fuzzified to a crisp value by the 'centre of gravity' method. The de-fuzzified Q is found as,

$$Q = defuzz(\tilde{Q}) = \frac{\int_R Q \mu_{\tilde{Q}}(Q)dQ}{\int_R \mu_{\tilde{Q}}(Q)dQ} \tag{21}$$

3.3 Derivation of discounted sell-price p

From (11), $p = \left[P - (\gamma A_s - \tilde{A}_b)t_1 + \frac{\tilde{h_b}}{\tilde{D}}t_2\right]$. Where, $t_1 = \left(\frac{1}{Q^*} - \frac{1}{Q}\right)\frac{1}{(\gamma+1)}$ and $t_2 = \frac{(Q^*-Q)}{2(\gamma+1)}$.
Using arithmetic operators,

$$^{\alpha}(\gamma A_s - \tilde{A}_b) = [\{(\gamma A_s - a_1) - \alpha(a_2 - a_1)\}t_1,\ \{(\gamma A_s - a_3) + \alpha(a_3 - a_2)\}t_2] \quad \text{for} \quad \alpha \in [0,1],$$

$$^{\alpha}\left(\frac{\tilde{h_b}}{\tilde{D}}\right) = \left[\left\{\frac{\alpha(h_{b2} - h_{b1}) + h_{b1}}{d_3 - \alpha(d_3 - d_2)}\right\}t_2,\ \left\{\frac{h_{b3} - \alpha(h_{b3} - h_{b2})}{\alpha(d_2 - d_1) + d_1}\right\}t_2\right] \quad \text{for} \quad \alpha \in [0,1],$$

Thus, for $\alpha \in [0,1]$

$$\begin{aligned} ^{\alpha}(p) = {}& ^{\alpha}\left[P - \left\{(\gamma A_s - \tilde{A}_b)t_1 + \frac{\tilde{h_b}}{\tilde{D}}t_2\right\}\right] \\ = {}& \left[P - \{(\gamma A_s - a_3) - \alpha(a_3 - a_2)\}t_1 + \left\{\frac{\alpha(h_{b2} - h_{b1}) + h_{b1}}{d_3 - \alpha(d_3 - d_2)}\right\}t_2\ ,\right. \\ & \left. P - \{(\gamma A_s - a_1) + \alpha(a_2 - a_1)\}t_1 + \left\{\frac{h_{b3} - \alpha(h_{b3} - h_{b2})}{\alpha(d_2 - d_1) + d_1}\right\}t_2\right], \end{aligned} \tag{22}$$

Similar to the earlier method, we put $\alpha = 0$ and $\alpha = 1$ in (22) and obtain an approximate triangular fuzzy number for (p).

Thus,

$$\begin{aligned} (p) = {}& \left[\left\{P - (\gamma A_s - a_3)t_1 + \left(\frac{h_{b1}}{d_3}\right)t_2\right\},\ \left\{P - (\gamma A_s - a_2)t_1 + \left(\frac{h_{b2}}{d_2}\right)t_2\right\},\right. \\ & \left.\left\{P - (\gamma A_s - a_1)t_1 + \left(\frac{h_{b3}}{d_1}\right)t_2\right\}\right] = [f_1, f_2, f_3] \end{aligned} \tag{23}$$

where $f_1 = \left\{P - (\gamma A_s - a_3)t_1 + \left(\frac{h_{b1}}{d_3}\right)t_2\right\}$, $f_2 = \left\{P - (\gamma A_s - a_2)t_1 + \left(\frac{h_{b2}}{d_2}\right)t_2\right\}$ and $f_3 = \left\{P - (\gamma A_s - a_1)t_1 + \left(\frac{h_{b3}}{d_1}\right)t_2\right\}$.

From (23), the membership function for $\mu_{\tilde{p}}(\tilde{p})$ is derived as,

$$\mu_{\tilde{p}}(\tilde{p}) = \begin{cases} 0, & \text{if} \quad p < f_1 \\ \frac{p-f_1}{f_2-f_1} & \text{if} \quad f_1 \le p < f_2 \\ \frac{f_3-p}{f_3-f_2} & \text{if} \quad f_2 \le p < f_3 \\ 0, & \text{if} \quad p \ge f_3 \end{cases} \tag{F}$$

p is de-fuzzified to a crisp value by the method of 'centre of gravity' method. The de-fuzzified p is found as,

$$p = defuzz(\tilde{p}) = \frac{\int_R p\mu_{\tilde{p}}(p)dp}{\int_R \mu_{\tilde{p}}(p)dp} \tag{24}$$

3.4 Derivation of total coordination benefit (J)

It has already been derived that,

$$J = \left\{\left(\tilde{A}_b + A_s\right)\tilde{D}\right\}t_3 + \tilde{h_b}t_4$$

where, $t_3 = \left(\frac{1}{Q^*} - \frac{1}{Q}\right)$ and $t_4 = \frac{(Q^*-Q)}{2}$.

Applying α-cut and arithmetic operators,

$$^{\alpha}(A_s+\tilde{A_b})t_3=[\{(A_s+a_1)+\alpha(a_2-a_1)\}t_3,\{(A_s+a_3)-\alpha(a_3-a_2)\}] \quad \text{for} \quad \alpha\in[0,1],$$
$$^{\alpha}(A_s+\tilde{A_b})\tilde{D}t_3=[\{(A_s+a_1)+\alpha(a_2-a_1)\}\{\alpha(d_2-d_1)+d_1\}t_3,$$
$$\{(A_s+a_3)-\alpha(a_3-a_2)\}\{d_3-\alpha(d_3-d_2)\}t_3] \text{ for } \alpha\in[0,1],$$
$$^{\alpha}(\tilde{h_b t_4})=[\{\alpha(h_{b2}-h_{b1})+h_{b1}\}t_4,\ \{h_{b3}-\alpha(h_{b3}-h_{b2})\}t_4] \text{ for } \alpha\in[0,1],$$

Thus,

$$^{\alpha}(J)=[\{(A_s+a_1)+\alpha(a_2-a_1)\}\{\alpha(d_2-d_1)+d_1\}t_3+\{\alpha(h_{b2}-h_{b1})+h_{b1}\}t_4],$$
$$[\{(A_s+a_3)-\alpha(a_3-a_2)\}\{d_3-\alpha(d_3-d_2)\}t_3+\{h_{b3}-\alpha(h_{b3}-h_{b2})\}t_4]=[J_1,J_2] \quad (25)$$

where, $J_1=[\{(A_s+a_1)+\alpha(a_2-a_1)\}\{\alpha(d_2-d_1)+d_1\}t_3+\{\alpha(h_{b2}-h_{b1})+h_{b1}\}t_4]$, and $J_2=[\{(A_s+a_3)-\alpha(a_3-a_2)\}\{d_3-\alpha(d_3-d_2)\}t_3+\{h_{b3}-\alpha(h_{b3}-h_{b2})\}t_4]$.

We put $\alpha=0$ and $\alpha=1$ in (25) and obtain an approximate triangular fuzzy number for J.

Thus, $^{\alpha}(J)=[\{(A_s+a_1)d_1t_3+h_{b1}t_4\},\ \{(A_s+a_2)d_2t_3+h_{b2}t_4\},\ \{(A_s+a_3)d_3t_3+h_{b3}t_4\}]=[j_1,j_2,j_3]$ (26)

where, $j_1=\{(A_s+a_1)d_1t_3+h_{b1}t_4\},\ j_2=\{(A_s+a_2)d_2t_3+h_{b2}t_4\},\ j_3=\{(A_s+a_3)d_3t_3+h_{b3}t_4\}$.

From (26), the membership function for $\mu_{\tilde{J}}(J)$ is derived as,

$$\mu_{\tilde{J}}(\tilde{J})=\begin{cases}0, & \text{if} \quad J<j_1\\ \frac{J-j_1}{j_2-j_1} & \text{if} \quad j_1\le J<j_2\\ \frac{j_3-J}{j_3-j_2} & \text{if} \quad j_2\le J<j_3\\ 0, & \text{if} \quad J\ge j_3\end{cases} \quad \text{(G)}$$

J is de-fuzzified to a crisp value by the 'centre of gravity' method. The de-fuzzified J is found as,

$$J=defuzz(\tilde{J})=\frac{\int_R J\mu_{\tilde{J}}(J)dJ}{\int_R \mu_{\tilde{J}}(J)dJ} \quad (27)$$

Thus, system-wide optimal lot-size Q and its corresponding discounted price p and related system savings J, can be derived from (14)-(27) and (D) $\sim$ (G).

4 Numerical example

In this section we first assume that all the parameter values are known to the seller and accordingly we calculate the benchmark solution. Next, we treat A_b, h_b and D to be imprecise and define them by triangular fuzzy numbers and calculate the optimal parameters. We then compare the solution with that of the benchmark solution derived under complete information scenario. The following data set is used for the numerical illustration which is mostly taken from Chakravarty and Martin (1988)[14].

Table 1. Data set under complete information

A_s	P_o	P	A_b	h_b	S	D	γ
500	20	25	100	5	30	1000	1.0

4.1 Complete information

Here, we derive the optimal solution under complete information scenario. The results are shown in the following table.

Table 2. Benchmark solution under complete information

Non-Coordinated Approach				*Coordinated Approach*					
Q^*	B^*_{nc}	S^*_{nc}	Total	Q	p	B_c	S_c	Total	J
200	4000	2500	6500	490	24.04	4530.92	3019.59	7550.51	1050.51

4.2 Incomplete information

Here, we assume that the annual demand (D), the ordering cost (A_b) and the holding cost (h_b) of the buyer, are not known precisely to the vendor and the vendor estimates the values by the following triangular fuzzy numbers based on his subjective judgment.

Table 3. Data set under incomplete information

A_b	h_b	D
[80 100 120]	[3 5 7]	[800 1000 1200]

In this case, the optimal values are shown in the following table. Accordingly, the system-wide extra profit (J) is also estimated and shown below.

Table 4. Solution under incomplete information

Q^*	Q	p	J
168	427	24.01	1409.31

Thus, the optimal $\{Q,\ p\}$ is calculated under incomplete information. Now, if this optimal $\{Q,\ p\}$ is implemented, the following table shows the individual actual pay-offs,

Table 5. Actual pay-offs under incomplete information

B_c	S_c	$Total$	b	s	J	γ
4688.31	2839.04	7527.35	688.31	339.04	1027.35	1.145

From Tab. 2 and Tab. 5, it is seen that the individual pay-offs received by the buyer and the seller and the extra channel profit (J) is very close to that of benchmark solution under complete information.

4.3 Sensitivity Analysis

We have conducted a simulation test by generating random data for the imprecise variables within a certain range as mentioned below. The test is conducted with 2500 randomly generated cases and the optimal Q and p are derived in each case which is again put in the original equation to find out the actual pay-offs for both buyer and the vendor. Tab. 7 shows the performance of the simulation test. It is found that even

Table 6. Data set for simulation test

A_b	h_b	D
[80 120]	[3 7]	[800 1200]

with uncertainty, considerable amount of extra savings can be generated system wide in each case with the application of fuzzy-set theory.

Table 7. Actual pay-offs under incomplete information: a simulation test

Parameter	B	S	b	s	J	γ
Benchmark	4530.92	3019.59	530.92	519.59	1050.51	1.02
Average	4416.74	3114.19	416.74	614.19	1030.93	0.82
Max.	4809.64	3529.38	809.64	1029.38	1050.51	5.08
Min.	4008.83	2659.34	8.83	159.34	968.98	0.01
Std. dev.	151.35	167.00	151.35	167.00	18.04	0.56

5 Conclusion

In this paper, a quantity discount policy to coordinate a buyer and a seller under asymmetric information has been studied. Fuzzy triangular membership function has been used to model the problem. Considering the annual demand, the ordering cost and the holding cost of the buyer to be fuzzy, and the joint economic lot size has been computed by fuzzy arithmetic operators. It is a well-established fact in the literature that under complete information, the system-wide coordination benefit is maximum and therefore the importance of sharing information is considered to be important criteria for a meaningful coordination. However, in this paper - based on a simulation study with 2500 random cases, it is shown that, even with incomplete information, it is possible to get a near-optimal solution which is very close to that of the benchmark solution under complete information.

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Appendix

Fuzzy set theory: We include a brief introduction on fuzzy set theory. More details are available with Klir et al. (2005), Lee (2005).

Definition 1. *A fuzzy set is a set where the members are allowed to have partial membership and hence the degree of membership varies from* 0 *to* 1*. It is expressed as,* $A = \{(x, \mu_{\tilde{A}}(x))|x \in X\}$ *where* X *is the universe of discourse and* $\mu_{\tilde{A}}(x)$ *is the universe of discourse and* $\mu_{\tilde{A}}(x) = 0$ *or* 1*, i.e.,* x *is a non-member in* A *if* $\mu_{\tilde{A}}(x) = 0$*, and* x *is a member in* A *if* $\mu_{\tilde{A}}(x) = 1$.

Definition 2. *If a fuzzy set* A *is defined on* X*, for any* $\alpha \in [0, 1]$*, the* α*-cuts* ${}^{\alpha}A$ *is represented by the following crisp set,*

$$\text{Strong } \alpha\text{-cuts: } {}^{\alpha+}A = \{x \in X|\mu_A(x) > \alpha\}; \quad \alpha \in [0, 1].$$
$$\text{Weak } \alpha\text{-cuts: } {}^{\alpha}A = \{x \in X|\mu_A(x) \geq \alpha\}; \quad \alpha \in [0, 1].$$

Therefore, it is inferred that fuzzy set A *can be treated as crisp set* ${}^{\alpha}A$ *in which all the members have their membership values greater than or at least equal to* α*. The concept of '*α*-cut' is one of the most important concept in fuzzy set theory. And here, we define support and height of a fuzzy set in terms of* α*-cut.*

Definition 3. *The support of a fuzzy set* A *is a crisp set represented as* $suppA(x)$ *such that,* $\forall\{x \in X|\mu(x) > 0\}$*. Thus, support of a fuzzy set is the set of all members with a strong* α*-cut where* $\alpha = 0$*.*

Definition 4. *The height of a fuzzy set* $h\{A(x)|x \in X\}$ *is the maximum value of its membership function* $\mu(x)$ *such that* ${}^{\alpha}A = \{x \in X|\mu_A(X) \geq \alpha\}$ *and* $0 \notin \alpha$*.*

A fuzzy set where Max $\{\mu(x)\} = 1$ *is called as a normal fuzzy set, otherwise, it is referred as sub-normal fuzzy set.*

Fuzzy Arithmetic Operations: We define fuzzy arithmetic operations on fuzzy numbers in terms of the α-cuts. Let, A and B are two fuzzy sets and if '$*$' denotes any of the four basic arithmetic operations $(+, -, *$ and $/)$ then a fuzzy set $Z = (A * B)$ and $Z \in R$, can be defined as, ${}^{\alpha}(A * B) = {}^{\alpha}A *^{\alpha} B$, such that $\forall\ \alpha \in (0, 1]$.

However, if '$*$' is a division operator, then ${}^{\alpha}(A * B) = {}^{\alpha}A *^{\alpha} B$, such that $\forall\ \alpha \in (0, 1]$ and $0 \notin^{\alpha} B$.

Theorem 1. *(First decomposition theorem) For every* $A \in X$,

$$A = \bigcup_{\alpha \in (0,1]} {}_{\alpha}A, \text{ where, } {}_{\alpha}A(x) = \alpha_{\alpha}A(x)$$

Proof is omitted.

From first decomposition theorem, if $Z = (A * B)$ and $Z \in R$, $(A * B) = \bigcup_{\alpha \in (0,1]} {}_{alpha}(A * B)$.

Since ${}^{\alpha}(A * B)$ is a closed interval for each $\alpha \in (0, 1]$ with both A and B fuzzy, $(A * B)$ is also a fuzzy number.

Definition 5. *For the de-fuzzification of a fuzzy set to a crisp value, 'centre of gravity' or 'moment method' is a popular and efficient approach.* $\tilde{A}(x)$ *is converted to a crisp value by the following operation,*

$$A = defuzz(\tilde{A}) = \frac{\int_R A\mu_{\tilde{A}}(x)dx}{\int_R \mu_{\tilde{A}}(x)dx}.$$