Group assets pricing and risk management in hedging based on multivariate partial distribution

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Abstract. This paper suggests the Multivariate Partial Distribution (MPD) for the first time, and establishes the new kinds of models for pricing the group assets (MPGA) based MPD, in which the competition and complementarity in cost resources are analyzed, and the way of dividing an asset into two assets is discussed. According to this study, integrated risk of group assets may be analyzed into two types - hedging risk and independent risk, for which corresponding representing models are given. Then, we could analyze the price risk and profit of the group assets in a more specific way. This study indicates that in most cases, a simple combination of assets hedging in the one-to-one mode cannot eliminate completely their market risks in any case, and puts forward a particular method of determining the optimal ratio between underlying asset and its derivative in hedging.

Keywords: multivariate partial distribution, pricing assets, group assets, risk management, optimal hedging

1 Introduction

Asset (capitals or commodities) pricing is of great importance in a modern economy. Up to now, many meaning studies and works including capital asset pricing model (CAPM)^[11, 15, 16], arbitrage pricing theory (APT, S.A. Ross 1976)^[13, 14], have been devoted to estimating and measuring of the price of capital asset. Through further examination and investigation, we can see that CAPM needs a group of risk capitals. It seems to be unlikely to make a whole samples indexes due to the huge size of the financial market. Thus CAPM is generally thought of as being an exclusive example of APT, because CAPM is a method for a single asset, and APT is a method for multiple assets. However, CAPM can also be extended to multiple assets, for example, the Consumer Service Model^[12], which takes into consideration the risk premium of assets group while APT does not. In addition, the consumption-based CAPM^[1] appears to be more imaginative.

It should be noted that CAPM is considered as being equilibrium in market as well as a result of investors' collective behavior. CAPM has to follow a series of assumptions, some of which are quite rigorous sometimes. APT is supportive to decision-making of investment in group assets. It accents the rule of no-arbitrage, which requires APT to be based on the assets group. Therefore it is not always right to price for single asset. On the other hand, CAPM and APT make the pricing on yield of asset or assets mainly. The asset price at a financial market is always changing. In many cases, we have to know both the yield of asset or assets and the current prices of asset, because they affect each other.

This study suggests a new model for asset pricing on the basis of the univariate Partial Distribution (UPD)^[2]. It also discusses the multivariate Partial Distribution (MPD)^[4]. Arguably, MPD is a new model for pricing group asset. Unlike CAPM and APT, the model offered here can price single asset, group assets, and other general commodities, virtual products and intangible assets. These kinds of models do not suppose

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equilibrium and no-arbitrage. Of course, there must be some of general assumptions. By these kinds of models, we can evaluate the average profit and risk after an asset is divided into two assets which are independent or correlated, discuss the best relation between the two assets if they are correlated, and analyze the hedging risk for group assets in market and give the optimal ratio between underlying asset and its derivative in hedging. Go a step further, by the models on UPD and MPD, some economic propositions, like "the more the risk is, the larger the possible profit is", "the new asset must be developed continuously in order to acquire the higher sale profits", "the competition results in the raise in cost of asset, and the cooperation results in descending the cost of asset", can be interpreted in analytic way.

It is worth saying that though the univariate partial distribution is the univariate left-truncated normal (Gaussian) distribution with truncation below zero, an approximate closed forms of integral $\int_0^x e^{-\frac{t^2}{2}} dt$ and $\int_0^x e^{-\frac{t-\mu^2}{2\sigma^2}} dt$, which are given in discussions on UPD, have not been given in the discussions on the univariate truncated normal distribution^[9], and the MDP is different in its definition from the current multivariate truncated normal distribution^[10]. So the partial distribution is still called.

2 Definitions and assumptions

2.1 Definitions of partial distribution

Definition 1. (Univariate Partial Distribution, UPD) Let X be a non-negative stochastic variable. It follows the distribution of density

$$f(x) = \begin{cases} e^{-\frac{(x-\mu)^2}{2\sigma^2}} / \int_0^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (1)$$

where $\mu \geq 0$ and $\sigma > 0$. Then, X is regarded as following a univariate Partial Distribution, noted as $X \in P(\mu, \sigma^2)$.

The univariate Partial Distribution is a kind of truncated Gaussian distribution. Partial Distribution is sharper than Gaussian and lognormal distribution as $x \rightarrow \mu$ and μ is less. If μ is big enough, Partial distribution tends to approximate Gaussian distribution.

Univariate Partial Distribution has two basic characteristics. One is that the probability is equal to zero when the variable is less than zero, which is even consistent with non-negative pricing of any asset like capitals, stocks, futures and commodity. The other is that probability is not equal to zero when the variable is zero. This is even corresponding that the prices of some asset may become zero in market, like the price of stock of a company closed down, the price of overdue food or medicine, etc. Both Gaussian and lognormal distributions do not have the two characteristics at the same time. The Levy distribution is better for fitting pricing behavior in financial market now, but it cannot be expressed in the form of an elementary function with the exception of Gaussian and Cauchy distributions. The Cauchy distribution has an infinite variance and thus is rather difficult in its application.

Moreover, we can get two important results in section 3, according to univariate Partial Distribution, which cannot be got by other probability distributions at the same time.

Definition 2. (Multivariate Partial Distribution, MPD for short) if $X_1, \dots, X_n (n \geq 2)$ are all non-negative stochastic variables, and follow the multivariate distribution of density

$$f(x_1, \dots, x_n) = \begin{cases} \frac{e^{-\frac{1}{2|M|} [\sum_{i=1}^n |M_{ii}|(x_i - \mu_i)^2 + \sum_{i,j=1, i \neq j}^n |M_{ij}|(\sigma_i(x_j - \mu_j))]} }{\int_0^\infty \dots \int_0^\infty e^{-\frac{1}{2|M|} [\sum_{i=1}^n |M_{ii}|(x_i - \mu_i)^2 + \sum_{i,j=1, i \neq j}^n |M_{ij}|(\sigma_i(x_j - \mu_j))]} dx_1 \dots dx_n} & 0 \leq x_1, \dots, x_n < \infty \\ 0 & \text{other cases} \end{cases} \quad (2)$$

where, $M = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}$, $\sigma_{ii} = \sigma_i^2$, $\sigma_{ij} = r_{ij}\sigma_i\sigma_j (i \neq j)$, $\sigma_i > 0$, $r_{ij} \leq 1$, $i, j = 1, \dots, n$.

Then we call $(X_1, \dots, X_n)$ to follow n -dimensions Partial Distribution, and note as $\mathbf{X} \in P(\mu', \sigma'^T \mathbf{R} \sigma')$. where, $\mathbf{X} = (X_1, \dots, X_n)^T$, $\mu' = (\mu_1, \dots, \mu_n)^T$, $\sigma' = (\sigma_1, \dots, \sigma_n)^T$, $\mathbf{R} = (r_{ij})_{n \times n}$, $\mu_1, \dots, \mu_n \geq 0$, $\sigma_1, \dots, \sigma_n > 0$, r_{ij} is called the correlation coefficient between X_i and X_j , $r_{ii} = 1$, $i, j = 1, \dots, n$.

We see that the definition of multivariate partial distribution is different from the current multivariate truncated normal distribution^[8, 10].

As a special example of MPD, if the non-negative stochastic variables X and Y follow the multivariate distribution of density:

$$f(x, y) = \begin{cases} \frac{e^{-\frac{1}{2(1-r^2)}[\frac{x-\mu_1}{\sigma_1} - 2r(\frac{x-\mu_1}{\sigma_1} + \frac{y-\mu_2}{\sigma_2}) + \frac{y-\mu_2}{\sigma_2}]^2}}{\int_0^\infty \int_0^\infty e^{-\frac{1}{2(1-r^2)}[\frac{x-\mu_1}{\sigma_1} - 2r(\frac{x-\mu_1}{\sigma_1} + \frac{y-\mu_2}{\sigma_2}) + \frac{y-\mu_2}{\sigma_2}]^2} dx dy} & 0 \leq x, y < \infty \\ 0 & x < 0 \text{ or } y < 0 \end{cases} \quad (3)$$

then, (X, Y) is said to have the bivariate Partial Distribution, and note as $(X, Y) \in P(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$, where, the constants $\mu_1, \mu_2 \geq 0$, $\sigma_1, \sigma_2 > 0$, $-1 < r < 1$. When $r = 1$, we know, according to the result about the correlation coefficient^[7], X is correlating with Y in linearity and on probability 1, i.e., the probability $P(Y = aX + b) = 1$, where, $a > 0$ if $r = 1$, and $a < 0$ if $r = -1$.

2.2 Estimating the parameters in MPD

Take the bivariate PD for example. The sample series of stochastic variable 1 and variable 2 are separately $x_{11}, x_{12}, \dots, x_{1n}$ and $x_{21}, x_{22}, \dots, x_{2n}$ ($x_{1i}, x_{2i} > 0, i = 1, \dots, n$).

With the modified maximum likelihood estimation (Dai, et al 2004), we can obtain $\hat{\mu}_k$ (estimated value of μ_k) and $\hat{\sigma}_k$ (estimated value of σ_k), $k = 1, 2$. Thus, the correlation coefficient can be estimated as expression 4:

$$\hat{r}_{1,2} = \frac{\sum_{i=1}^n (x_{1i} - \hat{\mu}_1)(x_{2i} - \hat{\mu}_2)}{\sqrt{\sum_{i=1}^n (x_{1i} - \hat{\mu}_1)^2 \cdot \sum_{j=1}^n (x_{2j} - \hat{\mu}_2)^2}} \quad (4)$$

2.3 Assumptions about the price of asset

Here we present some basic concepts and basic assumptions as a theoretical framework for discussion in this paper.

Definition 3. Asset price is made up of cost price and market price. Cost price is total payment for holding an asset and market price is the current exchange price of an asset in the market.

As to a financial or capital asset, cost price involves paying for buying asset, rate paying, interest paying and other payments concerned.

In the case of a consumed asset (like commodities), cost price includes payments for buying raw and processed materials, production, transportation, advertisement, rate-paying and other payments concerned.

Assumption 1. we suppose

(1) Asset prices (cost price and market price) fluctuates with time. Any price and fluctuation range (i.e., variance or standard variance) of prices are non-negative.

(2) Both cost price and fluctuation of cost price of an asset are basic elements to influence market prices of an asset and market prices come into being as market exchange occurs.

(3) The chances that market price of an asset is much lower or higher than its cost price are rather small.

This paper uses some basic notations:

μ : cost price of asset (capital asset or commodity asset), $\mu \geq 0$.

σ : fluctuation range of cost price, i.e., standard variance of cost price, $\sigma > 0$.

X : market price of an asset, $0 \leq X < \infty$.

r : correlation coefficient.

Assumption 2. If cost price and market price of an asset satisfy the assumption 1, then we suppose that its market price follows PD, i.e., $X \in P(\mu, \sigma^2)$.

If asset price changes along with time, then the market price of the asset $X(t) \in P(\mu(t), \sigma^2(t))$ (for any time $t \geq 0$), where $\mu(t)$ is cost price of the asset, and $\sigma(t)$ stands for standard variance of the cost price. If $\mu(t)$ and $\sigma(t)$ are continuous as to time t , the market price $X(t)$ will be continuous. If $\mu(t)$ or $\sigma(t)$ is dispersed as to time t , the market price $X(t)$ would be dispersed.

An asset might be regarded as a capital or a commodity, and $X(t) \in P(\mu(t), \sigma^2(t))$ as the asset or the market price of the asset.

3 The pricing models based on partial distribution

3.1 Basic results of partial distribution

According to the works^[2], we can get two basic results about Partial Distribution as follows:

Theorem 1. For any $x \in [0, \infty]$, the following formulas are approximately correct:

$$(1) \int_0^x e^{-\frac{t^2}{2}} dt = \sqrt{\frac{\pi}{2}} (1 - e^{-\frac{2}{\pi} x^2}).$$

$$(2) \int_0^x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sqrt{\frac{\pi}{2}} \sigma \left(\left(\sqrt{1 - e^{-\frac{2}{\pi} (\frac{\mu}{\sigma})^2}} + \operatorname{sgn}(x - \mu) \sqrt{1 - e^{-\frac{2}{\pi} (\frac{x-\mu}{\sigma})^2}} \right) \right).$$

$$\text{Where, } \operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}.$$

Theorem 2. Let X follow the PD, i.e., $X \in P(\mu, \sigma^2)$, thus

(1) The expected value $E(X)$ of X is

$$E(X) = \mu + \sqrt{\frac{2}{\pi}} \frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{1 - e^{-\frac{2}{\pi} (\frac{\mu}{\sigma})^2} + 1}} \quad (5)$$

(2) The variance $D(X)$ of X is

$$D(X) = \sigma^2 + E(X)[\mu - E(X)] \quad (6)$$

According to expression (1), theorem 1 and theorem 2, Partial Distribution also has these following properties:

$$(1) f(0) = \sqrt{\frac{2}{\pi}} \frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{1 - e^{-\frac{2}{\pi} (\frac{\mu}{\sigma})^2} + 1}}, \text{ which implies that the probability is non-zero at } x = 0.$$

$$(2) \text{ The expectation } E(X) = \mu + \sqrt{\frac{2}{\pi}} \frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{1 - e^{-\frac{2}{\pi} (\frac{\mu}{\sigma})^2} + 1}}, \text{ which means } E(X) > \mu.$$

$$(3) \text{ The variance } D(X) = 2 + E(X)(\mu - E(X)), \text{ which implies } D(X) < 2.$$

3.2 Pricing model for single asset

If the market price of an asset follows PD, i.e., $X \in P(\mu, \sigma^2)$, according to theorem 2, then,

(1) The average market price of the asset would be evaluated as

$$E(X) = \mu + \sqrt{\frac{2}{\pi}} \frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{1 - e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2} + 1}} \quad (7)$$

where, $R(X) = \sqrt{\frac{2}{\pi}} \frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{1 + \sqrt{1 - e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}}}$ is the average sale profit of the asset.

With $E(X) > \mu$, this means that the average market price is higher than the cost price of asset.

(2) The risk of market price of the asset can be evaluated as

$$D(X) = \sigma^2 + E(X)[\mu - E(X)] \quad (8)$$

Because of $D(X) < 2$, this means the trading risk is less than the cost risk of asset.

Thus, two economic propositions could be obtained according to theorem 2:

Proposition 1. *The more the risk is, the larger the possible profit would be.*

We can get the average sale profit from the expression (7) as the following:

$$R(X) = \sqrt{\frac{2}{\pi}} \frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{1 + \sqrt{1 - e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}}}$$

We can see that if the cost price μ is fixed, then the more the cost price risk σ , the larger the average sale profit $R(X)$.

Proposition 2. *The new asset will result in a higher sale margin.*

Suppose that X_i is the market price of an asset in i th trading and $X_i \in P[E(X_{i-1}), D(X_{i-1})]$, $E(X_0) = \mu$, $D(X_0) = \sigma^2$, $i = 1, \dots$, we have, from (7) and (8), $E(X_i) > E(X_{i-1}) > \dots > E(X_0)$, $R(X_i) < R(X_{i-1}) < \dots < R(X_0)$ and $D(X_i) < D(X_{i-1}) < \dots < D(X_0)$.

Obviously, if the economic environment and asset quality do not change, the average trading price of an asset would be increasing, the average sale profit will become lower and the price risk of the asset trading will go down, while the ranges within which the average trading price, the average sale profit and the price risk change will become narrower. Therefore, the new asset needs developing constantly with the aim of making higher sale profits.

3.3 Pricing model for group assets

From the related works in references [4], we could get theorems 3 and 4.

Theorem 3. *If X_1 and X_2 are both stochastic variables and follow the bivariate Partial Distribution, i.e., $(X_1, X_2) \in P(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$, thus*

$$(1) \int_0^\infty \int_0^\infty e^{-\frac{1}{2(1-r^2)} \left[\left(\frac{u-\mu_1}{\sigma_1} \right)^2 - 2r \left(\frac{u-\mu_1}{\sigma_1} \right) \left(\frac{v-\mu_2}{\sigma_2} \right) + \left(\frac{v-\mu_2}{\sigma_2} \right)^2 \right]} dudv$$

$$= \frac{\pi}{2} \sigma_1 \sigma_2 (1-r^2) e^{\frac{r^2}{1-r^2}} \left(1 + \sqrt{1 - e^{-\frac{2}{\pi} \left(\frac{\mu_1 + r\sigma_1}{\sigma_1 \sqrt{1-r^2}} \right)^2}} \right) \left(1 + \sqrt{1 - e^{-\frac{2}{\pi} \left(\frac{\mu_2 + r\sigma_2}{\sigma_2 \sqrt{1-r^2}} \right)^2}} \right)$$

$$(2) \int_0^{x_1} \int_0^{x_2} f(u, v) dudv = \frac{A_1(x_1) A_2(x_2)}{\left(1 + \sqrt{1 - e^{-\frac{2}{\pi} \left(\frac{\mu_1 + r\sigma_1}{\sigma_1 \sqrt{1-r^2}} \right)^2}} \right) \left(1 + \sqrt{1 - e^{-\frac{2}{\pi} \left(\frac{\mu_2 + r\sigma_2}{\sigma_2 \sqrt{1-r^2}} \right)^2}} \right)}, 0 \leq x_1, x_2 < \infty$$

where, $A_i(t) = \sqrt{1 - e^{-\frac{2}{\pi} \left(\frac{\mu_i + r\sigma_i}{\sigma_i \sqrt{1-r^2}} \right)^2}} + \text{sgn}(t - \mu_i) \sqrt{1 - e^{-\frac{2}{\pi} \left(\frac{t - (\mu_i + r\sigma_i)}{\sigma_i \sqrt{1-r^2}} \right)^2}}$, the meaning of $\text{sgn}(t)$ is similar to theorem 1, $i = 1, 2$.

If denoting

$$f_{1r}(x) = \int_0^\infty f(x, y) dy = \begin{cases} e^{-\frac{[x-(\mu_1+r\sigma_1)]^2}{2\sigma_1^2(1-r^2)}} / \int_0^\infty e^{-\frac{[u-(\mu_1+r\sigma_1)]^2}{2\sigma_1^2(1-r^2)}} du & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and

$$f_{2r}(x) = \int_0^\infty f(y, x) dy = \begin{cases} e^{-\frac{[x-(\mu_2+r\sigma_2)]^2}{2\sigma_2^2(1-r^2)}} / \int_0^\infty e^{-\frac{[u-(\mu_2+r\sigma_2)]^2}{2\sigma_2^2(1-r^2)}} du & x \geq 0 \\ 0 & x < 0 \end{cases}$$

thus have

Theorem 4. If X_1 and X_2 are both stochastic variables and follow bivariate Partial Distribution, i.e., $(X_1, X_2) \in P(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$, thus

(1) The expected values of each stochastic variable are

$$E_r(X_i) = \int_0^\infty x f_{ir}(x) dx = \mu_i + r\sigma_i + \sqrt{\frac{2}{\pi} \frac{\sigma_i \sqrt{1-r^2} e^{-\frac{1}{2} \left(\frac{\mu_i+r\sigma_i}{\sigma_i \sqrt{1-r^2}} \right)^2}}{1 + \sqrt{1 - e^{-\frac{2}{\pi} \left(\frac{\mu_i+r\sigma_i}{\sigma_i \sqrt{1-r^2}} \right)^2}}}} \quad (9)$$

where $i = 1, 2$.

Obviously, $E_r(X_i) = E(X_i)$ if $r = 0$.

(2) The variances of each stochastic variable are

$$D_r(X_i) = \int_0^\infty [x - E_r(X_i)]^2 f_{ir}(x) dx = \sigma_i^2(1-r^2) + E_r(X_i)[\mu_i + r\sigma_i - E_r(X_i)] \quad (10)$$

where $i = 1, 2$.

We may validate that $D_r(X_i) = D(X_i)$ ($i = 1, 2$) if $r = 0$.

If X_1 and X_2 in theorems 3 and 4 are two assets, then expression (9) are respectively the average market prices of X_1 and X_2 in group or correlation meaning, and expression (10) are separately the risks of market price of X_1 and X_2 in group or correlation meaning. There exist two important differences between the $E(X)$ in expression (7) and $E_r(X_i)$ in expression (9), and the $D(X)$ in expression (8) and $D_r(X_i)$ in expression (10):

Difference 1. The μ is replaced by the $\mu_1 + r\sigma_1$ or $\mu_2 + r\sigma_2$.

Difference 2. The σ is replaced by $\sigma_1 \sqrt{1-r^2}$ or $\sigma_2 \sqrt{1-r^2}$.

The correlation coefficient $r > 0$ means that cost prices of two assets are positively correlated. According to difference 1, if $r > 0$ is regarded as two assets requiring the same cost resource, then the two assets will compete for the same cost resource, as a result the cost prices of two assets will become higher. In contrast, $r < 0$ means that cost prices of two assets are negatively correlated. If $r < 0$ is regarded as two assets requiring the reverse cost resource, then the two assets will use the different cost resource; in this case, the cost prices of two assets become lower.

Whether $r > 0$ or $r < 0$, the risk of cost prices of two assets will become lower as difference 2 shows, which means that competition and cooperation tend to reduce the price risk of assets.

So we have the following economic propositions:

Proposition 3. The competition for resources results in higher cost price while complementarity in resource tends to lower cost price.

The cost prices of assets (especially to product and commodity) are closely related to the group of cost resources. The degree to which the group of cost resources influences the cost prices of assets can be valued by r in expressions (9).

We get the expressions of $E_r(X_i)$ from expressions (9) (the average sale price of correlated assets in market) and average sale profit for two correlated assets:

$$R_r(X_i) = \sqrt{\frac{2}{\pi}} \frac{\sigma_i \sqrt{1-r^2} e^{-\frac{1}{2} \left(\frac{\mu_i + r\sigma_i}{\sigma_i \sqrt{1-r^2}} \right)^2}}{1 + \sqrt{1 - e^{-\frac{2}{\pi} \left(\frac{\mu_i + r\sigma_i}{\sigma_i \sqrt{1-r^2}} \right)^2}}}, i = 1, 2.$$

With expression (10) we can value the risks of market prices for two correlated assets.

Proposition 4. When an asset (whose market price is $X \in P(\mu, \sigma^2)$) divide into two kinds of assets, their market prices are X_1 and X_2 .

(1) When the two assets are independent of each other, namely $X_1 \in P(\mu_1, \sigma_1^2)$ and $X_2 \in P(\mu_2, \sigma_2^2)$ (where, $\mu = \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$), we call them to satisfy uniform division on independence (uniform division for short) if following inequations are tenable:

$$\frac{\mu_i}{\sigma_i} < \frac{\mu}{\sigma} (i = 1, 2)$$

at this time, the total average market price and profit on sale will get higher.

(2) If the two assets are correlated with each another, namely $(X_1, X_2) \in P(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$, $\mu, \mu_1, \mu_2, \sigma, \sigma_1, \sigma_2 > 0, |r| \leq 1, \mu = \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$, and satisfy the uniform division on correlation

$$\frac{\mu_i + r\sigma_i}{\sqrt{1-r^2}\sigma_i} < \frac{\mu}{\sigma} (i = 1, 2)$$

where $|r| < 1$, thus the total average market price and profit on sale will get higher. If the cost prices of two assets are correlated with one another in linearity on probability being 1, i.e., $|r| = 1$, then the total average profit on sale will be zero.

(3) The total risk in market prices of assets will become lower whether the two assets are correlated or not.

The proof of proposition 4 is given in the appendix.

The uniform divisions mentioned previously are sufficient conditions. In fact, some results in proposition 4 may be right within a wider scope. A related analysis indicates that there are a number of r_{max} , the maximum of the total average profit, in the field $(-1, 0)$. Usually, $r_{max} \in (-0.2, -0.5)$. r_{max} can also be approximately obtained by computing the zero value of the derivative of $R_r(X_i)$ in the numeric way. This conclusion is useful in providing the way of dividing an asset and to control the total average profit of divided assets.

4 Risk management models for group assets based on mpd

4.1 The risk analysis model for group assets

If we have two assets X_1 and X_2 , and $(X_1, X_2) \in P(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$, the expressions (10) are separately computing formulas to evaluate the price risk of each asset in the meaning of correlation.

The computing formulas to evaluate the price risk of each asset on its independence are separately

$$\bar{D}(X_i) = \int_0^\infty [x - E(X_i)]^2 f_{ir}(x) dx = D_r(X_i) + [E_r(X_i) - E(X_i)]^2 \quad (11)$$

where, $E(X_i)$ comes from expression (7), $E_r(X_i)$ comes from expressions (9), and $D_r(X_i)$ comes from expressions (10), $i = 1, 2$.

The expressions (10) mean the price risk of asset X_1 and X_2 relating to the average market price before they are combined. $\bar{D}(X_1)$ and $\bar{D}(X_2)$ are called separately the independent risk of X_1 and X_2 .

Definition 4. If both X_1 and X_2 are stochastic variables and follow 2-dimensions Partial Distribution, i.e., $(X_1, X_2) \in P(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$, then we define the correlated coefficient between X_1 and X_2 as

$$\rho = \text{sgn}(r) \frac{[E_r(X_1) - E(X_1)] \cdot [E_r(X_2) - E(X_2)]}{\sqrt{\bar{D}(X_1) \cdot \bar{D}(X_2)}}$$

where, the function $\text{sgn}(x)$ is the same as in theorem 1.

If X is an underlying asset, Y is a derivative asset of X . Because there are always the differences in the price behaviors of underlying asset and its derivative asset, there should be an optimal ratio between underlying asset and its derivative in hedging. The optimal ratio is related to the correlation coefficient between underlying asset and its derivative asset. Thus, the holding group of X and Y should be $\gamma X + \alpha Y$. Where, γ is the correlation coefficient between the cost price of X and Y , α is the holding ratio of derivative asset Y in hedging. γX has a coequal measurement with Y in the meaning of price behavior. Also, α means the holding direction, $\alpha > 0$ means holding Y in the same direction with X , $\alpha < 0$ means holding Y in the reverse direction with X . Thus, the evaluating model for integrated risk on hedging is as follows:

$$Q^2 = \varphi[\bar{D}(X) + \bar{D}(Y)] + \psi \bar{D}(rX + \alpha Y) \quad \text{or} \quad Q = \varphi \sqrt{\bar{D}(X) + \bar{D}(Y)} + \psi \sqrt{\bar{D}(rX + \alpha Y)} \quad (12)$$

where, $\phi + \psi = 1$, $\phi, \psi > 0$. In general, $\psi = |r|^s$, $0 < s < \infty$. Specially, we have $s = 1$ or $s = 2$.

In expression (12), $\sqrt{\bar{D}(X) + \bar{D}(Y)}$ can be applied to evaluate the price risk for X and Y in independence, $\sqrt{\bar{D}(rX + \alpha Y)}$ can be applied to evaluate the price risk for hedging group of X and Y .

For convenience, we call Q in (12) the integrated risk, $\sqrt{\bar{D}(X) + \bar{D}(Y)}$ the independent risk, and $q = \sqrt{\bar{D}(rX + \alpha Y)}$ the hedging risk.

From the computing on the expression (12) and (11), we could know:

When $|r| = 1$, $\psi = 1$ and $\phi = 0$, thus, $Q = q = 0$ if putting $\alpha = -r \frac{[E_r(X) - E(X)]}{[E_r(Y) - E(Y)]}$, i.e., the both independent risk and hedging risk reach their minimum, i.e. all the integrated risk and the hedging risk reach zero.

When $r = 0$, i.e., X and Y are completely independent one with another, then putting $\alpha = 0$, so hedging risk is equal to zero. In this case, the assets hedged are not related one with another, so that hedging risk is zero, but the integrated risk reaches maximum. In fact, in expression (12), $\psi = 0$ and $\phi = 1$ when $r = 0$, these mean that the hedging risk is equal to zero, and independent risk reaches its maximum. As a result, establishing a hedging group for assets can reduce the integrated risk effectively.

4.2 Hedging risk analysis about one derivative asset to one underlying asset

Based on (12), and denoting $I = \bar{D}(rX + \alpha Y)$, then

$$\begin{aligned} I &= E_r\{[r(X - E(X))]^2\} + 2r\alpha E_r[(X - E(X))(Y - E(Y))] + \alpha^2 E_r\{[Y - E(Y)]^2\} \\ &= r^2 \bar{D}(X) + 2r\alpha [E_r(X) - E(X)][E_r(Y) - E(Y)] + \alpha^2 \bar{D}(Y) \end{aligned}$$

Let $\frac{dI}{d\alpha} = \frac{d\bar{D}(rX + \alpha Y)}{d\alpha} = 0$, we obtain

$$\alpha = -r \frac{[E_r(X) - E(X)][E_r(Y) - E(Y)]}{\bar{D}(Y)} \quad \text{or} \quad \alpha = -|r|\rho \sqrt{\frac{\bar{D}(X)}{\bar{D}(Y)}} \quad (13)$$

At this time the hedging risk reaches a minimum. It is

$$q = \sqrt{I_{\min}} = \sqrt{\frac{r^2}{\bar{D}(Y)} \{\bar{D}(X) \cdot \bar{D}(Y) - [E_r(X) - E(X)]^2 [E_r(Y) - E(Y)]^2\}} \quad (14)$$

Because pricing of underlying asset are not totally consistent with that of its derivative asset sometimes, the optimal ratio between underlying asset and its derivative in a hedging process tends to be existing, which is given by the expression (13).

We know, from (13), when $r > 0$, i.e. the underlying asset X is positively correlated with the derivative asset Y , the minimum risk group is holding in the reverse direction, which means buying underlying asset and selling the derivative asset at the same time, or vice versa. When $r < 0$, i.e. the underlying asset X is negatively correlated with the derivative asset Y , the minimum risk group is holding in the same direction; it is desirable to buy the underlying asset and derivative asset at the same time, or vice versa.

If $r = 0$, we see $\alpha = r = 0$ from (13), i.e. $rX + \alpha Y = 0$ and $\bar{D}(rX + \alpha Y) = 0$. This implies that we need not make a hedging on these two assets.

If $|r| = 1$, then X and Y are in a linear relationship and probability is 1. If $Y = cX + b$, the hedging risk is

$$\bar{D}(rX + \alpha Y) = \bar{D}[(r + \alpha c)X + \alpha b]$$

If $r + \alpha c = 0$, $\bar{D}(rX + \alpha Y) = 0$, hedging risk reaches minimum. At this time, $\alpha = -r/c$, i.e., $\alpha = -1/c$ when $r = 1$; $\alpha = 1/c$ when $r = -1$.

We can see that the minimum risk group always changes with r , thus to adjust in time in accordance with the real price of asset is necessary.

4.3 Management models for hedging risk about one derivative asset to more underlying assets

Let X be an underlying asset, $Y_1, \dots, Y_n$ be the derivative assets on X , and r_i be the correlation coefficient between X and Y_i , $i = 1, \dots, n$.

Model 1. The Risk management for hedging about one derivative asset to multiple underlying assets which are not correlated.

If denoting: $r = \sum_{i=1}^n r_i$, $\rho \neq 0$. And we know

$$E_r \{ [Y_i - E(Y_i)][Y_j - E(Y_j)] \} = \begin{cases} \bar{D}(Y_i) & i = j \\ [E_r(Y_i) - E(Y_i)][E_r(Y_j) - E(Y_j)] & i \neq j \end{cases}$$

Then, the holding group between X and $Y_1, \dots, Y_n$ is $rX + \sum_{i=1}^n \alpha_i Y_i$, α_i is the holding ratio of derivative asset Y_i in hedging. If Y_i ($i = 1, \dots, n$) are not correlated, have

$$\begin{aligned} I &= \bar{D} \left(rX + \sum_{i=1}^n \alpha_i Y_i \right) \\ &= r^2 \bar{D}(X) + 2r [E_r(X) - E(X)] \sum_{i=1}^n \alpha_i [E_r(Y_i) - E(Y_i)] + \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j E_r \{ [Y_i - E(Y_i)][Y_j - E(Y_j)] \} \end{aligned}$$

Let $\frac{\partial I}{\partial \alpha_k} = 0$ ($k = 1, \dots, n$), we obtain

$$2 \sum_{j=1}^n \alpha_j E_r \{ [Y_k - E(Y_k)][Y_j - E(Y_j)] \} + 2r [E_r(X) - E(X)][E_r(Y_k) - E(Y_k)] = 0,$$

i.e.

$$B\alpha + rc = 0 \tag{15}$$

where, $\alpha = (\alpha_1, \dots, \alpha_n)^T$, $c = (c_1, \dots, c_n)^T$, $c_i = [E_r(X) - E(X)][E_r(Y_i) - E(Y_i)]$, $B = (b_{ij})_{n \times n}$,

$$b_{ij} = E_r \{ [Y_i - E(Y_i)][Y_j - E(Y_j)] \}, i, j = 1, \dots, n.$$

The solution of equations group (15) is $\alpha_1, \dots, \alpha_n$. And α_i is the optimal holding ratio of Y_i , $i = 1, \dots, n$.

Model 2. Risk management for hedging about one derivative asset to correlated multiple underlying assets.

If $Y = (Y_1, \dots, Y_n)^T$ is the vector of derivative assets, $Q = (q_{ij})_{n \times n}$ is the matrix of correlation coefficients on $Y \times Y$, where $q_{ii} = 1, q_{ij} = q_{ji}, i, j = 1, \dots, n$. $\alpha = (\alpha_1, \dots, \alpha_n)^T$ is the vector of holding ratios to $Y = (Y_1, \dots, Y_n)^T$. Thus, the hedging risk of assets group is

$$I = \bar{D}(rX + \alpha^T QY)$$

i.e.,

$$I = r^2 \bar{D}(X) + 2r[E_r(X) - E(X)] \sum_{i,j=1}^n \alpha_i q_{ij} [E_r(Y_j) - E(Y_j)] + \sum_{i,j=1}^n b_{ij} E_r\{[Y_i - E(Y_i)][Y_j - E(Y_j)]\},$$

where, $b_{ij} = \sum_{s=1}^n \sum_{t=1}^n \alpha_s \alpha_t q_{is} q_{tj}$. Let $\frac{\partial I}{\partial \alpha_k} = 0 (k = 1, \dots, n)$, we obtain the equations group

$$B\alpha + rc = 0 \quad (16)$$

where, $B = (b_{ij})_{n \times n}, b_{ij} = \sum_{s=1}^n \sum_{t=1}^n q_{is} q_{tj} E_r\{[Y_s - E(Y_s)][Y_t - E(Y_t)]\},$

$$c = (c_1, \dots, c_n)^T, c_i = [E_r(X) - E(X)] \sum_{j=1}^n q_{ij} [E_r(Y_j) - E(Y_j)], i, j = 1, \dots, n.$$

The solution of equations group (16) is $\alpha_1, \dots, \alpha_n$. And α_i is the optimal holding ratio of $Y_i, i = 1, \dots, n$.

4.4 Hedging risk management for group assets

Suppose that we have m underlying assets like $X_1, X_2, \dots, X_m$ and n derivative assets like $Y_1, \dots, Y_n$. Denoting: $\mathbf{X} = (X_1, X_2, \dots, X_m)^T, \mathbf{Y} = (Y_1, Y_2, \dots, Y_n)^T$, and $\mathbf{S} = (s_{ij})_{m \times m}$ is the matrix of correlation coefficients on $X \times X, \mathbf{L} = (l_{ij})_{n \times n}$, is the matrix of correlation coefficients on $Y \times Y, \mathbf{R} = (r_{ij})_{m \times n}$ is the matrix of correlation coefficients on $X \times Y$, where, $s_{ii} = 1, s_{ij} = s_{ji}, i, j = 1, \dots, m; l_{jj} = 1, l_{ij} = l_{ji}, i, j = 1, \dots, n; 0 \leq |r_{ij}| \leq 1, i = 1, \dots, m, j = 1, \dots, n$.

Suppose $\mathbf{X} \in P(\mu_X, \sigma_X^T S \sigma_X), \mathbf{Y} \in P(\mu_Y, \sigma_Y^T L \sigma_Y), (\mathbf{X}, \mathbf{Y}) \in P(\mu_X, \mu_Y, \sigma_X^T S \sigma_X, \sigma_Y^T L \sigma_Y, R)$, where, $\mu'_X = (\mu_{1X}, \dots, \mu_{mX})^T, \mu'_Y = (\mu_{1Y}, \dots, \mu_{nY})^T, \sigma_X = (\sigma_{1X}, \dots, \sigma_{mX})^T, \sigma_Y = (\sigma_{1Y}, \dots, \sigma_{nY})^T, \mathbf{R} = (r_{ij})_{n \times n}, r_{ii} = 1, i = 1, \dots, n$.

Denoting: $r = (r_1, \dots, r_m)^T$, where, $r_k = \frac{1}{n} \sum_{j=1}^n r_{kj}, \alpha' = (\alpha_1, \dots, \alpha_n)^T$ is the vector of holding ratios to $Y = (Y_1, \dots, Y_n)^T$. Thus, the hedging risk of assets group $X = (X_1, \dots, X_m)^T$ and $Y = (Y_1, \dots, Y_n)^T$ is

$$I = D(r^T X + \alpha'^T Y) \quad (17)$$

Let $\frac{\partial I}{\partial \alpha_k} = 0 (k = 1, \dots, n)$, and seeking the solution, we obtain

$$\alpha' = -\mathbf{C}^{-1} \mathbf{b} \quad (18)$$

where, $\mathbf{C} = (c_{ij})_{n \times n}, c_{ij} = E_r\{[Y_i - E(Y_i)] \cdot [Y_j - E(Y_j)]\}, \mathbf{b} = (b_1, \dots, b_n)^T, b_i = \sum_{t=1}^m r_t E_r\{[Y_i - E(Y_i)][X_t - E(X_t)]\}$, the vector α' in (18) is the optimal holding ratios of $\mathbf{Y}$.

Replacing α' in (17) by the one in (18), we get the minimum hedging risk as follows

$$q = \sqrt{I_{\min}} = \sqrt{D(r^T X - \mathbf{C}^{-1} \mathbf{b} Y)} \quad (19)$$

And the integrated risk is:

$$Q = \varphi \sqrt{\sum_{i=1}^m \bar{D}(X_i) + \sum_{i=1}^n \bar{D}(Y_i) + \psi q} \quad (20)$$

where, $\phi + \psi = 1$, $\phi, \psi > 0$. In general, $\psi = \left(\frac{\sum_{i=1}^m \sum_{j=1}^n |r_{ij}|}{m \times n} \right)^s$, $0 < s < \infty$. Specially, $s = 1$ or $s = 2$.

5 Empirical research

In this part, we will take the Stocks Index as an asset. The relevant data of Stocks Index is sourced at <http://www.stockstar.com.cn..> The parameter μ and σ in PD or MPD are all estimated through modified maximum likelihood estimation^[3, 5]. In the following statistic testing, we adopt the significance level $\beta = 0.001$, i.e. the fiducial level $1 - \beta = 0.999$.

5.1 Some examples of risk analysis of one derivative asset to one underlying asset

5.1.1 Risk analysis of the integrated index of shanghai stock exchange in china (IISS)

Samples: closed prices of IISS.

Time field of Sampling: trading days in the field of April 12, 2004- June 17, 2005.

The estimated results of parameters are:

$$\mu = 1329.525414, \sigma = 153.5404930.$$

Because the number of divided fields on samples is $n = 40$, and the fiducial test: $\chi^2 = 55.58249506 < \chi^2(n - 2 - 1) = \chi^2(37) = 68.883$, that PD is applicable to IISS can be accepted; i.e., $X \in P(1329.525414, 153.5404930^2)$ in the time field of April 12, 2004- June 17, 2005.

From expression (7), $E(X) = 1329.525414 + R(X)$, where $R(X) = 0.3201535603 \times 10^{-14}$, we can see that the trading profit is little on average.

The trading data of the futures of IISS is not available due to the fact that up to now Shanghai Stock Exchange has not set up the index futures of IISS. In this case, we may suppose that the Futures of IISS is $F \in P(1329.525414e^{\delta(T-t)}, [153.5404930e^{\delta(T-t)}]^2)$ according to reference [6], where, $\delta = \gamma - y$, γ is the risk-free rate, y stands for the convenience yields.

If $e^{\delta(T-t)} = e^{0.01} = 1.010050167$, $F \in P(1342.887366, 155.083606^2)$, and if r is the correlation coefficient of X and F , according to formula (13), (14) and (12), then we can work out the results respectively with IISS: the optimal ratio of holding F corresponding to X , α , the minimum hedging risk q and the minimum integrated risk Q ($s = 2$). When $r \in (-1, 1)$, moving curves of α , q and Q are separately shown in (a), (b) and (c) of Fig. 1.

From (a) in Fig. 1, we can see that α changes from 1 to -1 as the correlation coefficient r changes from -1 to 1, which indicates that derivative asset holding should be in the reverse direction with its underlying asset in order to minimize integrated risk, and that the holding ratio to the derivative asset always changes with the correlation coefficient.

Also, we can see from (b) and (c) in Fig. 1 that the risk of group assets in financial market includes two parts — independent risk and hedging risk, both of which makes up integrated risk. When $|\rho|$ is near to zero, the hedging risk becomes smaller (see also (b) in Fig. 1). The reason is that hedging effect is not very good owing to the bad relativity between the two hedging assets. Meanwhile, the independent risk of two assets is so big that the integrated risk increases (see also the (c) in Fig. 1).

By (12) and other formula in section 4.1, we get $Q = q = 0$ when $|r| = 1$. This means that both the hedging risk and the integrated risk reach their minimum. Figure 1 does not show all results, because $|r| = 1$ is a strange point in formula (12) and other formulas in section 5.1, but, $\lim_{|r| \rightarrow 0} Q = \lim_{|r| \rightarrow 0} q = 0$.

Fig. 1. The risk structure and moving curves about IISS

In addition, (b) and (c) in Fig. 1 show that the hedging risk reaches zero when $r = 0$ i.e. $\alpha = 0$, but the integrated risk reaches its maximum. Therefore, to reduce the integrated risk, it is necessary to establish the hedging group of assets.

Despite the fact that real data of futures have not obtained, the accuracy of empirical results is not affected. So it is with the following empirical analysis.

5.1.2 The risk analysis about the components index of shenzhen stock exchange in china (cisz)

Samples: closed prices of CISZ.

Time field of Samples: trading days in the field of April 12, 2004- June 17, 2005.

The estimated results of parameters are:

$$\mu = 3245.116068, \sigma = 242.5451888.$$

Because the number of divided fields on samples is $n = 33$, and the fiducial test: $\chi^2 = 57.69646723 < \chi^2(n-2-1) = \chi^2(30) = 59.703$, we accept that CISZ follows PD, i.e., $X \in P(3245.116068, 242.5451888^2)$ in the time field of 2004.04.16-2005.06.17.

From expression (7), $E(X) = 3245.116068 + R(X)$, where $R(X) = 0.5470348045 \times 10^{-9427}$. So we see that there is almost no the trading profit in the average meaning.

Since Shenzhen Stock Exchange has not set up the index futures of CISZ so far, no trading data of the futures of CISZ is available. In this case let us suppose that the Futures of IISS is F , and $F \in P(3245.116068e^{\delta(T-t)}, [242.5451888e^{\delta(T-t)}]^2)$ according to reference [3], where, $\delta = \gamma - y$, γ is the risk-free rate, y is the convenience yields.

If $e^{\delta(T-t)} = e^{0.01} = 1.01005$, $F \in P(3277.730026, 244.982809^2)$, and if ρ is the correlation coefficient of X and F , according to formula (13), (14) and (12), then we can compute separately with CISZ. The optimal ratio of holding F corresponds to X , α , the minimum hedging risk q and the integrated risk Q ($s = 2$). When $r \in (-1, 1)$, moving curves of α, q and Q are drawn separately in (a), (b) and (c) of figure 2. Then we have the same discussions as in section 5.1.1. From the two examples above, we draw some conclusions as the following:

(1) The integrated risk gets smaller and smaller as $|r|$ is near to 1, but the integrated risk never clear away if $|r| \neq 1$. Thus it is important for us to know the optimal holding ratio.

(2) The higher the degree of correlation between the underlying asset and its derivative asset is, the lower the integrated risk and hedging risk would be. This tallies with the reality in financial market. It is proper that we choose the derivative asset which correlate highly with the underlying asset if making a hedging business in financial market. If like that, we will control the market furthest.

The two results above are important especially for the large scale of hedging business across the finance markets.

We have the same discussions as in section 5.1.1.

Fig. 2. The risk structure and moving curves about CISZ

5.2 Exemplification of risk analysis of group derivative assets to group underlying assets

In this part, we will exemplify IISS and CISZ as the underlying assets, and the futures of IISS and CISZ as the derivative assets. The notations are in accordance with those in section 4.4.

For the sake of elucidation, we suppose that the matrix of the correlation coefficient of the underlying assets is equal to zero, i.e., $\mathbf{S} = 0$, the matrix of the correlation coefficient of the derivative assets is equal to zero, i.e., $\mathbf{L} = 0$, and the matrix of the correlation coefficient of $\mathbf{X} \times \mathbf{Y}$ is

$$R = \begin{pmatrix} r_1 & r_1 \\ r_2 & r_2 \end{pmatrix} X$$

Thus, $(\mathbf{X}, \mathbf{Y}) \in P(\mu_X, \mu_Y, \sigma_X^T \sigma_X, \sigma_Y^T \sigma_Y, R)$. According to expression (14), we have $I = D(r^T X + \alpha^T Y)$, where, $\mathbf{r} = (r_1, r_2)^T$, $\alpha = (\alpha_1, \alpha_2)^T$. And from (18), have

$$\alpha_1 = -\frac{\begin{vmatrix} b_1 & c_{12} \\ b_2 & c_{22} \end{vmatrix}}{\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}}, \quad \alpha_2 = -\frac{\begin{vmatrix} c_{11} & b_1 \\ c_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}}$$

where,

$$\begin{aligned} c_{ij} &= E_R\{[Y_i - E(Y_i)] \cdot [Y_j - E(Y_j)]\}, \\ c_{ii} &= \bar{D}_R(Y_i) = \nu_i^2(1 - r_i^2) + E_R(Y_i)[\lambda_i + r_i \nu_i - E_R(Y_i)] + [E_R(Y_i) - E(Y_i)]^2, \\ c_{ij} &= [E_R(Y_i) - E(Y_i)] \cdot [E_R(Y_j) - E(Y_j)] (i \neq j), \\ b_i &= E_R\left\{[Y_i - E(Y_i)] \sum_{t=1}^2 r_t [X_t - E(X_t)]\right\}, \quad i, j = 1, 2. \end{aligned}$$

If the trading data of IISS and CISZ are the same as in section 6.1, then

$$\begin{aligned} \mu_X &= (1329.525414, 3245.116068)^T, \sigma_X = (153.5404930, 242.5451888)^T, \\ \mu_Y &= (1329.525414e^{\delta(T-t)}, 3245.116068e^{\delta(T-t)})^T = (1342.887366, 3277.730026)^T, \\ \sigma_Y &= (153.5404930e^{\delta(T-t)}, 242.5451888e^{\delta(T-t)})^T = (155.083606, 244.9828085)^T, \end{aligned}$$

where, $e^{\delta(T-t)} = e^{0.01} = 1.010050167$.

When $-1 < r_1, r_2 < 1$, the futures Y_1 and Y_2 should be held separately in the ratios α_1 and α_2 whose changing characteristics are shown in (a) and (b) of Fig. 3 respectively.

When $-1 < r_1, r_2 < 1$, q and Q , the minimum hedging risk and integrated risk of group assets are shown respectively in (a) and (b) of Fig. 4. Here, the s in expression (20) is equal to 1, i.e., $s = 1$. In most

Fig. 3. The optimal proportions for holding derivative assets.

cases, like $r_1 \neq 1$ or $r_2 \neq 1$, the risk of group assets can not be eliminated completely by the way of hedging. However, if assets hedging are not carried out, the risk will be higher. In order to minimize the integrated risk, we should hold futures Y_1 and Y_2 respectively in optimal ratios α_1 and α_2 , which tend to change along with the correlation coefficients γ_1 and γ_2 . The hedging risk q has a different movement characteristic with that of

Fig. 4. Risk characters of group assets.

integrated risk Q . When $\gamma_1 = \gamma_2 = 0$ or $\gamma_1 = \gamma_2 = 1$, the hedging risk $q = 0$, reaches its minimum; and when $|\gamma_1| = 1$ or $|\gamma_2| = 1$, the hedging risk q is also lower. But, when $\gamma_1 = \gamma_2 = 0$, the integrated risk Q reaches its maximum; when $\gamma_1 = \gamma_2 = 1$, the integrated risk $Q = 0$, reaches its minimum; when $|\gamma_1| = 1$ or $|\gamma_2| = 1$, the integrated risk Q is also lower. This confirms that the market risk of hedging group of assets can be surely reduced.

How we can eliminate risk through assets hedging is in close and complex relations to correlation coefficients among the group assets. Generally, the higher the related degree among the group assets is, the lower the integrated risk may be, before the risk is completely eliminated. In other words, the risk will be excluded by hedging (not just one-to-one hedging).

6 Conclusions

This paper presents a new approach to pricing group assets based on the multivariate Partial Distribution (MPD) as well as a method of pricing single asset on the basis of univariate Partial Distribution (UPD). This new approach deals with the correlation coefficient among the assets grouped when pricing and would be applied without so many assumptions as preconditions, which makes it different from other current methods of asset pricing.

Also, the MPD method enables us to evaluate the integrated risk on prices of group assets. The integrated risk in prices of group assets includes two kinds of risks of asset pricing -hedging risk and independent risk. The integrated risk has dynamics different from that of hedging risk.

By analysing integrated risk into hedging risk and independent risk, we could examine the risk of assets pricing in greater detail. The optimal ratio for hedging asset based upon correlation coefficient might be of great importance for current financial business.

Furthermore, on the basis of UPD or MPD and the correlated conclusions, we could analyze and testify five interesting economic propositions as the following:

- (1) The bigger the risk is, the larger the potential profit will be.
- (2) The new asset will bring a higher sale margin.
- (3) Competition for resources leads to higher cost price while complementarity in resources makes cost price lower.
- (4) The average profit of an asset is lower than that of two independent assets divided from the asset, and is lower than that of the two correlated assets divided from the asset under uniform division on correlation.
- (5) The total market price risk of two assets, which are divided from one asset, is smaller than that of the one asset, regardless of independence or correlation of two assets.

In the empirical analysis, we have investigated only two cases: one derivative asset to one underlying asset and group derivative assets to group underlying assets. It can be expected that we would get similar results if we inquire into other cases. In section 5.2, if the correlation coefficient matrix of the underlying assets is not equal to zero, i.e., $S \neq 0$, and the correlation coefficient matrix of the derivative assets is also not equal to zero, i.e., $L \neq 0$, then the case will be much more complex, which needs a detailed discussion in another paper.

The models and methods suggested in this paper need further verification and demonstration. Besides, group options pricing can be examined if those conclusion above are combined with reference [6].

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Appendix. (Proof of Proposition 4)

Making two assistant functions

$$g(t) = \frac{e^{-\frac{t^2}{2}}}{t(1 + \sqrt{1 - e^{-\frac{2}{\pi}t^2}})} \quad \text{and} \quad \bar{g}(t) = \left(t + \frac{e^{-\frac{t^2}{2}}}{(1 + \sqrt{1 - e^{-\frac{2}{\pi}t^2}})} \right) \frac{e^{-\frac{t^2}{2}}}{(1 + \sqrt{1 - e^{-\frac{2}{\pi}t^2}})}$$

It is easy to validate that two functions above are all monotone decreasing.

(1) Suppose that an asset (its market price $X \in P(\mu, \sigma^2)$) is divided to two independent assets ($X_1 \in P(\mu_1, \sigma_1^2)$ and $X_2 \in P(\mu_2, \sigma_2^2)$), $\mu, \mu_1, \mu_2, \sigma, \sigma_1, \sigma_2 > 0$. And $\mu = \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$. If the two assets satisfy the uniform division on independence, i.e.,

$$\frac{\mu_i}{\sigma_i} < \frac{\mu}{\sigma} \quad (i = 1, 2) \quad (A_1)$$

and by use of the inequalities (A_1), then

$$\frac{R(X_i)}{\mu_i} = g\left(\frac{\mu_i}{\sigma_i}\right) > g\left(\frac{\mu}{\sigma}\right) = \frac{R(X)}{\mu}$$

$$\text{where } R(X) = \sqrt{\frac{2}{\pi}} \frac{\sigma e^{-\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2}}{1 + \sqrt{1 - e^{-\frac{2}{\pi}\left(\frac{\mu}{\sigma}\right)^2}}} \quad \text{and} \quad R(X_i) = \sqrt{\frac{2}{\pi}} \frac{\sigma_i e^{-\frac{1}{2}\left(\frac{\mu_i}{\sigma_i}\right)^2}}{1 + \sqrt{1 - e^{-\frac{2}{\pi}\left(\frac{\mu_i}{\sigma_i}\right)^2}}, \quad i = 1, 2.$$

i.e., $R(X_1) + R(X_2) > R(X)$. These mean the total average profit on sale is higher than that of the original asset. According to the formula (7), we obtain

$$E(X_1) + E(X_2) = \mu_1 + R(X_1) + \mu_2 + R(X_2) > \mu + R(X) = E(X)$$

these mean the total average market price on sale is lower than that of the original asset.

On the other hand,

$$\frac{E(X_i)R(X_i)}{\sigma_i^2} = \bar{g}\left(\frac{\mu_i}{\sigma_i}\right) > \bar{g}\left(\frac{\mu}{\sigma}\right) = \frac{E(X)R(X)}{\sigma^2}$$

Because $\sigma^2 = \sigma_1^2 + \sigma_2^2$, then $E(X_1)R(X_1) + E(X_2)R(X_2) > E(X)R(X)$, and have

$$D(X_1) + D(X_2) = \sigma_1^2 - E(X_1)R(X_1) + \sigma_2^2 - E(X_2)R(X_2) < \sigma^2 - E(X)R(X) = D(X)$$

namely, the total risk of market price is lower than that of the original asset.

(2) Suppose that an asset (its market price $X \in P(\mu, \sigma^2)$) is divided to two correlated assets (their market prices are X_1 and X_2 separately), $(X_1, X_2) \in P(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$, $\mu, \mu_1, \mu_2, \sigma, \sigma_1, \sigma_2 > 0$, $|r| \leq 1$, and $\mu = \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$. As we know, the average market price and profits on sale of the two assets divided are

$$E_r(X_i) = \mu_i + r\sigma_i + R_r(X_i)$$

and

$$R_r(X_i) = \sqrt{\frac{2}{\pi}} \frac{\sigma_i \sqrt{1 - r^2} e^{-\frac{1}{2}\left(\frac{\mu_i + r\sigma_i}{\sigma_i \sqrt{1 - r^2}}\right)^2}}{1 + \sqrt{1 - e^{-\frac{2}{\pi}\left(\frac{\mu_i + r\sigma_i}{\sigma_i \sqrt{1 - r^2}}\right)^2}}, \quad i = 1, 2.$$

If the two assets satisfy the uniform division on correlation, namely the following inequalities are right:

$$\frac{\mu_i + r\sigma_i}{\sqrt{1 - r^2}\sigma_i} < \frac{\mu}{\sigma} (i = 1, 2) \quad (A_2)$$

When $r = 0$, the uniform division on independence (A_1) is satisfied, and the results obtained in the proof of section (1) are also tenable. According to inequalities (A_2), have

$$\left(1 + \left(\frac{\sigma}{\mu}\right)^2\right)r^2 + 2\left(\frac{\sigma}{\mu}\right)^2\left(\frac{\mu_i}{\sigma_i}\right)r + \left(\frac{\sigma}{\mu}\frac{\mu_i}{\sigma_i}\right)^2 - 1 < 0$$

namely

$$\left(r - \frac{-b + \sqrt{\Delta}}{a}\right)\left(r - \frac{-b - \sqrt{\Delta}}{a}\right) < 0 \quad (A_3)$$

where, $a = 1 + \left(\frac{\sigma}{\mu}\right)^2$, $b = \left(\frac{\sigma}{\mu}\right)^2\left(\frac{\mu_i}{\sigma_i}\right)$, $\Delta = b^2 - ac$, $c = \left(\frac{\sigma}{\mu}\frac{\mu_i}{\sigma_i}\right)^2 - 1$.

It is known by computing that $\Delta \geq 0$ when $\frac{\mu_i}{\sigma_i} \leq \sqrt{1 + \left(\frac{\mu}{\sigma}\right)^2}$. So the inequalities (A_3) have, according to (A_1), the real solutions always. Based on inequalities (A_3), when $r > 0$,

$$0 < r < s_i^+ = \frac{-b + \sqrt{\Delta}}{a} = \frac{\mu \sqrt{\mu^2\sigma_i^2 + \sigma^2(\sigma_i^2 - \mu_i^2)} - \mu_i\sigma^2}{\sigma_i(\mu^2 + \sigma^2)}, i = 1, 2.$$

We use the notion $r_1 = \min\{s_1^+, s_2^+\}$. And when $r \leq 0$,

$$0 > r > s_i^- = \frac{-b - \sqrt{\Delta}}{a} = -\frac{\mu \sqrt{\mu^2\sigma_i^2 + \sigma^2(\sigma_i^2 - \mu_i^2)} + \mu_i\sigma^2}{\sigma_i(\mu^2 + \sigma^2)}, i = 1, 2.$$

We use the notion $r_2 = \max\{s_1^-, s_2^-\}$.

When $r_2 < r < r_1$, we have

$$\frac{R_r(X_i)}{\mu_i + r\sigma_i} = g\left(\frac{\mu_i + r\sigma_i}{\sqrt{1 - r^2}\sigma_i}\right) > g\left(\frac{\mu}{\sigma}\right) = \frac{R(X)}{\mu},$$

i.e.,

$$R_r(X_1) + R_r(X_2) > R(X) + r\frac{\sigma_1 + \sigma_2}{\mu}R(X) \quad (A_4)$$

When $r > 0$,

$$R_r(X_1) + R_r(X_2) > R(X) + r\frac{\sigma_1 + \sigma_2}{\mu}R(X) > R(X).$$

And when $r \leq 0$, if

$$R(X) \geq R_r(X_1) + R_r(X_2),$$

let $r \rightarrow 0$, $R(X) \geq R(X_1) + R(X_2) > R(X)$, the contradiction appears according to the result in the proof of section 1). Thus, $R_r(X_1) + R_r(X_2) > R(X)$.

Again when $r_2 < r < r_1$,

$$\frac{E_r(X_i)R_r(X_i)}{\sigma_i^2(1 - r^2)} = \bar{g}\left(\frac{\mu_i + r\sigma_i}{\sqrt{1 - r^2}\sigma_i}\right) > \bar{g}\left(\frac{\mu}{\sigma}\right) = \frac{E(X)R(X)}{\sigma^2},$$

i.e.

$$E_r(X_1)R_r(X_1) + E_r(X_2)R_r(X_2) > (1 - r^2)E(X)R(X),$$

and

$$D_r(X_1) + D_r(X_2) = \sigma_1^2(1 - r^2) - E_r(X_1)R_r(X_1) + \sigma_2^2(1 - r^2) - E_r(X_2)R_r(X_2) < (1 - r^2)D(X).$$

Thus,

$$D_r(X_1) + D_r(X_2) < (1 - r^2)D(X) < D(X) \quad (A_5)$$

When $|r|=1$, we get $R_r(X_i)=0$ from the their expressions. This means that the total average profit on sale is equal to zero if an asset is divided to the two assets which are correlated in linearity on probability being 1. And at same time, the total risk of market price is equal to zero from the inequality (A_5) .

(3) To sum up the conclusions above, the total risk in market prices of assets will get lower whether the two assets are correlated or not if an asset is divided to the two assets.