New approach to fixed charges problems (FCP)

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Abstract. This paper introduces a new formulation for problems with fixed charges, such as knapsack and fixed charge linear programming problems (including transportation type), which until now have been treated as pure 0-1 integer, or mixed integer problems. In this paper the objective function is presented in a modified form, which is based on the number of fixed charges. The problems are defined in a novel way paving the way for better understanding of the local and global minima phenomena. Finally computational benefits of the formulation and a numerical example are shown.

Keywords: fixed charge, knapsack problem, linear programming

1 Introduction

Problems with fixed charges, such as knapsack (KP) and fixed charge linear programming problems (FCP), including the transportation type formulation (FCTP), have been historically treated as pure integer or mixed integer problems. The optimization literature presents several different methods for solving those problems (for the FCTP see Adlakha and Kowalski^[3]). The authors of those methods have different claims about the effectiveness of their procedures. Since the problems with fixed charges are usually NP-hard (Non-deterministic Polynomial-time hard), the computational time to obtain exact solution increases in a polynomial fashion and very quickly becomes extremely difficult and long as the size of the problem increases. The most common currently approaches to the fixed charges problems are through Branch-and-Bound, Tabu, Genetic and Evolutionary Algorithms. Kowalski^[6] and Adlakha^[1, 4] presented several novel features pertaining to problems with fixed charges. This paper builds on those determinations to introduce new formulations for those problems, which can be useful in explaining the local and global minima phenomena. The theory will be explained first on a FCP and then extended to the FCTP and KP. This paper assumes the reader is familiar with fixed charge and linear problems, and therefore doesn't cover the basic theory, which is covered in most of linear programming textbooks and in the referenced papers.

2 The fixed charge problem

The fixed charge linear programming problem can be presented as
FCP formulation:

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$$\text{minimize } Z = \sum_{j=1}^n (c_j x_j + f_j y_j) = Z_{var/cost} + Z_{fixed/cost} \quad (1)$$

$$\text{subject to } \begin{cases} \sum_{j=1}^n a_{ij} x_j = b_i, & \text{for } i = 1, 2, \dots, m \\ x_j \geq 0, & \text{for all } j \\ y_j = 0, & \text{if } x_j = 0 \\ y_j = 1, & \text{if } x_j > 0 \\ a_{ij}, b_i, c_j, f_j \geq 0 \\ n > m \end{cases} \quad (2)$$

If the number of basic non-zero variables is less than m , then the solution to a FCP is degenerate. Kowalski^[6] shows that the optimal solution to the FCPs can be affected by the degeneracy of the final distribution. Adlakha and Kowalski^[3] show that a constant value can be separated from the FCTP objective Z function without affecting the locations of local/global minima.

3 New formulations for FCP, FCTP and KP

3.1 The fixed charge problem

The FCP equation (1) can also be shown in the following form

$$\begin{aligned} \text{minimize } Z &= Z_{const/cost} + k\Delta + \sum_{j=1}^n (c_j x_j + f_j^* y_j) \\ \text{subject to } \sum_{j=1}^n a_{ij} x_j &= b_i \quad \text{for } i = 1, 2, \dots, m \end{aligned} \quad (3)$$

where $\Delta = \min f_j$ (the smallest fixed charge coefficient); $f_j^* = f_j - \Delta$ and the variable $k = \sum_{j=1}^n y_j$ and $k \leq m$ is a consolidation coefficient equal to the number of non-zero basic variables.

The $Z_{const/cost}$ - constant cost portion which represents the portion of the variable cost, which due to the constraint structure will appear in every distribution. Since the FCP has the same structure as a regular Simplex type linear problem this is equivalent to a situation when one of the variables is present in each basic solution/distribution.

Equation (3) shows that some FCPs can have several local minima based on the k factor.

Therefore, the new FCP formulation can be presented as:

$$\text{minimize } Z = Z_{const/cost} + Z_{var/cost} + Z_{fixed/cost} + Z_{consolid/cost} \quad (4)$$

where

$Z_{const/cost}$ - constant cost portion which represents the portion of the variable cost;

$Z_{var/cost}$ - variable cost portion which presents the cost generated from the variable coefficients ($\sum_{j=1}^n c_j x_j$);

$Z_{fixed/cost}$ - variable cost portion which represents the cost generated from the fixed charges ($\sum_{j=1}^n f_j^* y_j$);

$Z_{consolid/cost}$ - consolidation cost portion which can be separated from the original fixed charges $k\Delta$.

3.2 The fixed charge transportation problem

Since the FCTP is a variation of a general FCP, it can be also formulated as (4).

FCTP general formulation:

$$\text{minimize } Z = \sum_{i=1}^m \sum_{j=1}^n (c_{ij}x_{ij} + f_{ij}y_{ij}) \quad (5)$$

$$\text{subject to } \begin{cases} \sum_{j=1}^n x_{ij} = a_i & \text{for } i = 1, 2, \dots, m \\ \sum_{i=1}^m x_{ij} = b_j & \text{for } j = 1, 2, \dots, n \\ x_{ij} \geq 0 & \text{for all } (i, j) \\ y_{ij} = 0 & \text{if } x_{ij} = 0 \\ y_{ij} = 1 & \text{if } x_{ij} > 0 \\ \sum_{i=1}^m a_i = \sum_{j=1}^n b_j \\ a_i, b_j, c_{ij}, f_{ij} \geq 0 \end{cases} \quad (6)$$

From linear programming theory (also see [6]), we know that there are $m + n - 1$ basic variables in any transportation solution and that the cost matrix can be **hungarized** (the process of subtracting a constant number from columns and rows) without affecting the locations of local or global minima. Therefore equation (4) can also be shown in the following form

$$\text{minimize } Z = Z_{const/cost} + k\Delta + \sum_{i=1}^m \sum_{j=1}^n (c_{ij}^*x_{ij} + f_{ij}^*y_{ij}) \quad (7)$$

where

c_{ij}^* variable coefficients after hungarization;

$\Delta = \min f_{ij}$ (the smallest fixed charge coefficient) and $f_{ij}^* = f_{ij}\Delta$;

$Z_{const/cost}$ constant portion includes the constant portion generated thru hungarization;

k consolidation coefficient variable equals to the number of non-zero basic variables $k = \sum_{i=1}^m \sum_{j=1}^n y_{ij}$ and $k \leq m + n - 1$.

Equation (7) shows that some FCTPs can have several local minima based on the k factor.

3.3 The knapsack problem

The knapsack problem can be shown as

knapsack problem formulation:

$$\text{maximize } Z = Z_{fixed/cost} = \sum_{j=1}^n f_j y_j \quad (8)$$

$$\text{subject to } \begin{cases} \sum_{j=1}^n a_j y_j \leq b \\ y_j = 0 \text{ or } 1 \\ a_j, f_j, b \geq 0 \end{cases} \quad (9)$$

New KP formulation:

$$\text{maximize } Z = Z_{fixed/cost} + Z_{consolid/cost} \quad (10)$$

For the KP equation (10) can also be shown in the following form

$$\text{maximize } Z = k\Delta + \sum_{j=1}^n f_j^* y_j \quad (11)$$

subject to (9)

where

$\Delta = \min f_j$ (the smallest fixed charge coefficient) and $f_j^* = f_j - \Delta$;

$k = \sum_{j=1}^n y_j$ and $k \leq n$ consolidation coefficient variable equals to the number of non-zero variables.

4 Application of the new approach

While it is obvious that the new approach gives us a better insight into the problems with fixed charges, an immediate question arises – what is the practical use and advantage of it? The first benefit of the introduced material is a conclusion that since the value of the objective function depends on the value of the consolidation coefficient, the degeneracy of a given, fixed charge based problem can provide basis for identifying local or global minima. By separating the objective function into four basic components, it makes it easier to understand each of the four basic components and therefore the objective function is clearer. Also, as will be shown in the next paragraph, the separation of the constant portion and the introduction of the consolidation factor can be used to identify the optimal solution and to verify the optimality.

5 Computational example

For better comprehension of the introduced material a computational example is introduced. The selected example is a FCTP without the variable portion ($c_{ij} = 0$). This kind of problem was chosen, as it combines the features of all three discussed problems and introduces many of the benefits of the new approach. The problem is shown in Tab. 1. The first operation is a reduction of the cost matrix through a subtraction of the

Table 1. Fixed cost matrix f_j for numerical example ($c_{ij} = 0$)

	b_1	b_2	b_3	Supply
a_1	1	2	3	30
a_2	2	3	2	10
a_3	2	1	2	30
a_4	4	3	2	10
Demand	40	20	20	

smallest fixed charge value, which results in a modified matrix shown in Tab. 2.

Table 2. Cost matrix after primary reduction

	b_1	b_2	b_3	Supply
a_1	0	1	2	30
a_2	1	2	1	10
a_3	1	0	1	30
a_4	3	2	1	10
Demand	40	20	20	

Analyzing the supply and demand values from Table 1, we notice that the smallest load amount is 10, which is also equal to some of the supply values. This means that the corresponding rows have only one loaded location. This in turn allows to “hungarize” (subtracting a constant from rows and columns) those rows (or columns in reversed situation) by subtracting 1 (lowest fixed charge coefficient in the row) from each row element, without affecting the final optimal distribution. This operation results in the modified cost matrix shown in Tab. 3.

Following, a Balinski^[5] matrix is created. Such a matrix is obtained through formulating a linear version of FCTP by relaxing the integer restriction on y_{ij} as follows:

Table 3. Cost matrix after secondary reduction

	b_1	b_2	b_3	Supply
a_1	0	1	2	30
a_2	0	1	0	10
a_3	1	0	1	30
a_4	2	1	0	10
Demand	40	20	20	

$$y_{ij} = x_{ij}/m_{ij} \quad (12)$$

where $m_{ij} = \min(a_i, b_j)$.

So, the linear version of an FCTP thus formulated would simply be a standard TP with unit transportation costs as $C_{ij} = f_{ij}/m_{ij}$, as shown in Tab. 4.

Table 4. Balinski cost matrix

	b_1	b_2	b_3	Supply
a_1	0	0.05	0.1	30
a_2	0	0.1	0	10
a_3	0.033	0	0.05	30
a_4	0.2	0.1	0	10
Demand	40	20	20	

The optimal distribution for this linear version TP is identified in Tab. 5.

Table 5. Optimal solution to Balinski cost matrix

	b_1	b_2	b_3	Supply
a_1	30			30
a_2			10	10
a_3	10	20		30
a_4			10	10
Demand	40	20	20	

The optimal solution $\{x_{ij}\}$ to this TP can be easily modified into a feasible solution of FCTP as follows:

$$\begin{cases} y_{ij} = 0 & \text{if } x_{ij} = 0, \\ y_{ij} = 1 & \text{if } x_{ij} > 0. \end{cases}$$

Balinski^[5] shows that the optimal value of the linear version TP provides a lower bound on the optimal value Z of FCTP (Z_{FCTP})

$$\sum_{i=1}^m \sum_{j=1}^n C_{ij} x_{ij} \leq Z_{FCTP} \quad (13)$$

Knowing that Z is a discrete function, it can be concluded easily, see Adlakha and Kowalski^[2], that if the difference between Z_{FCTP} and the lower bound in (13) is less than the minimum cost function interval, then Z_{FCTP} represents the optimal distribution for the FCTP. In our case $Z_{Balinski} = 0.33$ and $Z_{FCTP} = 1$, therefore with the minimum cost function interval equal to 1, the optimal solution to Balinski matrix also represents the optimal solution to the problem from Table 1. The above successful application of the new approach to the presented example makes it tempting to apply it to really large problems, which cannot be handled by the Branch-and-Bound, Tabu, Genetic or Evolutionary Algorithms. Since such problems would have a significant number of local minimums, any applied gradient/simplex type algorithm would most probably terminate at

one of them. Balinski's approach is known to generate quite satisfactory results as well, and was used by the creators of the Tabu based algorithm (see Sun *et al*^[7]), for obtaining a good initial feasible solution. As shown above, by "extracting" the fixed and consolidation portions from the original problem, Balinski's method can be "calibrated" to bring even better results and in case of small problems can also provide a verification of the optimality.

6 Conclusion

The above work expands on the theory of problems with fixed charges, extending beyond the material currently available from textbooks and other sources. It introduces a novel formulation, which should contribute to better understanding of such problems. The new approach allows performing more thorough analysis, especially useful in identifying multiple minima. Finally an example is presented where the application of the new approach results in a quick identification and verification of a minimum.

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