ISSN 1 746-7233, England, UK
International Journal of Management Science and Engineering Management
Vol. 2 (2007) No. 1, pp. 3-22

The golden growth model and management strategy in macroeconomic structure

Feng Dai* , Jingxu Liu

Department of Management Science,
Zhengzhou Information Engineering University, Zhengzhou, Henan, P. R. China

(*Received October 15 2006, Accepted December 6 2006*)

Abstract. Based on the partial distribution, this paper establishes the golden growth model in economic process. The model describes the optimal relation between the economic investment and the economic growth, and could be taken as a basis to distinguish that the economic process is higher in developing efficiency or not, and a series of important economic constants are obtained on the golden growth model. Also, the management strategies are given to control the macroeconomic structure by the model. Finally, by the empirical researches, the golden growth model is explained to be existent and effective, and the strategies are proved to be useful to make decision in macroeconomic management.

Keywords: partial distribution, golden growth rule, macroeconomic structure, development and management

1 Introduction

There are many of the important studies in the economic theory, such as the business cycle theory[20, 21], the real business cycle theory[16, 18, 19, 23], and new growth theory[24, 25], etc. These theories availably propeled forward the economic development of world. In recent years, the economists pay attention to make the efficiency of economic development be higher[4], evaluate the government's economic aid to nation or district and their actual results[13], insure the economy to grow by establishing and choosing the policies[11] and perfecting the system of economic education[12], etc.

It is worthy to note the economic development is a large scale system itself, and we perhaps need a new and different theory for economic development and the systemic and structural models on it. According to the basic characters of economic growth and economic structure, here, try to make some different discussions as following based on partial distribution[6] [2] and developower[5, 8]:

• The concept of the developower and the way to evaluate it are given. It will be explained that the developower, which can push the economy to grow and advance the productivity to increase in its level, comes into being from the changes in economic environment, and the key problem of economic development is the interrelation and interaction between productivity and developower.

• The golden growth model is naturally obtained. In describing and analyzing the relation between the economic level and developower, the optimal correlation model, linked with golden ratio, is obtained on the

* Corresponding author. Tel.: +86-0371-6353 0975; *E-mail address*: fengdai@126.com.

[2] Though the univariate partial distribution (UPD) is the univariate left-truncated normal (Gaussian) distribution with truncation below zero, but some basic results, which are obtained in the discussion on UPD[6], have not been given in those on the univariate left-truncated normal distribution[15], and the multivariate partial distribution[7] is different in definition from the current multivariate truncated normal distribution[17]. So the partial distribution is still called.

correlation between the economic growth rate and the developower rate, which is called the golden growth rule.

The golden rule, as we know, is a mathematical relation in which the basic diameter is the limit of the ratio of consecutive Fibonacci numbers. The researches, applied the golden rule to capital accumulation,n are given earlier by E. S. Phelps[22] and D. Cass[1–3]. Christiaans has applied it to study the basic model of economic growth based on non-scale models of economic growth[3]. Being different from their researches, author here will derive naturally the result that the relation between economic growth rate and the developower rate follows the golden rule without special supposing it to be in advance.

• The model to design management strategy for macroeconomic structure is built on developower. The economic system is divided in two kinds of patterns, the producing type (economy is divided into the producing domains) and functional type (economy is divided into the functional realms), and then, the structural model of economic system is built. By the structural model, we could analyze, design and plan the economic growth.

• The empirical researches are given. Some of empirical researches on the results in this paper are done on the US GDP data (chained price index and in billions of dollars) from 1940 to 2004, in order to illuminate that the golden growth rule is existed in the process of economic growth, and the economic programming model is useful to make decisions for economic development.

On the other hand, a series of important constants in economy are obtained on the average growth and golden growth, like the coefficients of average growth and golden growth, the contribution quantity by developower in economic average growth and economic golden growth, and the stock coefficients of developower. These coefficients can reflect some of key quantifiable relations among the economic growth.

2 The economic level and the economic developower

2.1 The basic concepts and assumptions

The economic level is the state of economic development, also the evaluated value of economic status. The economic level includes the basic level of economy and the real level of economy. The basic level of economy, basic level for short, is the evaluated value of basic establishment for economy and the due ability in economy; the real level of economy, real level for short, is the real outcome presented in economic production. In order to describe the determinacy (the current basic level) and the randomicity (the future real level) in economic process, some of assumptions about the economic level are given here.

Assumption 1. Suppose the real level is a stochastic variable which has a determinate background, and this determinate background is based on the basic level which is a constant in a period of time.

(1) The values of basic level and real level are non-negative, i.e. the minimum values of basic level and real level are zero.

(2) Both the basic level and the real level are always fluctuated, and the fluctuation ranges are positive, i.e. the minimum values of the fluctuation ranges are larger than zero.

(3) The real level changes around the basic level, and the more the real level is apart from the basic level, the less its happening probability is.

Definition 1. *The economic developower, developower for short, is the motivity to push economy to progress. The economic development rate is the relative intensity of developower on unit of economic level.*

Developower is a kind of the invisible and potential force, like policy and system, science and technology, knowledge and education, market system, economic management, rule and regulation, cultural background, public idea, consumed desires, etc., which exist widely in economic field. Developower comes into being from the changes in the economic environments. The economic environments said here are composed of the environments of production, resource, investment, policy, science and technology, education, management, sale, etc. So developower reflects the uncertainty in economic process.

Based on the discussion above, we see that developower includes the developower in production which is caused by the changes of environment in economic production and the developower in nonproduction which is caused by the changes of environment in economic policies, science and technology, management, education, market system, etc.

There is the duality in the asset. The asset, which is physical (e.g. workshop, manpower, equipment, etc.) and then called the real assets, is the productivity if it has some kind of abilities to produce, and the asset, which is nonphysical (like money, bond, stock, etc.) and called the nonphysical asset, is the developower if it is the potential motivity to push economy to progress. For the financial asset, it could be taken as productivity if relating to real asset and it could be taken as developower if relating to nonphysical asset.

In any stage of economic production, the basic level could represent the original ability to produce, and the real level represents the closed ability to produce. In this stage, an investing asset regarded as the developower can not influence the basic level, but can influence the real level in the way to be converted to real abilities to produce and will be added to the basic level in the next stage. So we have

Assumption 2. Any of nonphysical asset will influence economy as a developower before it become a real ability to produce. Developower can be strengthened by adding nonphysical asset.

2.2 The measurement for developower and development rate

The economy is generally composed of producing economy and nonproducting economy. The producing economy is a series of economic activities to provide directly the commodities, products and serves for society and related realms. The nonproducting economy is a series of activities to ensure the producing economy to run normally.

In various cases, economic level could be measured by the total level of productivity (the quantity of all the effective assets in economic society), or the total quantity of output (GDP). The level of producing economy (the level of productivity) could be measured by the total quantity of productive assets, and the level of nonproducting economy could be measured by the total quantity of nonproductive assets.

Because the developower from the changes in economic environment, developower could be measured by the fluctuating range of economic level. The development rate could be measured by the ratio of developower to economic level, i.e.

$$\text{development rate } = \frac{\text{developower}}{\text{the economicl evel}}.$$

As we see, the development rate means developower contained in unit of economic level and the unit ability of conomic environment to push the rise in economic level. Development rate can describe the vitality of economic development.

3 The partial distribution and the related results

3.1 The univariate partial distribution

Definition 2. *(univariate partial distribution, UPD for short) Let X be a non-negative stochastic variable, and it follows the distribution of density*

$$f(x) = \begin{cases} e^{-\frac{(x-\mu)^2}{2\sigma^2}} / \int_0^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{1}$$

where $\mu \geq 0$ and $\sigma > 0$. Then, X is called to follow a univariate partial distribution, and note as $X \in P(\mu, \sigma^2)$. If all of μ, σ, and X are time-variant, then note as $X(t) \in P(\mu(t), v^2(t))$.

According to references[6, 9], we have two basic results[3] about UPD as follows:

[3] These two results have never appeared in the discussions on the univariate truncated normal distribution.

Theorem 1. *For any $x \in [0,\infty)$, the following formulas are correct approximately*

(1) $\int_0^x e^{-\frac{t^2}{2}}dt = \sqrt{\frac{\pi}{2}(1-e^{-\frac{2}{\pi}x^2})}$;

(2) $\int_0^x e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \sqrt{\frac{\pi}{2}}\sigma\left(\sqrt{1-e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}} + sgn(x-\mu)\sqrt{1-e^{-\frac{2}{\pi}(\frac{x-\mu}{\sigma})^2}}\right)$,

where, $sgn(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$.

Theorem 2. *Let X follow the partial distribution $P(\mu,\sigma^2)$, thus*

(1) The expected value $E(X)$ is as follows:

$$E(X) = \mu + \sqrt{\frac{2}{\pi}}\frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{1-e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}}+1}; \tag{2}$$

(2) The variance $D(X)$ is as follows:

$$D(X) = \sigma^2 + E(X)[\mu - E(X)] \tag{3}$$

Definition 3. *(Rightward Partial Distribution, RPD for short) If X is a non-negative stochastic variable, and it has the probability density function as follows:*

$$f(x) = \begin{cases} e^{-\frac{(x-\mu)^2}{2\sigma^2}} / \int_0^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}}d\mu & x \geq \alpha \\ 0 & x < \alpha \end{cases}$$

where, the constant $a > 0$, then X is called to follow the rightward partial distribution, and note as $X \in P_a(\mu,\sigma^2)$.

When $\mu > \sigma$ and $a = \mu - \sigma$, X is called to follow the standard rightward partial distribution, SRPD for short, and note as $X \in F_a(\mu,\sigma^2)$.

Corollary 1. *For any $x \in [a,\infty]$, a, μ and σ are constant, a, $\mu \geq 0$, $\sigma > 0$, then the following equations are correct approximately:*

$$\int_a^x e^{-\frac{(u-\mu)^2}{2\sigma^2}}du = \sqrt{\frac{\pi}{2}}\sigma\left[sgn(x-\mu)\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{x-\mu}{\sigma}\right)^2}} - sgn(a-\mu)\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{a-\mu}{\sigma}\right)^2}}\right],$$

where, $sgn(x)$ is the same as in Theorem 1.

If $X \in P_a(\mu,\sigma^2)$, and $a < \mu$, thus

$$E(X) = \mu + \sqrt{\frac{2}{\pi}}\sigma\frac{e^{-\frac{(\mu-a)^2}{2\sigma^2}}}{1+\sqrt{1-e^{-\frac{2}{\pi}\frac{(\mu-a)^2}{\sigma^2}}}} \tag{4}$$

$$D(X) = \sigma^2 - [E(X)-a][E(X)-\mu] = \sigma^2 - [E(X)-a]R(X) \tag{5}$$

3.2 The multivariate partial distribution

Definition 4. *(Multivariate Partial Distribution, MPD for short) If $X_1,\cdots,X_n (n \geq 2)$ are all the non-negative stochastic variables, and they have the multivariate probability density function as follows:*

$$f(x_1,\cdots,x_n) = \begin{cases} \frac{e^{-\frac{1}{2|M|}\left[\sum_{i=1}^n |M_{ii}|(x_i-\mu_i)^2 + \sum_{i,j=1;i\neq j}^n |M_{ij}|(\sigma_i(x_j-\mu_j))\right]}}{\int_0^\infty\cdots\int_0^\infty e^{-\frac{1}{2|M|}\left[\sum_{i=1}^n |M_{ii}|(x_i-\mu_i)^2 + \sum_{i,j=1;i\neq j}^n |M_{ij}|(\sigma_i(x_j-\mu_j))\right]}dx_1\cdots dx_n} & 0 \leq x_1,\cdots,x_n < \infty \\ 0 & othercases \end{cases} \tag{6}$$

where, $\boldsymbol{M} = (\sigma_{ij})_{n\times n}$, $\boldsymbol{R} = (r_{ij})_{n\times n}$, $\sigma_{ii} = \sigma_i^2$, $\sigma_{ij} = r_{ij}\sigma_i\sigma_j (i \neq j)$, $\sigma_i > 0$, $r_{ii} = 1$, $|r_{ij}| \leq 1 (i \neq j)$, r_{ij} *is called the correlation coefficient between* X_i *and* X_j, $i, j = 1, \cdots, n$. *Then* $X_1, \cdots, X_n$ *is called to follow multivariate partial distribution*[4] *(MPD), and note as* $\boldsymbol{X} \in P(\tilde{\mu}, \tilde{\sigma^T}, \tilde{\sigma}, \boldsymbol{R})$ *if denoting* $\boldsymbol{X} = (X_1, \cdots, X_n)^T$, $\tilde{\mu} = (\mu_1, \cdots, \mu_n)^T \geq \boldsymbol{0}$, $\tilde{\sigma} = (\sigma_1, \cdots, \sigma_n)^T > 0$.

As a special example of MPD, if the non-negative stochastic variables X and Y follow the bivariate distribution:

$$f(x,y) = \begin{cases} \dfrac{e^{-\frac{1}{2(1-r^2)}\left[(\frac{x-\mu_1}{\sigma_1})^2 - 2r\left[(\frac{x-\mu_1}{\sigma_1})+(\frac{y-\mu_2}{\sigma_2})\right]+(\frac{y-\mu_2}{\sigma_2})^2\right]}}{\int_0^\infty\int_0^\infty e^{-\frac{1}{2(1-r^2)}\left[(\frac{x-\mu_1}{\sigma_1})^2 - 2r\left[(\frac{x-\mu_1}{\sigma_1})+(\frac{y-\mu_2}{\sigma_2})\right]+(\frac{y-\mu_2}{\sigma_2})^2\right]} dxdy}, & 0 \leq x, y < \infty \\ 0 & x < 0 \quad or \quad y < 0 \end{cases} \tag{7}$$

then, (X, Y) is said to have the bivariate partial distribution, and note as $(X,Y) \in P(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$, where, the constants $\mu_1, \mu_2 \geq 0$, $\sigma_1, \sigma_2 > 0$, $-1 < r < 1$.

When $|r| = 1$, we know, according to reference [10], X is correlating with Y in linearity on probability 1, i.e., the probability $P(Y = dX + h) = 1$,where, both d and h are constant, $d > 0$ if $r = 1$, and $d < 0$ if $r = -1$.

Theorem 3. *If both* X_1 *and* X_2 *are stochastic variables and follow bivariate partial distribution, i.e.,* $(X_1, X_2) \in P(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$*, thus*

(1) $\int_0^\infty \int_0^\infty e^{-\frac{1}{2(1-r^2)}\left[(\frac{\mu-\mu_1}{\sigma_1})^2 - 2r\left[(\frac{\mu-\mu_1}{\sigma_1})+(\frac{\nu-\mu_2}{\sigma_2})\right]+(\frac{\nu-\mu_2}{\sigma_2})^2\right]} d\mu d\nu$

$$= \frac{\pi}{2}\sigma_1\sigma_2(1-r^2)e^{\frac{r^2}{1-r^2}}\left[1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_1+r\sigma_1}{\sigma_1\sqrt{1-r^2}}\right)^2}}\right]\left[1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_2+r\sigma_2}{\sigma_2\sqrt{1-r^2}}\right)^2}}\right];$$

(2) $\int_0^{x_1}\int_0^{x_2} f(\mu,\nu) = \dfrac{A_1(x_1)A_2(x_2)}{\left[1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_1+r\sigma_1}{\sigma_1\sqrt{1-r^2}}\right)^2}}\right]\left[1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_2+r\sigma_2}{\sigma_2\sqrt{1-r^2}}\right)^2}}\right]}, 0 \leq x_1, x_2 < \infty$

where, $A_i(t) = \sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_i+r\sigma_i}{\sigma_i\sqrt{1-r^2}}\right)^2}} + sgn(t-\mu_i)\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{t-(\mu_i+r\sigma_i)}{\sigma_i\sqrt{(1-r^2)}}\right)^2}}$, $sgn(x)$ *is the same as in Theorem 1,* $i = 1, 2$.

Denoting:

$$f_{1r}(x) = \int_0^\infty f(x,y)dy = \begin{cases} e^{-\frac{[x-(\mu_1+r\sigma_1)]^2}{2\sigma_1^2(1-r^2)}} \Big/ \int_0^\infty e^{-\frac{[\mu-(\mu_1+r\sigma_1)]^2}{2\sigma_1^2(1-r^2)}} d\mu & x \geq 0 \\ 0 & x < 0 \end{cases} \quad and$$

$$f_{2r}(y) = \int_0^\infty f(x,y)dx = \begin{cases} e^{-\frac{[y-(\mu_1+r\sigma_1)]^2}{2\sigma_1^2(1-r^2)}} \Big/ \int_0^\infty e^{-\frac{[\mu-(\mu_1+r\sigma_1)]^2}{2\sigma_1^2(1-r^2)}} d\mu & y \geq 0 \\ 0 & y < 0 \end{cases}.$$

Theorem 4. *If both* X_1 *and* X_2 *are stochastic variables and follow the bivariate Partial Distribution, i.e.,* $(X_1, X_2) \in P(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$*, thus*

(1) The expected values of each stochastic variable are

$$E_r(X_i) = \int_0^\infty x f_{ir}(x)dx = \mu_i + r\sigma_i + \sqrt{\frac{2}{\pi}}\frac{\sigma_i\sqrt{1-r^2}e^{-\frac{1}{2}\left(\frac{\mu_i+r\sigma_i}{\sigma_i\sqrt{1-r^2}}\right)^2}}{1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_i+r\sigma_i}{\sigma_i\sqrt{1-r^2}}\right)^2}}},$$

where $i = 1, 2$.

(2) The variances of each stochastic variable are

[4] The definition of multivariate partial distribution is different from the current multivariate truncated normal distribution, see references [16], [24] and [25].

$$D_r(X_i) = \int_0^{\infty} [x - E_r(X_i)]^2 f_{ir}(x)dx = \sigma_i^2(1-r^2) + E_r(X_i)\left[\mu_i + r\sigma_i - E_r(X_i)\right],$$

where $i = 1, 2$.

We can validate that $D_r(X_1) = D(X_1)$ *and* $D_r(X_2) = D(X_2)$ *if* $r = 0$.

3.3 Estimating the parameters in MPD

Here we take the bivariate partial distribution as an example. The samples series of stochastic variable 1 and variable 2 are separately $x_{11}, x_{12}, \cdots, x_{1n}$ and $x_{21}, x_{22}, \cdots, x_{2n}(x_{1i}, x_{2i} > 0, i = 1, \cdots, n)$.

According to the modified maximum likelihood estimation[23], we can obtain $\hat{\mu}_k$ (the estimate of μ_k) and $\hat{\sigma}_k$ (the estimate value of σ_k), $k = 1, 2$. Thus, the correlation coefficient can be estimated as:

$$\hat{r}_{1,2} = \frac{\sum\limits_{i=1}^{n}(x_{1i}-\hat{\mu}_1)(x_{2i}-\hat{\mu}_2)}{\sqrt{\sum\limits_{i=1}^{n}(x_{1i}-\hat{\mu}_1)^2 \cdot \sum\limits_{j=1}^{n}(x_{2j}-\hat{\mu}_2)^2}}.$$

4 The model of golden growth rule in economic process

We will use the following notations:

μ—the basic economic level, basic level for short, $\mu \geq 0$.

σ—the standard variance of basic level, $\sigma > 0$. σ could measure the developower because the standard variance can measure the fluctuation range of basic level.

$\nu = \sigma/\mu$—the fluctuation rate of basic level, ν could measure the development rate.

Z—the real economic level, real level for short. Z is a non-negative stochastic variable.

4.1 The basic conclusions

From assumption 1 and according to theorem 2, we have the following results about real level $Z \in P(\mu, \sigma^2)$:

(1) The expectation of Z, i.e. the average of Z, is $E(Z) = \mu + \sqrt{\frac{2}{\pi}}\sigma\frac{e^{-\frac{\mu^2}{2\sigma^2}}}{1+\sqrt{1-e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}}}$. Where, $R(Z) = \sqrt{\frac{2}{\pi}}\sigma\frac{e^{-\frac{\mu^2}{2\sigma^2}}}{1+\sqrt{1-e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}}}$ can valuates the average increment of real level to basic level, and this increment is caused by the original developower σ.

(2) The square of final developower is $D(Z) = \sigma^2 + E(Z)[\mu - E(Z)] = \sigma^2 - E(Z)R(Z)$. $E(Z) > \mu$ means the economy growing in its average level is a general trend though the economic recession may happen in some period of time. $D(Z) < \sigma^2$ means the economic growth needs to be pushed by the release in developower.

In reality, the real level will not be generally apart from the basic level if no serious accident happened. So we could suppose that the real level $Z \in P_a(\mu, \sigma^2)$. Specially, the real level follows SRPD, i.e., $Z \in F_a(\mu, \sigma^2)$.

The assumptions above are applicable to producing economy and nonproducting economy.

4.2 The average growth model in economic process

If the real level follows SRPD at a period of time, that is $Z(t) \in F_a(\mu(t), [\sigma(t)]^2)$, $t \in (t_0, T)$, t is omitted thereinafter. According to formula (4), the average of real level at the end of this period is

$$E(Z) = \mu + b\sigma, \tag{8}$$

where $b = \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{1}{2}}}{1+\sqrt{1-e^{-\frac{2}{\pi}}}}$=0.2869947990 is called the average growth coefficient. Formula (8) can also be expressed as $E(Z) = \mu(1 + b\nu)$, $\nu = \frac{\sigma}{\mu}$ is the original development rate.

According to formula (5), the square of developower at the end of this period is

$$D(Z) = [1 - (1 + b)b]\sigma^2 = b_0\sigma^2, \tag{9}$$

where $b_0 = [1 - (1 + b)b]$=0.6306391865, $\bar{b} = \sqrt{b_0} = \sqrt{1 - (1 - b(1 + b))}$=0.7941279409 is the stock coefficient of developower in average growth.

According to formula (8), the real economic growth rate in average is

$$g(v) = \frac{E(Z)}{\mu} - 1 = b\nu. \tag{10}$$

4.3 The golden growth model in economic process

If the real level $Z \in P(\mu, \sigma^2)$, according to reference (Dai, F., Xu, W..X., Liu, H. and H. Xu, 2003), then the optimal value of real level at the end of this period is $Z^* = \frac{\mu+\sqrt{\mu^2+4\sigma^2}}{2}$.

Correspondingly, the square of developower is $D^* = D(Z)+[E(Z)-Z^*]^2$. Where, the "optimal" means the product of the Z and its appearing probability reaches maximum.

If the real level $Z \in P_a(\mu, \sigma^2)$, the optimal value of real level at the end of this period is

$$Z^* = a + \frac{(\mu - a) + \sqrt{(\mu - a)^2 + 4\sigma^2}}{2}. \tag{11}$$

Further more, if $Z \in F_a(\mu, \sigma^2)$, i.e. $a = \mu - \sigma$, then

$$Z^* = \mu + c\sigma, \tag{12}$$

where, $c = \frac{\sqrt{5}-1}{2}$=0.618033989, is called the golden growth coefficient. Formula (12) can also be expressed as $Z^* = \mu(1 + cv)$, $v = \frac{\sigma}{\mu}$ is the original development rate.

Correspondingly, the square of developower is

$$D^* = \{[1 - (1 + b)b] + [b - c]^2\}\sigma^2 = c_1\sigma^2, \tag{13}$$

where, $c_1 = [1 - (1 + b)b] + [b - c]^2$=0.7402261316.
$\bar{c}^* = \sqrt{c_1} = \sqrt{1 - (1 - b(1 + b)) + (b - c)^2}$=0.8603639532 is the stock coefficient of developower in golden growth.

According to formula (12), the real economic growth rate in optimization is

$$g(\nu) = \frac{Z^*}{\mu} - 1 = c\nu \tag{14}$$

It is that, the growth coefficient $c = \frac{\sqrt{5}-1}{2}$=0.618033989 in expression (12) or (14) is just the golden ratio in mathematics. So, the economic process is called to follow the golden growth rule if its growth rate and development rate satisfies the equation (14).

In the meaning of expression (14), the development rate will be released to real growth rate according to its 61.8% under the golden growth rule. Here, the "release" means that the remained development rate does not reduce the same quantity as 61.8% from the original development rate, but $\bar{c}^*$=86.03639532% of the original development rate has been preserved according to formula (13). Similarly, the equation (10) means the development rate will be released to real growth rate according to its 28.7% under the average growth rule, and $\bar{b}$=79.41279409% of the original development rate has been preserved according to these the developower releasing is non-linearity and partly re-usable instead of linearity in economic growth.

4.4 The increment contribution of developower to economic growth

Let the basic level be constant and the economic process be divided to n stage, $Z_i \in F_a(\mu_i, \sigma_i^2)$ for stage i, $i = 0, 1, \cdots$. According to (8) and (9), denoting $\mu_0 = \mu$, $\sigma_0 = \sigma$, $\mu_{i+1} = E(Z_i)\mu_i + b\sigma_i$, $\sigma_{i+1} = \sqrt{D(Z_i)} = \bar{b}\sigma_i$, thus

$$E(Z_n) = \mu_n + b\sigma_n = \mu + b\sum_{i=0}^{n}\sigma_i = \mu + b\sigma\sum_{i=0}^{n}\left(\bar{b}\right)^i$$

This means the developower is released in the nonlinear way, i.e., total average increment contributions of original developower σ to economic growth on n times are added up as

$$R_n = b\sigma\sum_{i=0}^{n}\left(\bar{b}\right)^i = \frac{b\sigma[1-(\bar{b})^{n+1}]}{1-\bar{b}}$$

Let $n \to \infty$, then the total average increment contributions of developower σ is

$$R_\Sigma = \lim_{n\to\infty} R_n = \frac{b\sigma}{1-\bar{b}} = 1.394044438\sigma.$$

Correspondingly, the coefficient of total average increment contributions of developower is

$$I = 1.394044438. \tag{15}$$

Similarly, by use of (12) and (13), the developower is released also in the nonlinear way, i.e. the total optimal increment contributions of original developower σ to economic growth on n times are added up as

$$R_n^* = c\sigma\sum_{i=0}^{n}(\bar{c}^*)^i = \frac{c\sigma[1-(\bar{c}^*)^{n+1}]}{1-\bar{c}^*}$$

Let $n \to \infty$, the total optimal increment contributions of developower σ is

$$R_\Sigma^* = \lim_{n\to\infty} R_n^* = \frac{c\sigma}{1-\bar{c}^*} = 4.426034704\sigma.$$

Correspondingly, the coefficient of total optimal increment contributions of developower is

$$I^* = 4.426034704. \tag{16}$$

From the coefficients of increment contributions in expression (15) and (16), and if the real levels follow SRPD, we know the total average increment contribution and the total optimal increment contribution of unit developower are separately 1.394044438 and 4.426034704.

5 The strategic models for programming macroeconomic structure

Here we need to define two concepts: producing domain and functional realm. The economic producing domains, domain for short, are the domains to provide the products, commodities and serves to society or economic production, like industry, agriculture, business, tourism, etc. The economic functional realms, realm for short, are the different realms in economy which has its function, like production, policy, science and technology, education, management, sale, advertisement, etc. The economy can be divided into the various producing domains, called the producing type of economy, also can be divided into the different functional realms, called the functional type of economy.

5.1 Notation and basic relations

In addition to the notations in section 4, we also use the following notations for $i = 1, \cdots, n$, $j = 1, \cdots, m$:

μ_i—The basic level for ith producing domain, $\mu_i \geq 0$.

σ_i—The standard variance of basic level for ith producing domain, i.e. original development power of ith producing domain, $\sigma > 0$.

λ_j—The basic level for jth functional realm, $\lambda_j \geq 0$.

k_j—The standard variance of basic level for jth functional realm, i.e. the original development power of jth functional realm, $j > 0$.

Z—The real economic level, Z is a non-negative stochastic variable.

X_i—The real level of ith producing domain, X_i is a non-negative stochastic variable.

Y_j—The real level of jth functional realm, Y_j is a non-negative stochastic variable.

Suppose the real economic level is equal to the sum of all the real levels of producing domains, and the sum of all the real levels of functional realms, i.e.

$$Z = \sum_{i=1}^{n} X_i, \quad Z = \sum_{j=1}^{m} Y_j,$$

thus, when all the producing domains are independent one another, and all the functional realms are independent one another, have

$$E(Z) = \sum_{i=1}^{n} E(X_i) = \sum_{j=1}^{m} E(Y_j), \quad D(Z) = \sum_{i=1}^{n} D(X_i) = \sum_{j=1}^{m} D(Y_j).$$

If giving no special declaration thereinafter, all the producing domains are independent one another, and all the functional realms are independent one another. We will give a discussion on domains and realms in a certain period of time, i.e., the basic levels are their original values at the beginning of the period, and real levels are their final values at the end of the period.

Suppose the basic levels are constant in the discussed period of time thereinafter if we do not give a special declaration.

5.2 The analytic structure of economic growth under the golden model

We divide each of producing domains into the elementary components corresponding to the functional realms and each of functional realms into the elementary components corresponding to the producing domains. Suppose all the producing domains and the functional realms are independent one another, the real economic level $Z \in F_a(\mu, \sigma^2)$, the real level of ith producing domain $X_i \in F_a(\mu_i, \sigma_i^2)$, real level of jth functional realm $Y_j \in F_a(\lambda_j, \kappa_j^2)$, and real level of every elementary components follow SRPD, i.e., $X_{ij} \in F_a(\mu_{ij}, \sigma_{ij}^2)$, where,

$$X_i = \sum_{j=1}^{m} X_{ij}, Y_j = \sum_{i=1}^{n} X_{ij}, Z = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{ij} \quad i = 1, \cdots, n; j = 1, \cdots, m.$$

Combining the growth model in golden rule in (12) with the formula for computing developower on golden rule in (13), we can design the structural table for economic level, see Tab. 1.

The real levels can be valuated by real productive values in Tab. 1, like this, we can

• Compute the real economic productive value and developower, the real productive value and developower of producing domains, and functional realms contribution to the real economic productive value and their developowers on the golden rule.

• Compare the realistic productive values in national economy with those computed in Tab. 1, and judging that the realistic productive values are higher or lower.

Table 1. The analytic structure table for economic level based on golden growth rule

Realms / Domains	Y_1	$\cdots$	Y_m	The sum of real optimal levels of domains based on golden growth rule	The sum of domains developower based on golden growth rule
X_1	X_{11}	$\cdots$	X_{1m}	$X_1^* = \sum_{j=1}^{m} \mu_{1j}(1 + c\nu_{1j})$	$\sqrt{D^*(X_1)} = \sqrt{c_1 \sum_{j=1}^{m} \sigma_{1j}^2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_n	X_{n1}	$\cdots$	X_{nm}	$X_n^* = \sum_{j=1}^{m} \mu_{nj}(1 + c\nu_{nj})$	$\sqrt{D^*(X_n)} = \sqrt{c_1 \sum_{j=1}^{m} \sigma_{nj}^2}$
The sum of realms contribution to real optimal levels based on golden growth rule	$Y_j^* = \sum_{i=1}^{n} \mu_{ij}(1 + c\nu_{ij})$ $j = 1, \cdots, m$			$Z^* = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu_{ij}(1 + c\nu_{ij})$	
The sum of realms developower based on golden growth rule	$\sqrt{D^*(Y_j)} = \sqrt{c_1 \sum_{i=1}^{n} \sigma_{ij}^2}$ $j = 1, \cdots, m$				$\sqrt{D^*} = \sqrt{c_1 \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma_{ij}^2}$
Explanations	Real levels of elementary components $X_{ij} \in P(\mu_{ij}, \sigma_{ij}^2)$, $X_{ij}^* = \sum_{j=1}^{m} \mu_{ij}(1 + c\nu_{ij})$, $i = 1, \cdots, n; j = 1, \cdots, m. D^* = \sqrt{\sum_{i=1}^{n} D^*(X_i)} = \sqrt{\sum_{j=1}^{m} D^*(Y_j)}$. c, c_1 are given separately in formula (13) and (14), c=0.618033989, c_1=0.7402261316. $\nu_{ij} = \sigma_{ij}/\mu_{ij}$ is the development rate. The real levels can be valuated by productive values.				

• Compare the realistic economic growth rates with those computed by the formula in (14), and judge the real growth rates are higher or lower.

• Analyze the realistic economic structure is of equilibrium or not, and make realistic economic structure if needed.

We have a similar analysis for average economic growth according to the discussions above.

5.3 The model of controlling economic growth on the golden rule

If the real level $Z \in F_a(\mu, \sigma^2)$, the golden growth rate at the end of period discussed, from the expression (14), is

$$g(\nu) = \frac{Z^*}{\mu} - 1 = c\nu$$

The growth rate depends on the development rate v in above formula. If want the realistic growth rate to reach an expected value e, we need having a proper development rate. Let $g(\nu_e) = e$, then the proper development rate is

$$\nu_e = \frac{e}{c} \tag{17}$$

The corresponding developower is $\sigma_e = \mu \cdot v_e$. At the end of this period, the real value of development rate is

$$\nu(Z^*) = \frac{\sqrt{D^*}}{\mu}, \quad D^* \text{is determined by formula (13).}$$

Thus, the development rate needs to have an increment on its real value, i.e., the increment should be

$$\hat{\nu}^* = \sqrt{\nu_e^2 - \nu^2(Z^*)},$$

or the increment of developower should be

$$\hat{\sigma}^* = \mu \sqrt{\nu_e^2 - \nu^2(Z^*)}. \tag{18}$$

Here we suppose the nonphysical asset investing for increasing developower has been done at the moment after the beginning of this stage, to play the sufficient role of the asset developower.

If the basic level changes in the discussed period of time, the real value of development rate at the end of the period need to be modified as $v(Z^*) = \frac{\sqrt{D^*}}{Z^*}$, and the increment of developower should be $\hat{\sigma}^* = Z^* \sqrt{v_e^2 - v^2(Z^*)}$.

Based on the formula (10) and the method above, we can get a similar discussion on average growth rate.

5.4 The strategy model for macroeconomic programming based on developower

If knowing the value of $\hat{\sigma}^*$ determined by expression (18), we have separately the programming methods for producing domains and functional realms in economy as follows.

5.4.1 The programming methods for producing domains

Suppose the real economic level is as $Z \in F_a(\mu, \sigma^2)$, the economy includes n producing domains, their real levels are separately $X_i \in F_a(\mu_i, \sigma_i^2)(i = 1, \cdots, n)$, and each domain is divided into the elementary components corresponding to different functional realms, their real level are separately $X_{ij} \in F_a(\mu_{ij}, \sigma_{ij}^2)(j = 1, \cdots, m)$.

(1) programming economic developower according to domains

Let $\hat{\sigma}^* = \sqrt{\sum\limits_{i=1}^{n} \beta_i \left(\hat{\sigma}_i^*\right)^2}$ and $\beta_i = \frac{\sigma_i}{\sigma}$, the increments of developower for each domain[5], $\hat{\sigma}_i^*$, is given as following

$$\hat{\sigma}_i^* = \frac{\mu_i}{\beta_i} \frac{\hat{\sigma}^*}{\sqrt{\sum\limits_{i=1}^{n} \left(\frac{\mu_i^2}{\beta_i}\right)}} (i = 1, \cdots, n). \tag{19}$$

The expression (19) means the domain potential is larger if its basic level is higher and its quotient in original developower is smaller, this kind of domain should be paid more attention to invest.

(2) programming domain developower according to elementary components

If $\hat{\sigma}_i^*$ and $X_{ij} \in \quad P(\mu_{ij}, \sigma_{ij}^2)$ are known, then let $\hat{\sigma}_i^* = \sqrt{\sum\limits_{j=1}^{m} \gamma_{ij} \sigma_{ij}^2}$ be the increment of domain i, where $\sqrt{\gamma_{ij}}\sigma_{ij}$ is the increment of elementary components. The increment coefficients[6], γ_{ij}, is given as

$$\gamma_{ij} = \frac{\mu_{ij} \left(\hat{\sigma}_i^*\right)^2}{\sigma_{ij} \sum\limits_{j=1}^{m} \sigma_{ij} \mu_{ij}} (i = 1, \cdots, n; j = 1, \cdots, m), \tag{20}$$

and $\gamma_\sigma = \sqrt{\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} \gamma_{ij} \sigma_{ij}^2}$ is called the integrated index of elementary components on domains.

[5] The method of determining the increments of each domain is given in appendix.
[6] The method of determining the increment coefficients of each elementary component is given in appendix.

5.4.2 The programming methods for functional realms

Suppose the real economic level is as $Z \in F_a(\mu, \sigma^2)$, the economy includes m functional realms, their real levels $Y_j \in F_a(\lambda_j, \kappa_j^2)$, and each realm is divided into the elementary components in the various producing domains, their real level $X_{ij} \in F_a(\mu_{ij}, \sigma_{ij}^2)$.

(1) programming economic developower according to realms

Let $\hat{\sigma}^* = \sqrt{\sum_{j=1}^{m} \phi_j \left(\hat{\kappa}_j^*\right)^2}$ and $\phi_j = \frac{\kappa_j}{\sigma}$, the increments of developower for each realm[7], $\hat{\kappa}_j^*$, is given as following

$$\hat{\kappa}_j^* = \frac{\lambda_j}{\phi_j} \frac{\hat{\sigma}^*}{\sqrt{\sum_{j=1}^{m} \left(\frac{\lambda_j^2}{\phi_j}\right)}}, j = 1, \cdots, m.$$

The expression above means the realm potential is larger if its basic level is higher and its quotient in original developower is smaller, this kind of realm should be paid more attention to invest.

(2) programming realm developower according to elementary components

If $\hat{\kappa}_j$ and $X_{ij} \in P(\mu_{ij}, \sigma_{ij}^2)$ are known, then let $\hat{\kappa}_j^* = \sqrt{\sum_{i=1}^{n} \varphi_{ij} \sigma_{ij}^2}$ be the increment of realm j, where $\sqrt{\varphi_{ij}}\sigma_{ij}$ is the increment of elementary components. The increment coefficients[8], φ_{ij}, is given as

$$\varphi_{ij} = \frac{\mu_{ij} \left(\hat{\kappa}_j^*\right)^2}{\sigma_{ij} \sum_{i=1}^{m} \sigma_{ij} \mu_{ij}}, i = 1, \cdots, n; j = 1, \cdots, m.$$

$\varphi_\sigma = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} \varphi_{ij} \sigma_{ij}^2}$ is called the integrated index of elementary components on realms.

We will see, from the latter empirical researches, the integrated index of elementary components can be applied to evaluate the efficiency of investment for developower, so they could be taken as the basis to choose the final plan.

The methods above are also similar to the economic programming based on the average growth rule.

5.5 The analytic structure of economic growth under the meaning of correlation

If all the domains are correlated and all the realms are correlated, and let the real levels of domains follow the RPD, i.e., $\mathbf{X} \in P_a(\tilde{\mu}, \tilde{\sigma}^T\tilde{\sigma}, \mathbf{R})$, where $\mathbf{X} = (X_1, \cdots, X_n)^T$, $\tilde{\mu} = (\mu_1, \cdots, \mu_n)^T \geq \mathbf{0}$, $\tilde{\sigma} = (\sigma_1, \cdots, \sigma_n)^T \geq \mathbf{0}$, $\mathbf{a} = (a_1, \cdots, a_n)^T$, $\mathbf{R} = (r_{ij})_{n\times n}$, $| r_{ij} |\leq 1, i, j = 1, \cdots, n$. Taking the two domains for example, i.e., $(X_1, X_2) \in P_a(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$, and let $\mathbf{a} = \tilde{\mu} - \tilde{\sigma}$. From the section 3.2, we have the results for average growth rule as following

$$E_r(X_i) = \mu_i + \sigma_i[r + \sqrt{1 - r^2}b(r)] \text{ and } D_r(X_i) = [1 - (1 + b(r))b(r)]\sigma_i^2(1 - r^2), \tag{21}$$

where $b(r) = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{1}{2}\left(\frac{1+r}{1-r}\right)}}{1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{1+r}{1-r}\right)}}}$, $i = 1, 2$.

[7] The method of determining the increments of each realm is given in appendix.
[8] The method of determining the increment coefficients of each elementary component is given in appendix.

In the golden growth model, replacing μ and σ in expression (12) separately by $\mu_i + r\sigma_i$ and $\sqrt{1-r^2}\sigma_i$, and let a be replace separately by $a_i = \mu_i + r\sigma_i - \sqrt{1-r^2}\sigma_i$, thus

$$\begin{aligned} X_i^* &= \mu_i + \sigma_i[r + c\sqrt{1-r^2}] \\ D_i^* &= \sigma_i^2(1-r^2)\{[1-(1+b(r))b(r)] + [b(r)-c]^2\}, \end{aligned} \tag{22}$$

where, $c = \frac{\sqrt{5}-1}{2}$, $b(r)$ is the same in (21), $i = 1, 2$.

We can discuss the real levels and developowers on correlated realms in the same way as those of correlated domains. And we can discuss the economic programming under the meaning of correlation. According to section 5.2 — section 5.4.

For the average growth, denoting $g(r) = r + \sqrt{1-r^2}b(r)$ in expression (21), the $g(r)$ is monotone increasing and $g(0) = b(b = 0.2869947990$ is determined by explanation in expression (8)). When the two domains or realms are developing in the same direction one another, i.e., $r > 0, E_r(X_i)$ will be larger as σ_i become larger, that means both the real levels of two domains will increase as their developower increases. In reverse, When the two domains or realms are developing in the reverse direction one another, i.e., $r < 0, E_r(X_i)$ will be smaller as σ_i become larger, that means both the real levels of two domains will decrease as their developower increases. So, the effective developower investment to one domain or realm will be able to push the economic level to rise when the domains or realms in economy are in the same developing direction.

For the golden growth and denoting $g(r) = r + c\sqrt{1-r^2}$ in expression (22), $g(r)$ will reache its maximum at $r = \frac{1}{\sqrt{1+c^2}}$= 0.8506508081. So $g(r)$ is monotone increasing when $r \in [-1, 0.8506508081]$.

And in this field, if the two domains or realms are developing in the same direction one another, i.e., $r > 0$, then the larger the developower of domains or realms is, the higher their real level is. However, if the two domains or realms are developing in the reverse direction one another, i.e., $r < 0$, then the larger the developower is, the lower the real level is. When $r > 0.8506508081$, $g(r)$ is monotone decreasing. At this time, if the two domains or realms are developing in the same direction one another, then the larger the developower is, the lower the real level is. This means the two domains or realms should be amalgamated and in one.

6 The empirical researches and analysis

Here we take US GDP (chained) price index and GDP in billions of dollars (Fiscal Year 2000 = 1.000, http://www.whitehouse.gov) in the period of 1940-2004 as the scales to evaluate the economic level and the empirical samples.

6.1 The notations and descriptions

We have the notations and expressions as follows:

$\mu_1(t)$—GDP (Chained) price index of the year t, t=1940,$\cdots$, 2004.

$\mu_2(t)$—GDP in billions of dollars of the year t, t=1940,$\cdots$, 2004.

$\sigma_1(t)$—The fluctuation range of GDP (Chained) price index, $\sigma_1(t) = |\mu_1(t) - \mu_1(t-1)|$, t=1941,$\cdots$, 2004.

$\nu_1(t)$—The fluctuation rate of GDP (Chained) price index, $\nu_1(t) = |\mu_1(t) - \mu_1(t-1)|/\mu_1(t)$, t, t=1941,$\cdots$, 2004.

$\sigma_2(t)$—The fluctuation range of GDP in billions of dollars, $\sigma_2(t) = |\mu_2(t) - \mu_2(t-1)|$, t=1941,$\cdots$, 2004.

$\nu_2(t)$—The fluctuation rate of GDP in billions of dollars, $\nu_2(t) = |\mu_2(t) - \mu_2(t-1)|/\mu_2(t)$, t, t=1941,$\cdots$, 2004.

Here, $\sigma_1(t)$ and $\sigma_2(t)$ are separately the developowers scaled on GDP (Chained) price index and GDP in billions of dollars at year t, and $v_1(t)$ and $v_2(t)$ are separately the development rates scaled on GDP (Chained) price index and GDP in billions of dollars at year t. Because the interval unit of GDP data is one year, the economic status can be reflected stably and economic fluctuation can be nicely described by the difference between GDP value on one year and that of last year. So we adopt the formulas of $\sigma_1(t)$ and $\sigma_2(t)$ mentioned above to measure the developower, and $v_1(t)$ and $v_2(t)$ to measure the development rate. The curves of developower, development rate and economic level in US economy on GDP (Chained) price index are drawn in Fig. 1. In Fig. 1(a), the proportion of the index drawn to the real GDP (Chained) price index $\mu_1(t)$ is 1:20. In Fig. 1(b), the proportion of the index drawn to the real GDP (Chained) price index $\mu_1(t)$ is 1:10.

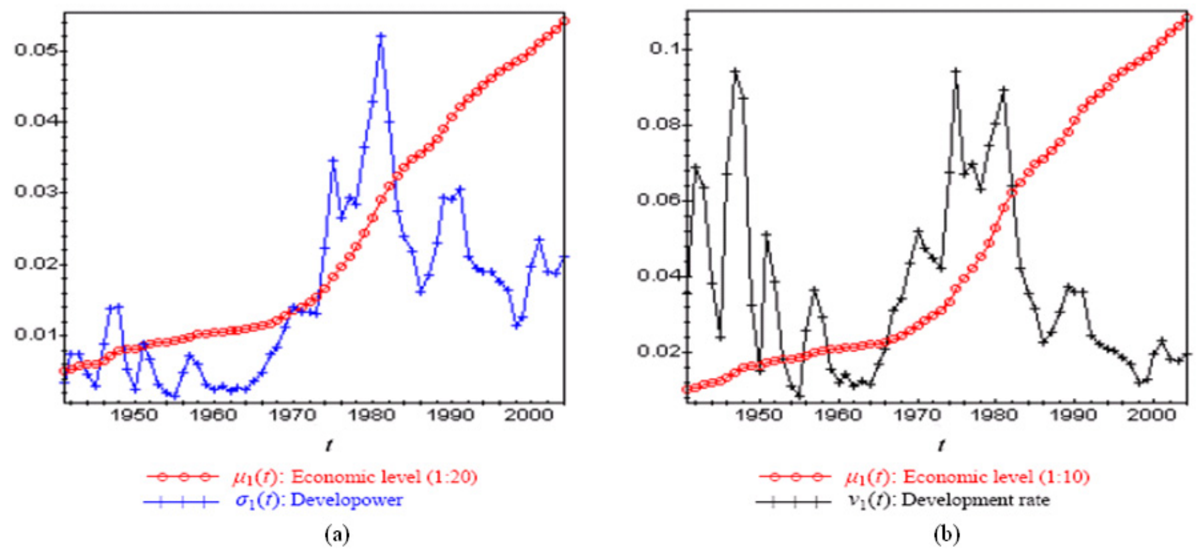


Fig. 1. The curves of US economic levels $\mu_1(t)$, the developowers $\sigma_1(t)$ and developower rates $\nu_1(t)$ in the time period from 1941 to 2004. $\mu_1(t)$ is valued by GDP (chained) price index (Fiscal Year 2000=1.000). In (a), the proportion of the drawn indexes of 1(t) to the real indexes is 1:20, and the proportion of the drawn indexes of 1(t) to the real indexes is 1:10 in (b). As we see, both the developower and the developower rate fluctuate always though the economic level 1(t) has been growing.

6.2 The fitness analysis for economic growth in golden rule and average rule

Here, we make a fitting on US GDP (Chained) price index according to the expression (12) and (8). And the fitting models are separately as follows

$$\text{the model on golden rule}: X_1^*(t) = \mu_1(t-1) + c\sigma_1(t-1)$$
$$\text{the model on average rule}: E_1(t) = \mu_1(t-1) + b\sigma_1(t-1)$$

where, c and b are separately determined by (12) and (8), t=1942, $\cdots$, 2004.

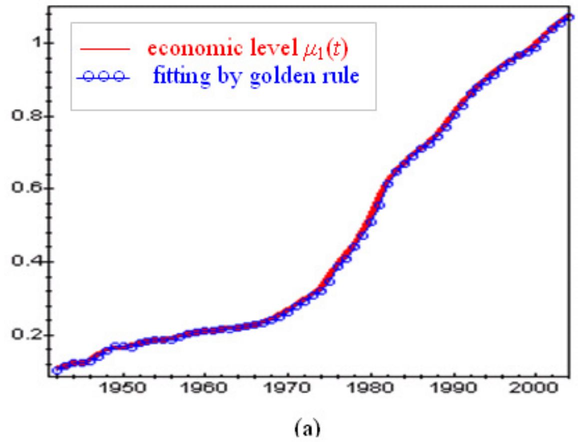
The curves of fitting are drawn in Fig. 2, the fitting of model on golden rule is in Fig. 2(a), and the fitting of model on average rule is in Fig. 2(b).

If we analyze the effect of fitting by average error and relative error, then the average error are separately $e_1^* = \frac{\sqrt{\sum_{t=1942}^{2004}[X_1^*(t)-\mu_1(t)]^2}}{2004-1942}$ and $e_1 = \frac{\sqrt{\sum_{t=1942}^{2004}[E_1(t)-\mu_1(t)]^2}}{2004-1942}$. And the relative error are separately $e_2^* = \frac{\sqrt{\sum_{t=1942}^{2004}[X_1^*(t)-\mu_1(t)]^2}}{\sum_{i=1942}^{2004}\mu_1(t)}$ and $e_2 = \frac{\sqrt{\sum_{t=1942}^{2004}[E_1(t)-\mu_1(t)]^2}}{\sum_{i=1942}^{2004}\mu_1(t)}$.

And then the error values on US GDP (Chained) price index are separately as

$$e_1^* = 0.00847006861, e_2^* = 0.002234007407; e_1 = 0.01420798692, e_2 = 0.003747401525.$$

Because $e_1^* < e_1$ and $e_2^* < e_2$, the golden growth model matches US economic process rather than the average growth model. Again, the error e_2^*=0.002234007407 is so small that we can think the golden growth model to be effective in US real economic process.



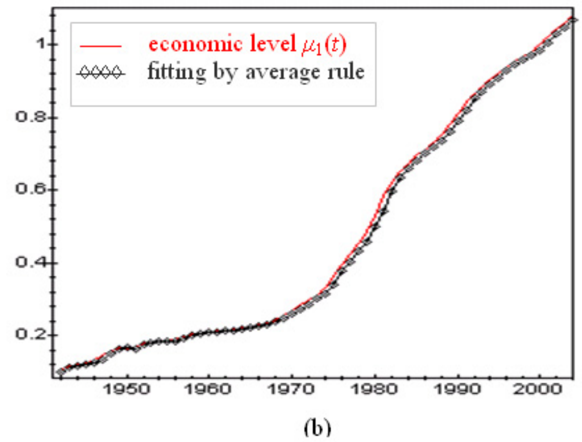


Fig. 2. The curves of US economic levels $\mu_1(t)$ and fitting for it in the period from 1941 to 2004. $\mu_1(t)$ is valued by GDP (chained) price index (Fiscal Year 2000=1.000). There is the comparison between the curves of real US economy and fitting by the golden growth rule in (a), and the comparison between the curves of real US economy and fitting by the average growth rule is in (b).

As for US GDP in billions of dollars, we can replace separately $X_1^*(t)$, $\mu_1(t)$ and $E_1(t)$ in the error models above by $X_2^*(t)$, $\mu_2(t)$ and $E_2(t)$. The error values are

$$e_1^* = 120.4741818, e_2^* = 0.004904053749; e_1 = 189.0383114, e_2 = 0.007695043250.$$

We also have $e_1^* < e_1$and $e_2^* < e_2$. These interpret further that US economic process follows the golden growth model rather than the average growth model. In the period of 1950-2000, US economy develops in a higher efficiency, and its economic growth is durative and in higher speed in macro meaning. So we think that the golden growth rule should be a basic rule in economic process which is of high-efficiency. If like this, the golden growth rule may be taken as a criterion by which we can judge the economic growth is high-effective or not.

6.3 Controlling and programming the structure for economic growth

Suppose the economy includes three producing domains: primary industry (agriculture, etc.), secondary industry (manufacture, etc), tertiary industry (service, etc), and five functional realms: production, science & technology, education, management and policy. And $Z \in F_a(\mu, \sigma^2)$, $X_i \in F_a(\mu_i, \sigma_i^2)$, $Y_j \in F_a(\lambda_j, \kappa_j^2)$ and $X_{ij} \in F_a(\mu_{ij}, \sigma_{ij}^2)$ are all known. The basic data and computed data are listed in Tab. 2.

According to Tab. 2 and the results in section 5.3, we know the real value of development rate at the end of this period is

$$\nu(Z^*) = \frac{\sqrt{D^*}}{\mu} = 0.08726680459, \tag{23}$$

i.e., the real growth rate on the golden model is $cv(Z^*)$ =0.05393385135.

If we hope the final growth rate is e= 0.07, then the development rate expected is $v_e = \frac{e}{c}$=0.1132623792 ($c = \frac{\sqrt{5}-1}{2}$). So, the increment of development rate is $\hat{v}^* = \sqrt{v_e^2 - v^2(Z^*)}$=0.07220021707 in order to reachv_e=0.1132623792, or the increment of developower is $\hat{\sigma}^* = \sqrt{\mu^2 v_e^2 - D^*}$=3.898811722.

If the data of industrial structure and three domains are known as in Tab. 2, the developing strategies for macroeconomic structure have been given in Tab. 3 by the methods in section 5.4.1.

If the data of functional structure and five realms are known as in Tab. 2, the developing strategies for macroeconomic structure have been given in Tab. 4 by the methods in section 5.4.2.

Comparing the results in Tab. 3 and Tab. 4, we know, the integrated indexes of elementary components on domains and realms are separately

Table 2. The data of economic structure on golden growth rule

Producing domains in economy	Functional realms in economy					Total productive value of domain on golden growth	Total developower of domain on golden growth
	Production Y_1	$S\&T$ Y_2	Education Y_3	Management Y_4	Policy Y_5		
Primary industry $X_1 \in F_a(12,6)$	$X_{11} \in F_a(5,3)$	$X_{12} \in F_a(3,1.2)$	$X_{13} \in F_a(2,0.8)$	$X_{14} \in F_a(1,0.4)$	$X_{15} \in F_a(1,0.6)$	$X_1^* =$ 15.16988107	$\sqrt{D^*(X_1)} =$ 2.107452678
Secondary industry $X_2 \in F_a(18,10)$	$X_{21} \in F_a(7,5)$	$X_{22} \in F_a(4,2)$	$X_{23} \in F_a(3,1.4)$	$X_{24} \in F_a(2,0.9)$	$X_{25} \in F_a(2,0.7)$	$X_2^* =$ 22.09066859	$\sqrt{D^*(X_2)} =$ 2.720709709
Tertiary industry $X_3 \in F_a(24,14)$	$X_{31} \in F_a(4,2)$	$X_{32} \in F_a(7,4.5)$	$X_{33} \in F_a(4,2.2)$	$X_{34} \in F_a(3,1.2)$	$X_{35} \in F_a(2,1.1)$	$X_3^* =$ 28.93492846	$\sqrt{D^*(X_3)} =$ 3.219187141
Total productive value of realms on golden growth	Y_1^* =23.83439829;Y_2^* =16.86210244; Y_3^* =11.20074663;Y_4^* =7.654219852; Y_5^* =6.644010919					Z^*= 66.19547812	
Total developower of realm on golden growth	$\sqrt{D^*(Y_1)}$ =3.102086350; $\sqrt{D^*(Y_2)}$ =2.387413080; $\sqrt{D^*(Y_3)}$ =1.804714653; $\sqrt{D^*(Y_4)}$ =1.360354854; $\sqrt{D^*(Y_5)}$ =1.332870105						$\sqrt{D^*}$= 4.712407448
Explanation	The real economic level $Z \in F_a(54,30), \mu = 54, \sigma^2 = 30$						

$$\gamma_\sigma = \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{5}\gamma_{ij}\sigma_{ij}^2} = 5.076044940 \text{ and } \varphi_\sigma = \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{5}\varphi_{ij}\sigma_{ij}^2} = 5.653075310$$

Table 3. The data for management strategies on realms

The increments of developower of each domains	The increment coefficients of developower for elementary components				
$\hat{\sigma_1^*} =$ 2.517870230	$\gamma_{11} =$ 1.208588886	$\gamma_{12} =$ 1.146568091	$\gamma_{13} =$ 0.9361689254	$\gamma_{14} =$ 0.6619713956	$\gamma_{15} =$ 0.5404973812
$\hat{\sigma_2^*} =$ 2.925500842	$\gamma_{21} =$ 0.9424140590	$\gamma_{22} =$ 0.8514785504	$\gamma_{23} =$ 0.7632836423	$\gamma_{24} =$ 0.6346546395	$\gamma_{25} =$ 0.7196307193
$\hat{\sigma_3^*} =$ 3.296665978	$\gamma_{31} =$ 0.8825974083	$\gamma_{32} =$ 0.8140469364	$\gamma_{33} =$ 0.6652828287	$\gamma_{34} =$ 0.6755979061	$\gamma_{35} =$ 0.4704259994
Explanation	The final growth rate : e =0.07		Development rate in expectancy: v_e =0.1132623792		
	The real growth rate : $cv(Z^*)$ =0.05393385135		Real development rate: $v(Z^*)$ =0.087266805		
	Difference for growth rates : $e - cv(Z^*)$ =0.016066149		Developower increment: $\hat{\sigma^*}$=3.898811722		
	Developower increment of elementary components: $\sqrt{\gamma_{ij}}\sigma_{ij}$, $\gamma_\sigma = \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{5}\gamma_{ij}\sigma_{ij}^2}$ =5.076044940				

The increment of developower caused by way on domains (in section 5.4.1) is equal to that of realms (in section 5.4.2), but $\gamma_\sigma < \varphi_\sigma$, this means more investment needs to increase the same developower by the way of realms than that of domains, so the way in domains should be better because it is more sparing in investing. By the discussion above on comparing the way in domains with that in realms in their integrated index of elementary components, we could know which one is better at any time and should be applied to carry.

Now we have introduced the application process about the methods to program the economic growth on golden model in empirical way. It is similar to the methods on average model. The difference between the methods on golden rule and on average rule is that the proper developower rate in expression (17) and the increment of developower in expression (18) are all different.

Table 4. The data for management strategies on realms

The increments of developower of each realms	$\hat{k}_1^* =$ 3.144428434	$\hat{k}_2^* =$ 2.860000232	$\hat{k}_3^* =$ 2.432201582	$\hat{k}_4^* =$ 2.151120016	$\hat{k}_5^* =$ 1.829564727
The increment coefficients of developower for elementary components	$\varphi_{11} =$ 0.6763433864	$\varphi_{12} =$ 0.9415078202	$\varphi_{13} =$ 1.173556278	$\varphi_{14} =$ 1.257949279	$\varphi_{15} =$ 0.9506813665
	$\varphi_{21} =$ 0.7334506679	$\varphi_{22} =$ 0.9723850958	$\varphi_{23} =$ 1.330687741	$\varphi_{24} =$ 1.677265704	$\varphi_{25} =$ 1.760319835
	$\varphi_{31} =$ 0.8382293348	$\varphi_{32} =$ 1.134449278	$\varphi_{33} =$ 1.415362126	$\varphi_{34} =$ 2.178832062	$\varphi_{35} =$ 1.404249442
Explanation	The final growth rate : e =0.07		Development rate in expectancy: v_e =0.1132623792		
	the real growth rate : $cv(Z^*)$ =0.05393385135		Real development rate: $v(Z^*)$ =0.087266805		
	Difference for growth rates : $e - cv(Z^*)$ =0.016066149		Developower increment: $\hat{\sigma}^*$=3.898811722		
	Developower increment of elementary components: $\sqrt{\varphi_{ij}}\sigma_{ij}$, $\varphi_\sigma = \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{5}\gamma_{ij}\sigma_{ij}^2}$ =5.6530753				

6.4 Analysis for developower to raise the productive value

In Tab. 2, the sum of basic levels on production realms at the end of this period is

$$\mu_1 = \sum_{i=1}^{3} \mu_{i1} = 20$$

and the total productive value (i.e., real level) on golden growth model is

$$Y_1^* = \sum_{i=1}^{3} (\mu_{i1} + c\sigma_{i1}) = 23.83439829$$

and the corresponding increment of total productive value is

$$Y_1^* - \mu_1 = 3.83439829$$

In another hand, the total productive value on the integrated developower (with all of realms) is

$$Z_1^* = \sum_{i=1}^{3} \left(\mu_{i1} + c\sqrt{\sum_{j=1}^{5} \sigma_{ij}^2} \right) = 25.78073443,$$

and the corresponding increment of total productive value is

$$Z_1^* - \mu_1 = 5.78073443.$$

We see, the increment of productive value brought by the developower from the functional realms of non-production is 5.78073443-3.83439829=1.94633614, and the percentage of total increment is 33.6693575%. So the functional realms of nonproduction can make great contributions to raise productive value, and this contribution is in sustained existence even it will be smaller and smaller.

In general, if the real level of production realm is $Y_1 \in P(\lambda_1, \kappa_1^2)$, the total real level of all the nonproduction realm is $Y \in P(\lambda, \kappa^2)$, and the real level of industrial economy $Z_1 \in P(\lambda_1, \sigma^2)$, where $\sigma^2 = \kappa_1^2 + \kappa^2$, then we have the following inequation according to formula (2)

$$E(Z_1) > E(Y_1).$$

Namely the real average output of industrial economy is larger than that of production, this means the factors of nonproduction is redound to raise the real level of economy and its contribution for average value can be valuated as

$$\varDelta = E(Z_1) - E(Y_1).$$

7 Conclusions

This paper has mainly done the works as follows:

• The developower and developower rate are given and applied to measure the uncertainty and fluctuation. And the key interrelation and interaction between productivity and developower have been described for economic growth.

• By means of the partial distribution (the univariate and multivariate partial distribution), the author has built the models for the relation between developower, basic economic level and real economic level, including the models in average relation and in optimal relation, where the "optimal" means the product of the economic level and its occurred probability is maximum. The basic character of these models is the real economic level is composed of basic economic level and economic increment, and the economic increment is caused from developower. In the process of developower pushing economy to grow, developower is released in the nonlinear not the linear (see also section 4.4).

• It is important for us to get a kind of the economic growth mode which is called the golden growth rule here. We analyze the contribution of developower to economic increment, and compute the coefficient of total quantity of contribution in average rule and in the golden rule. And they are separately as I =1.394044438 and I^*=4.426034704.

• Based on the partial distribution and division in producing domains and functional realms, we build the economic structure model, a kind of controlling and programming model for economic system in golden rule (see in Tab. 1). According to the division, the management strategies to control and program the economic growth structure are given.

• The empirical researches are made on the US GDP data (chained price index and in billions of dollars) from 1940 to 2004. It is illustrated that the golden growth model is better than the average growth model in the fitting with real US economic process. So the golden growth rule should be existed in the process of economic growth, and could be taken as a criterion by which we can judge the economic growth is high-effective or not.

Besides the coefficients of total quantity of contribution I and I^* (section 4.4), we also get the some of important constants as follows:

► The average growth coefficient b=0.2869947990 in formula (8).

► The coefficient of stock developower in average growth $\bar{b}$=0.7941279409 in formula (9).

► The golden growth coefficient c=0.618033989 in formula (12).

► The coefficient of stock developower in golden growth $\bar{c}^*$=0.8603639532 in formula (13).

These coefficients may reflect some of important numeric relations in economic growth.

Of course, we need to test further in practice whether the assumptions in this paper are suitable or not, all the basic conclusions are right or not, and many of the details in models are perfect or not. The authors hope the works in this paper to be useful to policymaking for economic planning.

References

[1] D. Cass. Optimum growth in an aggregative model of capital accumulation. *Review of Economic Studies*, 1965, **32**: 233–240.

[2] D. Cass, K.Shell. The structure and stability of competitive dynamical systems. *Journal of Economic Theory*, 1976, **12**: 31–70.

[3] T. Christiaans. Economic growth, the mathematical pendulum, and a golden rule of thumb. University of Siegen, 2001.

[4] P. Collier, D. Dollar. Development Effectiveness: What Have We Learnt? *Economic Journal*, 2004, **114**: 244–271.

[5] F. Dai. Developower: The Potential Motivity in Economic Process. *Working paper*, 2006. Http://mpra.ub.uni-muenchen.de/115/.

[6] F. Dai, G. P. Ji. A New Kind of Pricing Model for Commodity and Estimating Indexes System for Price Security. *Chinese Journal of Management Science*, 2001, **9**: 62–69.

[7] F. Dai, H. Liu, Y. Wang. Multivariate Partial Distribution: A new method of pricing group assets and analyzing the risk for hedging. *Working paper*, 2005. Http://econwpa.wustl.edu:80/eps/em/papers/0507/0507012.pdf.

[8] F. Dai, B. H. Sun, J. Sun. Derivative process model of development power in industry: Empirical research and forecast for Chinese software industry and us economy. *China-USA Business Review*, 2004, **3**: 1–17.

[9] F. Dai, W. X. Xu, et al. A new kind of optimal pricing for commodity. *Chinese Journal of Management Science*, 2003, **10**: 33–37.
[10] M. Fisz. *The probability Theory and Mathematical Statistics*. Shanghai Sci.& Tech. Press, 1978.
[11] B. Graziella. The Politics of Co-optation. *Journal of Comparative Economics*, 2001, **29**(561-579).
[12] B. Graziella. The Evolution of Modern Education Systems: Technical vs. General Education, Distributional Conflict and Growth. *Journal of Development Economics*, 2004, **73**: 559–582.
[13] P. Guillaumont, L. Chauvet. Aid and Performance: A Reassessment. *Journal of Development Studies*, 2001, **37**: 66–87.
[14] W. C. Horrace. Some results on the multivariate truncated normal distribution. *Journal of Multivariate Analysis*, 2005, **94**: 209–221.
[15] N. L. Johnson, S. Kotz, N. Balakrishnan. *Continuous Univariate Distributions*, 2nd edn. John Wiley and Sons, New York, 1994.
[16] R. King, C. Plosser, S. Rebelo. Growth and Business Cycles: I & II. *Journal of Monetary Economics*, 1988, **21**: 195–232, 309–341.
[17] S. Kotz, N. L. Johnson, N. Balakrishnan. *Continuous Multivariate Distributions: Models and Applications*. John Wiley and Sons, New York, 2000.
[18] F. Kydland, E. Prescott. Time to Build and Aggregate Fluctuations. *Econometrica*, 1982, **50**: 1345–1370.
[19] J. Long, C. Plosser. Real Business Cycles. *Journal of Political Economy*, 1983, **91**: 39–69.
[20] R. E. Lucas. *Understanding business cycles, Studies in Business Cycle Theory*, 215–239. MIT Press, Cambridge, 1981.
[21] R. E. Lucas. *Models of Business Cycles*. Basil Blackwell, New York, 1987.
[22] E. S. Phelps. Golden rule of capital accumulation: A fable for growthmen. *American economic review*, 1961, **52**: 638–643.
[23] C. Plosser. Understanding Real Business Cycles. *Journal of Economic Perspectives*, 1989, **3**: 51–77.
[24] P. M. Romer. Increasing Returns and Long-run Growth. *Journal of Political Economy*, 1986, **94**: 1002–1037.
[25] P. M. Romer. Endogenous technological change. *Journal of Political Economy*, 1990, **71-102**.

Appendix—the method to determine the coefficients by normal vector

If $(\hat{\sigma}^*)^2 = \sum_{i=1}^{n} \beta_i (\hat{\sigma}_i^*)^2$, then let $\beta_i = \frac{\sigma_i}{\sigma}$ (i.e., weights of branch of developower should correspond to its current proportion in total developower). When $\hat{\sigma}^*$ is given, the normal vector of hypersphere $(\hat{\sigma}^*)^2 = \sum_{i=1}^{n} \beta_i (\hat{\sigma}_i^*)^2$ is $(2\beta_i\hat{\sigma}_i^*, \cdots, 2\beta_n\hat{\sigma}_n^*)^T$. Let the normal vector be in the same direction with the vector $(\mu_1, \cdots, \mu_n)^T$ (this means the incremental structure of developower is consistent to the distributed structure of basic economic level, like this, the investment for developower reaches its maximum efficiency in increasing economic level), then

$$\frac{2\beta_i\hat{\sigma}_i^*}{\mu_i} = t \ \ (i = 1, \cdots, n),$$

i.e. $\hat{\sigma}_i^* = \frac{\mu_i}{2\beta_i} t$, and put it into $(\hat{\sigma}^*)^2 = \sum_{i=1}^{n} \beta_i (\hat{\sigma}_i^*)^2$, we get $t = \frac{2\hat{\sigma}^*}{\sqrt{\sum_{i=1}^{n} \frac{\mu_i^2}{\beta_i}}}$, thus

$$\hat{\sigma}_i^* = \frac{\mu_i}{\beta_i} \frac{\hat{\sigma}^*}{\sqrt{\sum_{i=1}^{n} \left(\frac{\mu_i^2}{\beta_i}\right)}}, \ \ i = 1, \cdots, n.$$

Further, when $\hat{\sigma}_i^*$ and $X_{ij} \in P(\mu_{ij}, \sigma_{ij}^2)$ are given, the normal vector of hypersphere $(\hat{\sigma}_i^*)^2 = \sum_{j=1}^{m} \gamma_{ij}\sigma_{ij}^2$ is $(2\gamma_{i1}\sigma_{i1}, \cdots, 2\gamma_{im}\sigma_{im})^T$, let the normal vector be in the same direction with the vector $(\mu_{i1}, \cdots, \mu_{im})^T$, then

$$\frac{2\gamma_{ij}\sigma_{ij}}{\mu_{ij}} = t \ \ (i = 1, \cdots, n; j = 1, \cdots, m,$$

i.e. $\gamma_{ij} = \frac{\mu_{ij}}{2\sigma_{ij}} t$, and put it into $(\hat{\sigma}_i^*)^2 = \sum\limits_{j=1}^{m} \gamma_{ij}\sigma_{ij}^2$, we get $t = \frac{2(\hat{\sigma}_i^*)^2}{\sum\limits_{j=1}^{m} \sigma_{ij}\mu_{ij}}$, thus

$$\gamma_{ij} = \frac{\mu_{ij}\,(\hat{\sigma}_i^*)^2}{\sigma_{ij}\sum\limits_{j=1}^{m} \sigma_{ij}\mu_{ij}} \ \ (i = 1, \cdots, n; j = 1, \cdots, m).$$