ACADIMIC
World Academic Union

ISSN 1479-3889 (print), 1479-3897 (online)
International Journal of Nonlinear Science
Vol.1 (2006) No.3, pp. 187-192

On Numerical Results of an Elliptic Boundary Value Problem

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(Received 9 December 2005, accepted 5 April, 2006)

Abstract: In this work we present a numerical approach for finding positive solutions of the type $-\Delta u = \lambda f(u)$ for $x \in \Omega$, with Dirichlet boundary condition, where f is a superlinear function of u. We will show in which range of λ, this problem achieves multiple numerical solutions and what the behavior of the branches of solutions is.

Keywords: elliptic boundary value problems; multiple solutions; finite difference method
AMS Subject Classification: 35J60, 35B30, 65N99

1 Introduction

In this paper we investigate numerically positive solutions of superlinear elliptic equations on bounded domains.

Numerical techniques based on finite difference schemes leading to parallel algorithms have been developed for obtaining approximate solutions of a boundary value problem of elliptic type in any dimension. This class of PDEs plays a very important role in many branches of science and engineering. For example such problems arise in a variety of situations, in the theory of nonlinear diffusion generated by nonlinear sources, in the theory of thermal ignition of gases (see [5,6]), in quantum field theory and mechanical statistics ([4,9]), and in the theory of gravitational equilibrium of stars ([6]). Existence and multiplicity results are proved by topological degree arguments variational techniques due to *A. Ambrosetti* and *P.H. Rabinowitz* in ([3,8]). We also refer to works of *H. Amann* (1,2]) for multiplicity results. So it is essential to approximate the solution of these partial differential equation numerically in order to investigate the predictions of the mathematical models, as the exact solutions are usually unavailable.

In this paper we shall consider the three dimensional superlinear elliptic equation of the type

$$\begin{cases} -\Delta u(x) = \lambda f(x, u(x)) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^N$ $(N \geq 3)$ with boundary $\partial\Omega$, and $f(u) = u - u^2 + au^3 - bu^5$, $0 < a << b << 1$ where $<<$ means very less than. And we show that the first eigenvalue of the problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{2}$$

is a bifurcation point of the branch of numerical solutions that all of these solutions are less than 1.Also there is a positive $\lambda^* > \lambda_1$ that for any $\lambda > \lambda^*$ we have two different numerical positive solutions.

This paper is organized in the following way: the main idea behind the finite difference technique is given in section 2 and Section 3 contains some numerical experiments of problem (1) for varying λ and a, b. At the end of this section the numerical results are presented.

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IJNS.2006.06.15/025

2 The central difference technique

The main idea behind the finite difference methods for obtaining the solution of a given partial differential equation is to approximate the derivatives appearing in the equation by a set of values of the function at a selected number of points. The most useful way to generate these approximations is through the use of Taylor series. The numerical techniques developed here are based on the modified equivalent partial differential equation as are described by [7]. This approach allows the simple determination of the theoretical order of accuracy, thus the allowing methods to be compared with one another. Also from the trunction error of the modified equivalent equation, it is possible to eliminate the dominant error terms associated with the finite difference equations that contain free parameters, thus leading to more accurate methods.

The algorithm presented in this section depends on the choice of grid and differentiation method. For simplicity we consider a simple region Ω, a regular grid and central differencing. One expects that increased accuracy and efficiency would be gained by using more sophisticated methods, and this is possible by using more grid points of the region Ω.

We will use the notation $\mathbf{u}$ to present an array of real numbers agreeing with u on a grid $\mathbf{\Omega} \subset \Omega$. In our experiment, we will take the grid to be regular, but it is clear that irregular grids with finer meshes would increase accuracy and efficiency. Also more sophisticated grid techniques will be necessary to investigate solutions when Ω itself is more complicated than an interval or a square.

For an interior net points (x, y, z) we can substitute the Laplacian operator the finite difference formula. In fact, let us assume that the function u has bounded derivatives of order up to four in Ω. Let us also assume that the points $(x+h, y, z)$, $(x-h, y, z)$, $(x, y+h, z)$, $(x, y-h, z)$, $(x, y, z+h)$, $(x, y, z-h)$ as well as the segments between the point (x, y, z) and the points $(x+h, y, z)$, $(x-h, y, z)$, $(x, y+h, z)$, $(x, y-h, z)$, $(x, y, z+h)$, $(x, y, z-h)$, respectively, lie in Ω. Then

$$u(x+h, y, z) = u(x, y, z) + hu_x(x, y, z) + \frac{1}{2}h^2 u_{xx}(x, y, z) + \frac{1}{6}h^3 u_{xxx}(x, y, z) + \frac{1}{24}h^4 u_{xxxx}(\tilde{\tilde{x}}, y, z)$$

$$u(x-h, y, z) = u(x, y, z) - hu_x(x, y, z) + \frac{1}{2}h^2 u_{xx}(x, y, z) - \frac{1}{6}h^3 u_{xxx}(x, y, z) + \frac{1}{24}h^4 u_{xxxx}(\tilde{x}, y, z)$$

$$u(x, y+h, z) = u(x, y, z) + hu_y(x, y, z) + \frac{1}{2}h^2 u_{yy}(x, y, z) + \frac{1}{6}h^3 u_{yyy}(x, y, z) + \frac{1}{24}h^4 u_{yyyy}(x, \tilde{\tilde{y}}, z)$$

$$u(x, y-h, z) = u(x, y, z) - hu_y(x, y, z) + \frac{1}{2}h^2 u_{yy}(x, y, z) - \frac{1}{6}h^3 u_{yyy}(x, y, z) + \frac{1}{24}h^4 u_{yyyy}(x, \tilde{y}, z)$$

$$u(x, y, z+h) = u(x, y, z) + hu_z(x, y, z) + \frac{1}{2}h^2 u_{zz}(x, y, z) + \frac{1}{6}h^3 u_{zzz}(x, y, z) + \frac{1}{24}h^4 u_{zzzz}(x, y, \tilde{\tilde{z}})$$

$$u(x, y, z-h) = u(x, y, z) - hu_z(x, y, z) + \frac{1}{2}h^2 u_{zz}(x, y, z) - \frac{1}{6}h^3 u_{zzz}(x, y, z) + \frac{1}{24}h^4 u_{zzzz}(x, y, \tilde{z})$$

Here $\tilde{x}$ and $\tilde{\tilde{x}}$ denote numbers contained between x and $x+h$, x and $x-h$, and $\tilde{y}$, $\tilde{\tilde{y}}$ and $\tilde{z}$, $\tilde{\tilde{z}}$ is defined in this way. Clearly,

$$u(x+h, y, z) + u(x-h, y, z) + u(x, y+h, z) + u(x, y-h, z) + u(x, y, z+h) + u(x, y-h) - 6u(x, y, z)$$

$$= h^2[u_{xx} + u_{yy} + u_{zz}] + \tfrac{1}{6}h^4 M_1 \theta$$

where $-1 \leq \theta \leq 1$ and M_1 denotes the least upper bound of the values of $|u_{xxxx}|$ and of $|u_{yyyy}|$ and of $|u_{yyyy}|$. Therefore the equation

$$\frac{\sum_{j=1}^{3}(u[(x, y, z) + he_j] + u[(x, y, z) - he_j]) - 6u(x, y, z)}{h^2}$$

is equivalent to the expression $u_{xx} + u_{yy} + u_{zz}$ to whithin terms of order h^2. In fact the essence of the method of differences for the solution of differential equations is that instead of solving a differential equation one solves a corresponding finite difference equation that is obtained by substituting differences expressions with higher or lower level of accuracy for the derivatives.

We substitute above approximation in ordinary and partial differential equation for $N = 1$ and $N > 1$ respectively. It is important to note, moreover, that on substitution of the differential equation, a difference equation is obtained that combines the value of the required function only in individual, discretely distributed points. The points are usually chosen so to form a quadrate network, i.e., we find an array $\mathbf{u}$ of real numbers agreeing with solution u on a grid $\mathbf{\Omega} \subset \Omega$ and then one can study the behavior of solution by considering this numerical solution.

The method of differences is especially suitable for the solution of boundary value problems, for instance, the problem of determining a function that satisfies the Laplace equation in the interior of a given field Ω and possesses given values at the boundary of the field; such problems arise in the exploration of stationary temperature distribution when the temperature at the boundary of the field is known, in investigating the tension in a twisted rod of prismetic section, etc. In this cases the procedure is as above.

3 Numerical results

In this section we consider problem (1) and use all of discussions in the previous section to find numerical solutions.

At first we note that to solve problem (1) we consider $N \geq 3$. Let $N = 3$ and $\Omega = [0,1] \times [0,1] \times [0,1]$ and the grid $\mathbf{\Omega} \subset \Omega$ be a division of Ω and $h = \frac{1}{4}(n_1 = n_2 = n_3 = 4)$.

We solve numerically the problem

$$\begin{cases} -(u_{xx} + u_{yy} + u_{zz}) = \lambda[u(x,y,z) - u(x,y,z)^2 + au(x,y,z)^3 - bu(x,y,z)^5] & (x,y,z) \in \Omega \\ u(x,y,z) = 0 & (x,y,z) \in \partial\Omega \end{cases} \tag{3}$$

for $a = 10^{-6}$ and $b = 10^{-3}$. Dirichlet boundary condition lead us to have

$$u_{0,j,k} = u_{i,0,k} = u_{i,j,0} = 0 \quad \forall 1 \leq i,j,k \leq 3$$

where $u_{i,j,k} = u(x_i, y_j, z_k)$.

By using the approximation of u_{xx} and u_{yy} and u_{zz} we have a system of equations of this type

$$-\frac{u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j+1,k} + u_{i,j-1,k} + u_{i,j,k+1} + u_{i,j,k-1} - 6u_{i,j,k}}{h^2} = \lambda(u_{i,j,k} - u^2_{i,j,k} + au^3_{i,j,k} - bu^5_{i,j,k}).$$

Some of the equations of this system are mentioned as follows:

$$16(u_{211} + u_{121}u_{112} - 6u_{111}) + \lambda(u_{111} - 3u^2_{111} + au^3_{111} - bu^5_{111}) = 0 \qquad for\ \ i = j = k = 1$$

$$16(u_{212} + u_{122} + u_{113} + u_{121} - 6u_{112}) + \lambda(u_{112} - u^2_{112} + au^3_{112} - bu^5_{112}) = 0 \qquad for\ \ i = j = 1, k = 2$$

$$\vdots$$

After solving this system we can obtain $\mathbf{u}$ in grid $\mathbf{\Omega}$ that guide us to understand the behavior of solution branches. For brevity we express just some of values of u_{ijk}s, w_{ijk}s and v_{ijk}s in the following tables. It is easy to see that λ^+ (the first eigenvalue of the problem (2)) in this case is 26.7 with decimal accuracy and λ^* is around 100 so that before it we don't have any positive solution greater than 1.

Note that the first table is for the branch of solution bifurcates from λ^+ and the second and third is for another branch of solutions. We mention them by $\mathbf{u}$, $\mathbf{w}$ and $\mathbf{v}$.

λ	**u**		
1	$u_{111}=2.05\times10^{-16}$	$u_{121}=3.69\times10^{-16}$	$u_{131}=2.06\times10^{-16}$
	$u_{211}=2.89\times10^{-16}$	$u_{221}=4.89\times10^{-16}$	$u_{231}=2.91\times10^{-16}$
	$u_{311}=1.69\times10^{-16}$	$u_{321}=2.78\times10^{-16}$	$u_{331}=1.72\times10^{-16}$
26.6	$u_{111}=3.75\times10^{-8}$	$u_{121}=5.68\times10^{-8}$	$u_{131}=3.77\times10^{-8}$
	$u_{211}=4.92\times10^{-8}$	$u_{221}=3.12\times10^{-8}$	$u_{231}=4.95\times10^{-8}$
	$u_{311}=3.34\times10^{-8}$	$u_{321}=4.79\times10^{-8}$	$u_{331}=3.39\times10^{-8}$
26.7	$u_{111}=0.0013$	$u_{121}=0.0020$	$u_{131}=0.0013$
	$u_{211}=0.0017$	$u_{221}=0.0025$	$u_{231}=0.0017$
	$u_{311}=0.0012$	$u_{321}=0.0017$	$u_{331}=0.0012$
50	$u_{111}=0.3197$	$u_{121}=0.4201$	$u_{131}=0.3200$
	$u_{211}=0.3986$	$u_{221}=0.5065$	$u_{231}=0.3992$
	$u_{311}=0.3100$	$u_{321}=0.3964$	$u_{331}=0.3111$
100	$u_{111}=0.5965$	$u_{121}=0.6954$	$u_{131}=0.5965$
	$u_{211}=0.6843$	$u_{221}=0.7835$	$u_{231}=0.6844$
	$u_{311}=0.5932$	$u_{321}=0.6838$	$u_{331}=0.5935$
1000	$u_{111}=0.9519$	$u_{121}=0.9673$	$u_{131}=0.9519$
	$u_{211}=0.9671$	$u_{221}=0.9823$	$u_{231}=0.9671$
	$u_{311}=0.9519$	$u_{321}=0.9671$	$u_{331}=0.9519$

λ	**w**		
100	$w_{111}=35.059$	$w_{121}=35.059$	$w_{131}=35.059$
	$w_{211}=35.059$	$w_{221}=35.6$	$w_{231}=35.059$
	$w_{311}=35.059$	$w_{321}=35.059$	$w_{331}=35.059$
500	$w_{111}=29.735$	$w_{121}=29.735$	$w_{131}=29.735$
	$w_{211}=29.735$	$w_{221}=29.735$	$w_{231}=29.735$
	$w_{311}=29.735$	$w_{321}=29.735$	$w_{331}=29.735$
1000	$w_{111}=28.667$	$w_{121}=28.668$	$w_{131}=28.667$
	$w_{211}=28.668$	$w_{221}=28.668$	$w_{231}=28.668$
	$w_{311}=28.667$	$w_{321}=28.668$	$w_{331}=28.667$

λ	**v**		
100	$v_{111}=38.599$	$v_{121}=38.599$	$v_{131}=38.599$
	$v_{211}=38.599$	$v_{221}=38.599$	$v_{231}=38.599$
	$v_{311}=38.599$	$v_{321}=38.599$	$v_{331}=38.599$
500	$v_{111}=39.307$	$v_{121}=39.307$	$v_{131}=39.307$
	$v_{211}=39.307$	$v_{221}=39.307$	$v_{231}=39.307$
	$v_{311}=39.307$	$v_{321}=39.307$	$v_{331}=39.307$
1000	$v_{111}=40.014$	$v_{121}=40.014$	$v_{131}=40.014$
	$v_{211}=40.014$	$v_{221}=40.014$	$v_{231}=40.014$
	$v_{311}=40.014$	$v_{321}=40.014$	$v_{331}=40.014$

Now we compare the numerical results for constants $a = 10^{-6}$ and $b = 0.001$ with $a = 10^{-4}$, $b = 0.1$ that is mentioned in the following tables.

λ	**u**		
1	u_{111}=2.08×10^{-16}	u_{121}=3.73×10^{-16}	u_{131}=2.09×10^{-16}
	u_{211}=2.92×10^{-16}	u_{221}=4.95×10^{-16}	u_{231}=2.95×10^{-16}
	u_{311}=1.71×10^{-16}	u_{321}=2.81×10^{-16}	u_{331}=1.74×10^{-16}
26.6	u_{111}=4.62×10^{-8}	u_{121}=7.00×10^{-8}	u_{131}=4.64×10^{-8}
	u_{211}=6.06×10^{-8}	u_{221}=8.77×10^{-8}	u_{231}=6.10×10^{-8}
	u_{311}=4.12×10^{-8}	u_{321}=5.90×10^{-8}	u_{331}=4.17×10^{-8}
26.7	u_{111}=0.0013	u_{121}=0.0020	u_{131}=0.0013
	u_{211}=0.0017	u_{221}=0.0025	u_{231}=0.0017
	u_{311}=0.0012	u_{321}=0.0017	u_{331}=0.0012
50	u_{111}=0.3169	u_{121}=0.4156	u_{131}=0.3172
	u_{211}=0.3949	u_{221}=0.5006	u_{231}=0.3955
	u_{311}=0.3075	u_{321}=0.3928	u_{331}=0.3086
100	u_{111}=0.5815	u_{121}=0.6731	u_{131}=0.5816
	u_{211}=0.6638	u_{221}=0.7539	u_{231}=0.6638
	u_{311}=0.5789	u_{321}=0.6634	u_{331}=0.5791
1000	u_{111}=0.8899	u_{121}=0.9020	u_{131}=0.8899
	u_{211}=0.9019	u_{221}=0.9137	u_{231}=0.9019
	u_{311}=0.8899	u_{321}=0.9019	u_{331}=0.8899

λ	**w**		
100	w_{111}=35.183	w_{121}=35.183	w_{131}=35.183
	w_{211}=35.183	w_{221}=35.183	w_{231}=35.183
	w_{311}=35.183	w_{321}=35.183	w_{331}=35.183
500	w_{111}=29.905	w_{121}=29.905	w_{131}=29.905
	w_{211}=29.905	w_{221}=29.905	w_{231}=29.905
	w_{311}=29.905	w_{321}=29.905	w_{331}=29.905
1000	w_{111}=28.849	w_{121}=28.849	w_{131}=28.849
	w_{211}=28.849	w_{221}=28.849	w_{231}=28.849
	w_{311}=28.849	w_{321}=28.849	w_{331}=28.849

λ	**v**		
100	v_{111}=38.701	v_{121}=38.701	v_{131}=38.701
	v_{211}=38.701	v_{221}=38.701	v_{231}=38.701
	v_{311}=38.701	v_{321}=38.701	v_{331}=38.701
500	v_{111}=39.405	v_{121}=39.405	v_{131}=39.405
	v_{211}=39.405	v_{221}=39.405	v_{231}=39.405
	v_{311}=39.405	v_{321}=39.405	v_{331}=39.405
1000	v_{111}=40.109	v_{121}=40.109	v_{131}=40.109
	v_{211}=40.109	v_{221}=40.109	v_{231}=40.109
	v_{311}=40.109	v_{321}=40.109	v_{331}=40.109

So by using the results in these tables we can draw the bifurcation diagram of the solutions in the plane $(\lambda, ||u||)$, where

$$||u|| = ||u||_\infty = \sup_{(x,y,z)\in[0,1]\times[0,1]\times[0,1]} u(x,y,z)$$

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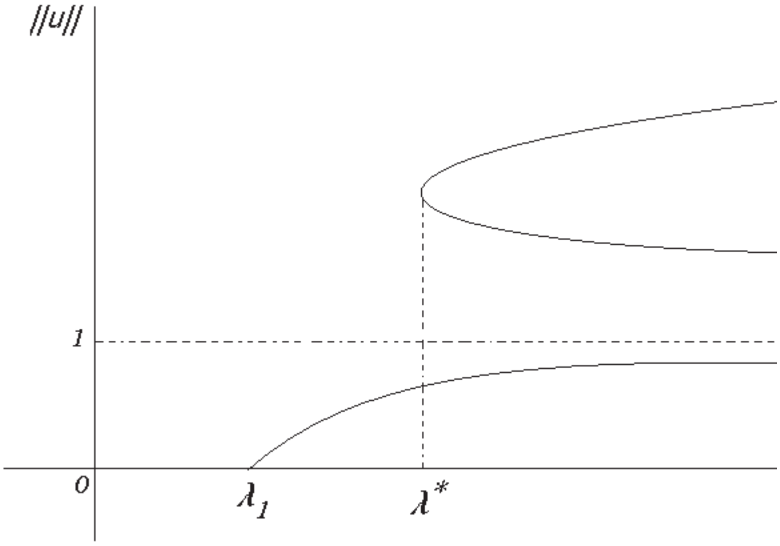

Figure 1: Bifurcation diagram

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