On the Well-posedness Problem for the Generalized Dullin-Gottwald-Holm Equation

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Abstract. In this paper, we study the local well-posedness of the Cauchy problem for the generalized DGH equation. By applying some Sobolev’s inequalities and related knowledge of PDE and using Kato’s theory, we prove that there is a unique local solution to this problem which continuously depends on the initial value.

Keywords: generalized DGH equation; Kato’s theory; local well posedness

1 Introduction

Dullin, Gottwald, Holm\cite{1} discussed the following 1 + 1 quadratically nonlinear equation (we call it DGH equation for short) as

\begin{equation}
m_t + c_0 u_x + um_x + 2mu_x = -\gamma u_{xxx}, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}
\end{equation}

where \(m = (1 - \alpha^2 \partial_x^2)u\) is a momentum variable, the constants \(\alpha\) and \(\gamma\) are squares of length scales, and \(c_0 = \sqrt{gh}\) (where \(c_0 = 2\omega\)) is the linear wave speed for undisturbed water at rest at spatial infinity. Eq. (1.1) was derived by using asymptotic expansions directly in the Hamiltonian for Euler’s equations in the shallow water regime and thereby was shown to be bi-Hamiltonian and has a Lax pair. Eq. (1.1) combines the linear dispersion of the Korteweg-de Vries equation with the nonlinear dispersion of the Camassa-Holm equation, yet still preserves integrability via the inverse scattering transform (IST) method. This IST-integrable class of equations contains both the KdV equation and CH equation as limiting cases.

Using the notation \(m = (1 - \alpha^2 \partial_x^2)u\), one can rewrite the DGH equation as

\begin{equation}
u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3mu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), \quad t > 0, \quad x \in \mathbb{R}
\end{equation}

In\cite{4} Lixin Tian, Guilong Gui and Yue Liu have recently studied the Cauchy problem and the scattering problem of Eq. (1.2). \cite{5}-\cite{6} researched the global theory blowup phenomenon and the Painleve analysis symmetry reductions of Eq.1.2 with strong dispersive term respectively.

In this paper, we are interested in the study of the Cauchy problem for a generalized case of Eq. (1.2). Thus replacing the term 3\(mu_x\) of Eq. (1.2) with 3\(um_x\), we can write the initial value problem of the generalized DGH equation as

\begin{equation}
\begin{cases}
u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3mu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), & t > 0, \quad x \in \mathbb{R} \\
u(0, x) = u_0(x)
\end{cases}
\end{equation}

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2 Notations

We shall use the following notations without further comments: \( \| \cdot \|_X \) for the norm of the Banach space \( X \); \( \mathcal{B}(X,Y) \) denotes the space of all bounded linear operator from \( X \) to \( Y \) \((\mathcal{B}(X)\) if \( X=Y \)); \( \mathcal{D}(A) \) for the domain of the operator \( A \), \( \partial = \partial_x = \partial_x^2 \); \( \Lambda^s = (1-\partial^2_x)^{s/2}, s \in \mathbb{R} ; H^s = H^s(\mathbb{R}) \) with norm \( \| f \|_{H^s} = \| f \|_{s} = (\int_{\mathbb{R}} (1+|\xi|^s)|\hat{f}(\xi)|^2d\xi)^{1/2} \) and \( \langle \cdot, \cdot \rangle_s \) for its inner product; \([A,B]\) denotes the commutator of the linear operators \( A \) and \( B \); and \( C(I;X) \) for the space of all continuous functions on an interval \( I \) into Banach space \( X \); if \( I \) is compact, it is seen as a Banach space with the sup norm.

3 Some useful lemmas

In this paper we apply some well-know lemmas. So we list them here without proof.

Lemma 3.1. Let \( s, t \in \mathbb{R} \) such that \( -s < t \leq s \). then
\[
\| fg \|_t \leq c \| f \|_s \| g \|_t \quad \text{if } s > \frac{1}{2}
\]
and
\[
\| fg \|_{s+t-m/2} \leq c \| f \|_s \| g \|_t \quad \text{if } s < \frac{1}{2}
\]
where \( c \) is a positive constant depending on \( s, t \).

Lemma 3.2 (kato’s). Let \( f \in H^r \), \( r > \frac{3}{2} \); \( M_f \) is the multiplication operator by \( f \). Then
\[
\Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f] \Lambda^{-\tilde{t}} \in \mathcal{B}(L^2(\mathbb{R})) \quad \text{if } \lvert \tilde{s} \rvert, \lvert \tilde{t} \rvert \leq r - 1
\]
Moreover
\[
\| \Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f] \Lambda^{-\tilde{t}} \omega \|_0 \leq c \| f' \|_{r-1} \| \omega \|_0
\]
where \( c > 0 \) is a constant.

Lemma 3.3. Let \( f, g \in H^s \) and \( s > \frac{1}{2} \), then
\[
\| fg \| \leq c \| f \|_s \| g \|_s
\]
That’s because \( H^s \) is a Banach algebra for \( s > \frac{1}{2} \).

Lemma 3.4. Let \( s > \frac{3}{2} \), then
\[
\| u_x \|_{L^\infty} \leq \| u \|_s
\]
This lemma derives directly from the Sobolev embedding theorem.

4 Kato’s theory

Consider the Cauchy problem associated to a quasilinear evolution equation
\[
\begin{align*}
\frac{\partial u}{\partial t} + A(u)u &= f(u) \in X, \ t \geq 0 \\
u(0) &= u_0 \in Y
\end{align*}
\]
where $A(u)$ is a linear operator depending on the unknown $u$, and $\phi$ the initial value. To study the Cauchy problem (local in the time) associated to (4.1) we will make the following assumptions:

(X) $X$ and $Y$ are reflexive Banach spaces where $X \subset Y$ with the inclusion continuous and dense, and there is an isomorphism $S$ from $Y$ onto $X$ such that $\|\phi\|_X = \|S\phi\|_X$ for all $\phi \in Y$.

(A1) Let $W$ be an open ball centered in 0 and contained in $Y$. The linear operator $A(u) \in G(X, 1, \beta)$ where $\beta$ is a real number, i.e., $-A(u)$ generates a $c_0$-semigroup such that

$$\|e^{-sA(u)}\|_{\mathcal{L}(X)}$$

Note that if $X$ is a Hilbert space, then $A(u) \in G(X, 1, \beta)$ if and only if

(a) $\langle A\phi, \phi \rangle_X \geq -\beta \|\phi\|_X^2$, $\forall \phi \in D(A)$

(b) $(A + \lambda)$ is an onto mapping for some (all) $\lambda > \beta$ Under these conditions $A(u)$ is said to be quasi-m-accretive.

(A2) The map

$$w \in W \to B(w) = [S, A(w)]S^{-1} \in \mathcal{B}(X)$$

is uniformly bounded and Lipchitz continuous, that is, there exist constants $\lambda_1, \mu_1 > 0$, such that

$$\|B(w)\|_{\mathcal{L}(X)} \leq \lambda_1$$
$$\|B(w) - B(y)\|_{\mathcal{L}(X)} \leq \mu_1 \|w - y\|_Y$$

for all $w, y \in W$.

(A3) $Y \subseteq D(A(w))$ for each $w \in W$ (so that $A|_Y \in \mathcal{B}(X, Y)$) by the Closed Graph theorem. Moreover, the map $w \in W \to A(u) \in \mathcal{B}(Y, X)$ satisfies the following Lipschitz condition:

$$\|A(w) - A(y)\|_{\mathcal{L}(Y, X)} \leq \mu_2 \|w - y\|_X$$

for all $w, y \in W$, where $\mu_2$ is a non-negative constant.

(f1) The function $f : W \to Y$ is bounded, i.e. there is a constant $\lambda_2 > 0$ such that

$$\|f(w)\|_Y \leq \lambda_2$$

for all $w \in W$, and the function $w \in W \to f(w)$ is Lipschitz in $X(Y)$, i.e.

$$\|f(w) - f(y)\|_X \leq \mu_3 \|w - y\|_X, \forall w, y \in W$$
$$\|f(w) - f(y)\|_Y \leq \mu_4 \|w - y\|_Y, \forall w, y \in W$$

where $\mu_3, \mu_4$ is non-negative constant.

In practice, as we will see, the value of $\beta, \lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3, \mu_4$ are functions of $r$, the radius of ball $W$.

**Theorem 1 (Kato).** Assume conditions (X), (A1) $\sim$ (A3), (f1). Given $u_0 \in Y$, there is $T > 0$ and unique solution $u \in C(I, Y) \cap C^1(I, X)$ to (4.1) with $u(0) = u_0$, where $I = [0, T]$. Moreover, the map $u_0 \in Y \to u \in C([0, T], Y)$ is continuous in the following sense:

Assume that $s- \sup \limits_{n \to \infty} A_n(w) \stackrel{\mathcal{L}(X,Y)}{=} A_\infty(w)$, $s- \sup \limits_{n \to \infty} B_n(w) \stackrel{\mathcal{L}(X)}{=} B_\infty(w)$, $\lim \limits_{n \to \infty} f_n(w) \stackrel{\mathcal{L}(Y)}{=} f_\infty(w)$, $\lim \limits_{n \to \infty} u_{0,n} = u_{0,\infty}$ where $s - \lim$ denotes the strong limit. Consider the sequence of Cauchy problems

$$\begin{cases}
\partial_t u_n + A_n(u_n)u_n = f_n(u_n) \\
u_n(0) = u_{0,n}, \quad n \in Z \cup \{\infty\}
\end{cases}$$

(4.7)

Suppose also that (X), (A1) $\sim$ (A3), (f1) holding for all equations in (4.6) with the same $X, Y, S, W$, and that the constants $\beta, \lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3, \mu_4$ can be chosen independently of $n$. Let $T_n$ be the time of existence of the solutions $u_n$. Then all $u_n$’s with $n < \infty$ sufficiently large, can be extended (if necessary) to $[0, T_\infty]$ and

$$\lim \sup \limits_{n \to \infty} \|u_n(t) - u_\infty(t)\|_Y = 0$$

Conditions (4.2) and (4.6) are not needed in the proof of existence and uniqueness of solutions. They are used only in the proof of continuous dependence.
5 Local theory

In this section we will apply Kato’s theory to establish local well posedness for the Cauchy problem associated to the generalized Dullin-Gottwald-Holm equation. The equation (1.3) can be rewritten in the following way

\[
\begin{cases}
   u_t + uu_x - \frac{\gamma}{\alpha^2} u_x = -\partial_x (1 - \alpha^2 \partial_x^2)^{-1} \left( \frac{\alpha^2}{2} (u_x)^2 - \frac{1}{2} u^2 (2w + \frac{\gamma}{\alpha^2}) u + \frac{3}{m + 1} u^{m+1} \right) \\
u(0, x) = u_0(x)
\end{cases}
\]  

(5.1)

**Theorem 2.** Let \( u_0 \in H^s, s > \frac{3}{2} \) then there exist \( T > 0 \) depending on \( \|u_0\|_s \), and unique solution \( u \) to (1.3) (or (5.1)) such that

\[ u \in C([0, T], H^s) \cap C^1([0, T], H^{s-1}) \]

Moreover, the map

\[ u_0 \in H^s \rightarrow u \in C([0, T], H^s) \]

is continuous in the sense described in theorem 1

To prove this result we will apply Theorem 1, with

\[ X = H^{1/2}, Y = H^s, S = \Lambda^{s-1/2}, \Lambda = (1 - \partial_x^2)^{1/2}, \]

\[ A(u) = (u - \frac{\gamma}{\alpha^2}) \partial_x f(u) = -\partial_x (1 - \alpha^2 \partial_x^2)^{-1} \left( \frac{\alpha^2}{2} (u_x)^2 - \frac{1}{2} u^2 (2w + \frac{\gamma}{\alpha^2}) u + \frac{3}{m + 1} u^{m+1} \right) \]

and

\[ W = \{ \varphi \in H^s | \|\varphi\|_s \leq R \} = \overline{B}(0, R) \]

We begin with the following lemma.

**Lemma 5.1.** The operator \( A(u) = (u - \frac{\gamma}{\alpha^2}) \partial_x \in G(H^{1/2}, 1, \beta) \) with \( u \in H^s, s > \frac{3}{2} \).

**Proof:** First, we claim that

\[ \langle (u - \frac{\gamma}{\alpha^2}) \partial_x \varphi, \varphi \rangle_{1/2} \geq -\beta \|\varphi\|^2_{1/2} \]  

(5.2)

we write the left hand side of (5.2) as follows:

\[
\langle (u - \frac{\gamma}{\alpha^2}) \partial_x \varphi, \varphi \rangle_{1/2} = \langle \Lambda^{1/2} (u - \frac{\gamma}{\alpha^2}) \partial_x \varphi, \Lambda^{1/2} \varphi \rangle_0 \\
= \langle [\Lambda^{1/2}, u] \partial_x \varphi + u \partial_x \Lambda^{1/2} \varphi - \Lambda^{1/2} \frac{\gamma}{\alpha^2} \partial_x \varphi, \Lambda^{1/2} \varphi \rangle_0 \\
= \langle \Lambda^{1/2} \partial_x \varphi, \Lambda^{1/2} \varphi \rangle_0 - \langle \partial_x u, (\Lambda^{1/2} \varphi)^2 \rangle_0 - \frac{\gamma}{\alpha^2} \langle \Lambda^{1/2} \partial_x \varphi, \Lambda^{1/2} \varphi \rangle_0
\]

we will show that each of the terms on the RHD of above can be estimated by \( c \|u\|_s \|\varphi\|^2_{1/2} \), where \( c \) is a positive constant. We have

\[
\left| \langle \Lambda^{1/2}, u \rangle \partial_x \varphi, \Lambda^{1/2} \varphi \rangle_0 \right| \leq \|\Lambda^{1/2}, u \|_{1/2} \|\Lambda^{1/2} \partial_x \varphi \|_0 \|\Lambda^{1/2} \varphi \|_0 \\
\leq \|u\|_s \|\varphi\|^2_{1/2}
\]

(5.3)

where we applied Lemma 3.2 with \( \tilde{s} = 0, \tilde{t} = -\frac{1}{2} \). According to lemma 3.4 the second term is bounded by

\[
\left| \langle \partial_x u, (\Lambda^{1/2} \varphi)^2 \rangle_0 \right| \leq \|u_x\|_{\infty} \|\Lambda^{1/2} \varphi \|^2_0 \leq \|u\|_s \|\varphi\|^2_{1/2}
\]

(5.4)

and the last term of the RHS can be easily estimated as follows

\[
\left| \frac{\gamma}{\alpha^2} \langle \Lambda^{1/2} \partial_x \varphi, \Lambda^{1/2} \varphi \rangle_0 \right| \leq \frac{\gamma}{2\alpha^2} \|\varphi\|^2_{1/2}
\]

(5.5)

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so combining (5.3) (5.4) (5.5) we get (5.2).

To complete the proof, we need to show that \((A(u) + \lambda)\) is onto \(H^{1/2}\) for some \(\lambda > \beta\). The fact that \(A(u)\) is a closed operator, together with inequality (5.2) shows that \((A(u) + \lambda)\) has closed range for \(\lambda > \beta\). Thus, it suffices to prove that \((A(u) + \lambda)\) has dense range for \(\lambda > \beta\). Let \(\varphi \in H^{1/2}\) be such that \(\langle (A(u) + \lambda)\phi, \varphi \rangle = 0\) for all \(\phi \in \mathcal{D}(A(u)) = \{ \phi \in H^{1/2} | (u - \frac{\gamma}{\alpha^2}) \partial_x \phi \in H^{1/2} \}\). Then

\[
\varphi \in \mathcal{D}(((u - \frac{\gamma}{\alpha^2}) \partial_x)^*) = \{ g \in H^{1/2} | (u - \frac{\gamma}{\alpha^2}) \Lambda g \in H^{1/2} \} \subseteq \mathcal{D}(A(u))
\]

and satisfies the equation

\[-\Lambda^{-1} \partial_x((u - \frac{\gamma}{\alpha^2}) \Lambda \varphi) + \lambda \varphi = 0.\]

Applying \(\Lambda\) to this equation, multiplying by \(\Lambda^{1/2} \varphi\) integrating by parts and invoking (5.2), we obtain

\[0 = \langle \Lambda^{1/2}((u - \frac{\gamma}{\alpha^2}) \partial_x \varphi) + \lambda \Lambda^{1/2} \varphi, \Lambda^{1/2} \varphi \rangle_0 \geq (\lambda - \beta) \| \varphi \|^2_{1/2}, \lambda > \beta
\]

since \((\lambda - \beta) > 0\), we conclude that \(\varphi = 0\).

**Lemma 5.2.** (i) \(B(\Lambda) = [\Lambda^{1/2}, (u - \frac{\gamma}{\alpha^2}) \partial_x] \Lambda^{1/2-s} \in \mathcal{B}(H^{1/2})\) for \(u \in H^s, s > \frac{3}{2}\)

(ii) \(\| (B(u) - B(v)) w \|_{1/2} \leq \| w \|_{1/2} \| u - v \|_s, s > \frac{3}{2}\)

**Proof:** (i) Note that

\[[\Lambda^{1/2}, (u - \frac{\gamma}{\alpha^2}) \partial_x] \Lambda^{1/2-s} = [\Lambda^{1/2}, u \partial_x] \Lambda^{1/2-s} - [\Lambda^{1/2}, \frac{\gamma}{\alpha^2} \partial_x] \Lambda^{1/2-s} = [\Lambda^{1/2}, u] \Lambda^{1/2-s} \partial_x - 0
\]

Therefore

\[
\| B(u) w \|_{1/2} = \| \Lambda^{1/2}[\Lambda^{1/2-s} u \Lambda^{1-s} \Lambda^{-1/2} \partial_x w] \|_0 \\
\leq \| u \|_s \| \Lambda^{-1/2} \partial_x w \|_0 \\
\leq \| u \|_s \| w \|_{1/2}
\]

where we used Lemma 3.2 once again (with \(s = -\frac{1}{2}, \ell = s - 1\)).

(ii) Note that, replacing \(u\) by \(u - v\) in inequality (5.6) we get

\[
\| (B(u) - B(v)) w \|_{1/2} \leq \| w \|_{1/2} \| u - v \|_s
\]

We complete the proof of lemma 5.2.

**Lemma 5.3.** (i) \(H^s \subseteq \mathcal{D}((u - \frac{\gamma}{\alpha^2}) \partial_x) = \{ f \subseteq H^{1/2} | (u - \frac{\gamma}{\alpha^2}) \partial_x f \in H^{1/2} \}, s > \frac{3}{2}\)

(ii) \(A(u) \in \mathcal{B}(H^s, H^{1/2}), s > \frac{3}{2}\)

(iii) \(\| A(u) - A(v) \|_{\mathcal{B}(H^s, H^{1/2})} \leq \mu \| u - v \|_{1/2}\)

**Proof:** Let \(w \in H^s, s > \frac{3}{2}\). Then

\[
\| (u - \frac{\gamma}{\alpha^2}) \partial_x w \|_{1/2} \leq \| u - \frac{\gamma}{\alpha^2} \|_{1/2} \| \partial_x w \|_{s-1} \leq \| u - \frac{\gamma}{\alpha^2} \|_{1/2} \| w \|_{s}
\]

where we have used Lemma 3.1. This proves (i) and (ii). Part (iii) follows at once from this inequality, replacing \(u\) by \(u - v\).

**Lemma 5.4.** Let \(f(u) = -\partial_x(1 - \alpha^2 \partial_x^2)^{-1}(\frac{\gamma^2}{\alpha^2} u_x^2 - \frac{1}{2} u^2 + 2w + \frac{\gamma}{\alpha^2} u + \frac{3}{m+1} u^{m+1})\). Then

(i) \(\| f(u) \|_s \leq \mu, s > \frac{3}{2}\)

(ii) \(\| f(u) - f(v) \|_{1/2} \leq c \| u - v \|_{1/2}\)

(iii) \(\| f(u) - f(v) \|_s \leq c \| u - v \|_s, s > \frac{3}{2}\)

where \(\mu\) and \(c\) are positive constants.

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Proof: We begin our proof with (ii). Note that
\[
\|f(u) - f(v)\|_{1/2} \leq \|\Lambda^{-1/2}(u^2 - v^2)\|_0 + \frac{\alpha^2}{2}\|\Lambda^{-1/2}\partial_x(u - v)\partial_x(u + v)\|_0 \\
+ \left(2w + \frac{\gamma}{\alpha^2}\right)\|\Lambda^{-1/2}(u - v)\|_0 + \frac{3}{m+1}\|\Lambda^{-1/2}(u^{m+1} - v^{m+1})\|_0
\]
\[
\leq \|(u + v)(u - v)\|_0 + \frac{\alpha^2}{2}\|\partial_x(u + v)\partial_x(u - v)\|_{-1/2} + \left(2w + \frac{\gamma}{\alpha^2}\right)\|u - v\|_{1/2}
\]
\[
+ \frac{3}{m+1}\|u^{m+1} - v^{m+1}\|_0
\]

The RHS of this inequality can be estimated as follows
\[
\left\| (u - v)(u + v) \right\|_0 \leq \left\| u + v \right\|_{L^\infty} \| u - v \|_0 \leq \left\| u + v \right\|_s \| u - v \|_{1/2} \quad (5.7)
\]
\[
\left\| \partial_x(u + v)\partial_x(u - v) \right\|_{-1/2} \leq \left\| \partial_x(u - v) \right\|_{-1/2} \left\| \partial_x(u + v) \right\|_{s-1} \leq \left\| u + v \right\|_s \| u - v \|_{1/2} \quad (5.8)
\]
\[
\left\| u^{m+1} - v^{m+1} \right\|_0 = \left\| (u - v)(u^m + u^{m-1}v + \ldots + v^m) \right\|_0
\]
\[
\leq \left\| u - v \right\|_0 \left\| u^m + u^{m-1}v + \ldots + v^m \right\|_s
\]
In these terms, we applied lemma 3.1 and lemma 3.4. But for the third term we have to make further estimates. Note that \( s > \frac{3}{2} \), so we can use lemma 3.3. Therefore we obtain
\[
\left\| u^m + u^{m-1}v + \ldots + v^m \right\|_s \leq \left\| u^m \right\|_s \left\| v \right\|_s + \ldots + \left\| v^m \right\|_s \stackrel{\Delta}{=} K < \infty
\]

Then
\[
\left\| u^{m+1} - v^{m+1} \right\|_0 \leq K\left\| u - v \right\|_{1/2} \quad (5.9)
\]

Combining (5.8)-(5.9) we complete the proof of (ii)
For (iii) we can directly apply lemma 3.3 and get the following conclusion
\[
\left\| f(u) - f(v) \right\|_s \leq \left\| \partial_x\left(1 - \alpha^2\partial_x^2\right)^{-1}\left(\frac{\alpha^2}{2}(u_x^2 - v_x^2) - \frac{1}{2}(u^2 - v^2)\right)\right\|_s
\]
\[
+ \left\| \partial_x\left(1 - \alpha^2\partial_x^2\right)^{-1}(2w + \frac{\gamma}{\alpha^2})(u - v) + \frac{3}{m+1}(u^{m+1} - v^{m+1})\right\|_s
\]
\[
\leq \left\| \frac{\alpha^2}{2}(u_x^2 - v_x^2) \right\|_{s-1} + \frac{1}{2}(u^2 - v^2)\right\|_{s-1} + \left(2w + \frac{\gamma}{\alpha^2}\right)(u - v)\right\|_{s-1}
\]
\[
+ \frac{3}{m+1}(u^{m+1} - v^{m+1})\right\|_{s-1}
\]

Where
\[
\left\| u_x^2 - v_x^2 \right\|_{s-1} \leq \left\| (u + v)(u - v) \right\|_s \leq \left\| u + v \right\|_s \| u - v \|_s
\]
\[
\left\| u^2 - v^2 \right\|_{s-1} \leq \left\| (u + v)(u - v) \right\|_s \leq \left\| u + v \right\|_s \| u - v \|_s
\]
\[
\left\| u - v \right\|_{s-1} \leq \left\| (u - v)(u + v) \right\|_s \leq \left\| u + v \right\|_s \| u - v \|_s
\]
\[
\left\| u^{m+1} - v^{m+1} \right\|_{s-1} \leq K\| u - v \|_s
\]

Therefore, there is constant \( c > 0 \) satisfying (iii). We can easily find that (i) is an immediate consequence of (iii), since we choose \( v = 0 \). So far, we have verified conditions of theorem 1. That is to say the proof of theorem 2 complete.

Theorem 3. The existence time for DGH may be chosen independently of \( s \) in the following sense. If
\[
u \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})
\]

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is the solution of equation (1.2) with $u(0) = u_0 \in H^\gamma$ for some $r \neq s, r > \frac{3}{2}$ then,

$$u \in C([0, T], H^\gamma) \cap C^1([0, T], H^{\gamma - 1})$$

with the same time $T$. In particular, if $u_0 \in H^\infty$ then

$$u \in C([0, T], H^\infty).$$

The proof of this result is essentially the same as that of part (c) of Theorem I in [13]. For this reason we will only sketch it briefly.

Observe that it is sufficient to prove case $r > s$ since the other case is a consequence of uniqueness. Also we can suppose that $s < r \leq s + 1$, since if $r > s + 1$, we obtain the result by iterated application of the below argument.

Equation (5.1) can be rewritten in the following way:

$$v_t + A(t)v + B(t)v = f(t)$$

(5.10)

where

$$A(t)v = \partial_x (u(t)v - \frac{\gamma}{\alpha x^2}), B(t)v = u_x v,$$

$$f(t) = u_x (3p^{-1}v - \frac{\gamma}{\alpha^2} - 2w - 3(p^{-1}v)^m), v(t) = pu(t),$$

$$p = (1 - \alpha^2 \partial_x^2)$$

and

$$u \in C([0, T], H^s)$$

is viewed as a known function.

Note also that

$$v \in C([0, T], H^{s-2})$$

and

$$v(0) = (1 - \alpha^2 \partial_x^2)u_0 \in C([0, T], H^{r-2}).$$

Our objective is to prove that $v \in C([0, T], H^{r-2})$ which will imply $u \in C([0, T], H^r)$. To accomplish this we will prove that the Cauchy problem for the linear equation (5.10) is well posed in $H^k(1 - s \leq k \leq s - 1)$.

Lemma 5.5. There exists a unique propagator $\{U(t, s)\}$ associate to the family $\{A(t)\}$ with the corresponding spaces $X = H^h, Y = H^k$, where

$$-s \leq h \leq s - 2, 1 - s \leq k \leq s - 1 \text{ and } k \geq h + 1$$

(5.11)

In particular, $U(t, s)$ maps $H^r$ into itself for $-s \leq r \leq s - 1$.

Proof: It is similar to theorem 2. we only have to verify three conditions

(i) $A(t) \in G(X, 1, \beta)$, that is, we will show that $\langle A(t)v, v \rangle_h \geq -\beta \|v\|_h^2$, i.e.

$$\langle A(t)v, v \rangle_h \geq -\beta \|v\|_h^2$$

(5.12)

We begin estimating the term on the left-hand side of this inequality, which can be written in the following way:

$$\langle A(t)v, v \rangle_h = -\langle \Lambda^h(u(t)v - \frac{\gamma}{\alpha x^2}), \Lambda^h \partial_x v \rangle_0$$

$$= -\langle \Lambda^h((\Lambda^{-h}(u(t) - \frac{\gamma}{\alpha x^2})), \Lambda^h v) - [\Lambda^{-h}, (u(t) - \frac{\gamma}{\alpha x^2})] \Lambda^h v, \Lambda^h \partial_x v \rangle_0$$

$$= \langle \Lambda^{h+1}[\Lambda^{-h}, u] \Lambda^h v, \Lambda^{h-1} \partial_x v \rangle_0 + \langle \partial_x (u(t) - \frac{\gamma}{\alpha^2}) v, (\Lambda^h v)^2 \rangle_0$$
Therefore
\[ |(\Lambda^h \partial_x (u(t)v) - \frac{\gamma}{\alpha^2} v), \Lambda^h v)\|_0 \leq \|u\|_k \|\Lambda^h v\|_0 \|\Lambda^{h-1} \partial_x v\|_0 + \|\partial_x (u(t) - \frac{\gamma}{\alpha^2})\|_{L^\infty} \|\Lambda^h v\|^2_0 \]
\[ \leq \|u\|_k \|v\|_h + \|u(t) - \frac{\gamma}{\alpha^2} \|_k \|v\|^2_h \]
thus (5.12) is proved.
(ii) $\Lambda^h \partial_x [\Lambda^{k-h}, (u(t) - \frac{\gamma}{\alpha^2})] \Lambda^{-k}$ is uniformly $L^2$-bounded, but this is again a consequence from lemma 5.2. Here we have used $\Lambda^{k-h}$ which is the isomorphism of $Y$ to $X$.
(iii) $A(t) \in \mathcal{B}(Y, X)$ is strongly continuous in $t$, but this is a consequence of the continuity of $u$ and of the following inequality:
\[ \|\partial_x ((u(t+h) - u(t))w\|_h \leq \|(u(t+h) - u(t))w\|_{h+1} \]
\[ \leq \|u(t+h) - u(t)\|_{s-1} \|w\|_{h+1} \]
\[ \leq \|u(t+h) - u(t)\|_s \|w\|_k \]

**Lemma 5.6.**
\[ v(t) = U(t, 0)v(0) + \int_0^t U(t, \tau)[-B(\tau)v(\tau) + f(\tau)]d\tau \quad (5.13) \]

**Proof:** In lemma 5.5 , choose $h = s - 3, k = s - 2$ (which satisfy (5.11)). Since $v \in C([0, T]: H^{s-2}) \cap C^1([0, T]; H^{s-3})$, as is easily verified by $u \in C([0, T]; H^s)$, we can carry out the standard computation
\[ \frac{dU(t, \tau)v(\tau)}{d\tau} = U(t, \tau)[\frac{dv(\tau)}{d\tau} + A(\tau)v(\tau)] = U(t, \tau)[-B(\tau)v(\tau) + f(\tau)] \]
and we can obtain (5.13) on integration $\tau \in [0, t]$. This is essentially a uniqueness proof of the solution $v$ to (5.10) in the class $C([0, T]; H^{s-2})$.

**Lemma 5.7.** $v \in C([0, T]; H^r)$

**Proof:** Note that $v(0) = (1 - \alpha^2 \partial_x^2)u_0 \in C([0, T]; H^{r-2})$ and $v \in C([0, T]; H^{s-2})$ implying that $f(t) = u_x(3p^{-1}v - \frac{\gamma}{\alpha^2} - 2w - 3(p^{-1}v)^m) \in C([0, T]; H^{r-1})$ if $r \leq s + 1$, since $u_x$ is in the same class and $s - 1 > \frac{1}{2}$. For the same reason, $B(t) = u_x \in \mathcal{B}(H^{r-2})$ is strongly continuous in $t \in [0, T]$ if $r \leq s + 1$; note that $H^{s-1}, H^{r-2} \in H^{r-2}$ by $s - 1 > \frac{1}{2}$ Since the family \{U(t, s)\} is strongly continuous on $H^{r-2}$ to itself, the required result follows from (5.13); we have only to regard (5.13) as an integral equation of Volterra type, which can be solved for $v$ by successive approximation.

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**References**

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