Attractor of Cauchy Problem in Dissipative KdV Type Equation

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(Received 22 February 2006, accepted 30 March 2006)

Abstract: In this paper we studied the dynamic behavior of Cauchy problem of dissipative KdV equation, and obtain the existence of global attractor in the phase space.

Key words: dissipative KdV equation; global attractor; absorbing set; strong weak convergence

1 Introduction

Our aim in this work is to study the existence of the global attractor of the following type of dissipative KdV equation in the phase place $H^2(R^1)$,

$$u_t + \alpha u u_x + u_{xxx} - u_{xx} + \beta u = f(x)$$  \hspace{1cm} (1.1)

$$u(x,0) = u_0(x), x \in R^1.$$  \hspace{1cm} (1.2)

Where $\alpha, \beta$ are real constants, and $\alpha > 0, \beta > 0$.

If the weak dispersive term $u_{xxx}$ is rewritten as the strong dispersive term $u_{xxx} + \gamma u_{xxxx}$, then we get

$$u_t + \alpha uu_x + u_{xxx} + \gamma u_{xxxx} - u_{xx} + \beta u = f(x)$$  \hspace{1cm} (1.3)

$$u(x,0) = u_0(x), x \in R^1$$  \hspace{1cm} (1.4)

The existence of the compact global attractor of Eq. (1.3)-(1.4) in the phase space $H^2(R^1)$ is proved by the Theorem in [5]. In order to prove the asymptotic compactness of the solution operator of $S(t)$, the kuratowskii measure of non-compactness [2, 6, 7] is used in conjunction with a suitable splitting of the solutions operator, since $H^s(R') \rightarrow H^s(R')$ is not a compact imbedding. In fact, taking $\gamma \rightarrow 0$ in Eq. (1.3)-(1.4), we obtain the existence of the compact global attractor of Eq. (1.1)-(1.2) in the phase space $H^2(R^1)$.

For convenience, we introduce some notations:

$$\Delta = \frac{\partial^2}{\partial x^2}, \nabla = \frac{\partial}{\partial x}, H^s(R^1) = H^{s,2}(R^1),$$ and we denote $(\bullet, \bullet)$ as the inner product on $L^2(R^1), ||\bullet||$ as the norm of the space $L^2(R^1), ||\bullet||_{s,p}$ as the norm of the space $H^{s,p}(R^1)$.

2 Preliminary estimates and the existence of the solution

We rewrite (1.3)-(1.4), i.e.

$$u_t + \gamma \Delta^2 u + \alpha u \nabla u + \nabla \Delta u - \Delta u + \beta u = f(t, x), x \in R^1, t > 0,$$  \hspace{1cm} (2.1)

$$u(x,0) = u_0.$$  \hspace{1cm} (2.2)

where $\gamma, \alpha, \beta$ are real constants and $\alpha > 0, \beta > 0, \gamma > 0$.

For the global existence of solutions to problem (2.1), (2.2), we have

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Lemma 2.1 Given $f \in L^\infty (R^+; L^2 (R^1))$, there exists a unique solution $u$ to Eq. (2.1)-(2.2) in $L^\infty (R^+; L^2 (R^1))$, such that

$$\sup \|f\|^2 \leq \|u_0\| \exp (-\beta t) + \frac{t}{\beta} (1 - \exp (-\beta t)).$$

Therefore, there exists a $t_1 (R) > 0, \forall t > t_1 (R), \|u_0\| \leq R, such that

$$\|u (t)\|^2 \leq \|u_0\|^2 \exp (-\beta t) + \frac{t}{\beta} (1 - \exp (-\beta t)).$$

Proof. Multiplying (2.1) with $u$ and integrating on $R$ with respect $x$

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \gamma \|\Delta u\|^2 + \|\nabla u\|^2 + \beta \|u\|^2 \leq \|f\| \|u\| \leq \frac{1}{2} \beta \|u\|^2 + \frac{1}{2} \|f\|^2.$$

Using the Lemma of Gronwall

$$\|u (t)\|^2 \leq \|u_0\|^2 \exp (-\beta t) + \frac{t}{\beta} (1 - \exp (-\beta t)).$$

Lemma 2.2 Given $f \in L^\infty (R^+; L^2 (R^1))$, $u_0 \in H^1 (R^1)$, then the solution $u$ to Eq. (2.1)-(2.2) in $L^\infty (R^+; H^1 (R^1))$, and there exists a $t_2 (R) \geq 0, \forall t > t_2 (R), \|\nabla u_0\|^2 \leq R, such that

$$\|\nabla u (t)\|^2 \leq c.$$

Proof: Multiplying (2.1) with $-\Delta u$ and integrating on $R$ with respect $x$

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \gamma \|\nabla \Delta u\|^2 + \frac{1}{2} \alpha (u^2, \nabla u) + \|\Delta u\|^2 + \beta \|\nabla u\|^2 \leq \|f\| \|\nabla u\|. \tag{2.3}$$

In fact

$$\frac{1}{2} \alpha (u^2, \nabla u) \leq \epsilon_1 \|\nabla \Delta u\|^2 + \epsilon_2 \|\nabla u\|^2 + c \|u\|^6,$$

$$\|f\| \|\nabla u\| \leq c \|f\| (\|u\| + \|\nabla \Delta u\|) \leq \epsilon_1 \|\nabla \Delta u\|^2 + c \|f\|^2 + c \|u\|^2,$$

let $\epsilon_1 = \frac{\gamma}{2}, \epsilon_2 = \frac{\alpha}{2}$, we have

$$\frac{d}{dt} \|\nabla u\|^2 + \gamma \|\nabla \Delta u\|^2 + \|\Delta u\|^2 + \beta \|\nabla u\|^2 \leq \sup_t \|f\|^2 + c \|u\|^6 + c \|u\|^2 \Delta K \tag{2.4}$$

Using the Lemma of Gronwall, then the lemma 2.2 is proved

Corollary 2.3 Given $f \in L^\infty (R^+; L^2 (R^1))$, $\forall u_0 \in H^1$ such that

$$u (t) \in L^\infty (R^+ \times R^1).$$

Lemma 2.4 Given $f \in L^\infty (R^+; L^2 (R^1))$, $\forall u_0 \in H^2 (R^1)$, then the solution $u$ to Eq. (2.1)-(2.2) in $u (t) \in L^\infty (R^+; H^2 (R^1))$, and there exists a $t_3 (R) > 0, \forall t > t_3 (R), \|u_0\|_{2,2} \leq R, such that $\|\Delta u\|^2 \leq c$.

Proof: Multiplying (2.1) with $\Delta^2 u$ and integrating on $R$ with respect $x$

$$\frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 + \gamma \|\Delta^2 u\|^2 + \alpha (u \nabla u, \Delta^2 u) + \|\nabla \Delta u\|^2 + \beta \|\Delta u\|^2 = (f, \Delta^2 u).$$

In fact
\begin{equation}
\alpha \left( u \nabla u, \Delta^2 u \right) \leq \|u\|_{\infty} \|\nabla u\| \|\Delta^2 u\| \leq \epsilon_1 \|\Delta^2 u\|^2 + c \|u\| \|\nabla u\|^3;
\end{equation}

\begin{equation}
(f, \Delta^2 u) \leq \epsilon_1 \|\Delta^2 u\|^2 + c \|f\|^2,
\end{equation}

let \( \epsilon_2 = \frac{\gamma}{4} \), we have
\[
\frac{d}{dt} \|\Delta u\|^2 + \gamma \|\Delta^2 u\|^2 + \|\nabla \Delta u\|^2 + \beta \|\Delta u\|^2 \leq \|u\| \|\nabla u\| \|\nabla^2 u\|^2 + c \sup_t \|f\|^2 \triangleq K_2.
\] (2.5)

Using the Lemma of Gronwall, then the lemma 2.4 is proved.

**Lemma 2.5** Given \( f \in L^\infty (R^+; H^1 (R^1)) \), \( u_0 \in H^2 (R^1) \), such that
\[
\|\nabla u\|^2 \leq c \quad \forall t \geq t_2 + 1.
\]

Proof: Multiplying (2.1) with \( \Delta^2 u \) and integrating on \( R \) with respect \( x \)
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \gamma \|\Delta^2 u\|^2 + \beta \|\nabla u\|^2 + \|\Delta^2 u\|^2 + \alpha (\Delta u^2, \nabla \Delta^2 u) = (\nabla f, \nabla \Delta^2 u).
\]

In fact
\[
|\alpha (\Delta u^2, \nabla \Delta^2 u)| \leq \frac{1}{4} \gamma \|\Delta u^2\|^2 + c \|\Delta u^2\|^2;
\]
\[
|\nabla f, \nabla \Delta^2 u| \leq \frac{1}{4} \gamma \|\nabla \Delta^2 u\|^2 + c \|\nabla f\|^2.
\]

Form the above Lemma 2.1-2.4, we obtain \( \|\Delta u^2\|^2 < \infty \), such that
\[
\frac{d}{dt} \|\nabla u\|^2 + \gamma \|\nabla \Delta u\|^2 + 2 \beta \|\nabla u\|^2 + \|\Delta^2 u\|^2 \leq c \sup_t \|\nabla f\|^2 + c \|\Delta u^2\|^2 \triangleq K_3.
\] (2.6)

Combing with (2.4), we obtain
\[
\frac{d}{dt} \|\nabla u\|^2 + \gamma \|\nabla u\|^2 \leq K_1,
\]
\[
\gamma \int_t^{t+1} \|\nabla \Delta u\|^2 d\tau + \|\nabla u (t+1)\|^2 \leq K_1 + \|\nabla u (t)\|^2.
\]

Form the Lemma 2.2, we have
\[
\int_t^{t+1} \|\nabla \Delta u\|^2 d\tau \leq K_4
\] (2.7)

By using Gronwall’s inequality, we get the result that
\[
\|\nabla u\|^2 \leq c, \quad \forall t > t_2 (R) + 1.
\]

**Theorem 2.6** Given \( f \in L^\infty (R^+; L^2 (R^1)) \), \( \forall u_0 \in H^2 (R^1) \), there exists a unique solution \( u \) to Eq. (2.1)-(2.2) in \( L^\infty (R^+; H^2 (R^1)) \); the solution operator \( S(t) \) is continue in \( H^2 \) and have a bounded absorbing set \( B \subset H^2 (R^1) \). Given \( f \in L^\infty (R^+; H^1 (R^1)) \), such that the bounded absorbing set \( B \subset H^2 (R^1) \).
3 The smoothness of the solution

Given \( f = f(x) \in H^1(R^1) \), and \( \lambda_L(x) \in C_0(R^1), 0 \leq \lambda_L \leq 1 \), which satisfies

\[
\lambda_L = \begin{cases} 
1, |x| \leq L, \\
0, |x| > 1 + L, 
\end{cases}
\]

Then \( \forall \eta \in (0, 1) \), there exists a \( L(\eta) > 0 \), such that

\[
\|f - f_\eta\|_{1,2}^2 \leq \eta, f_\eta = f \times \lambda_L(\eta), \|\nabla u^2 - \nabla u^2 \lambda_L(\eta)\|_{1,2}^2 \leq \eta.
\]

If \( u_\eta \) is a solution to the following Eq. (3.1)-(3.2)

\[
u_{\eta t} + \gamma \Delta^2 u_\eta + \nabla \Delta u_\eta + \Delta u_\eta + \beta u_\eta = f - f_\eta - \frac{1}{2} \alpha \nabla u^2 (1 - \lambda_L(\eta)), \tag{3.1}
\]

\[
u_\eta(x, 0) = u_0. \tag{3.2}
\]

let \( S_1(\eta) u_0 = u_\eta, w_\eta = S_2(\eta) u_0 = S(t) u_0 - S_1(\eta) u_0 \) is a solution to the following Eq. (3.3)-(3.4)

\[
w_{\eta t} + \nu \Delta^2 w_\eta + \nabla \Delta w_\eta + \Delta w_\eta + \beta w_\eta = f_\eta - \frac{1}{2} \alpha \nabla u^2 \lambda_L(\eta), \tag{3.3}
\]

\[
w_\eta(x, 0) = 0. \tag{3.4}
\]

Lemma 3.1 Under the conditions of Lemma 2.5, there exists a real constant \( c > 0 \), such that

\[
\|u_\eta\|^2, \|\nabla u_\eta\|^2, \|\Delta u_\eta\|^2, \|\nabla \Delta u_\eta\|^2 \leq c, \forall \eta \in (0, 1), \forall t \geq 0;
\]

\[
\|u_\eta\|^2, \|\nabla u_\eta\|^2, \|\Delta u_\eta\|^2 \leq c \eta, \forall \eta \in (0, 1), \forall t \geq t_\eta > 0 (\exists t_\eta > 0).
\]

Proof: Multiplying (3.1) with \( u_\eta \) and integrating on \( R \) with respect \( x \)

\[
\frac{1}{2} \frac{d}{dt} \|u_\eta\|^2 + \gamma \|\Delta u_\eta\|^2 + \|\nabla u_\eta\|^2 + \beta \|u_\eta\|^2
\]

\[
\leq \|f - f_\eta\| \|u_\eta\| + \frac{1}{2} \alpha \|\nabla u^2 - \nabla u^2 \lambda_L(\eta)\| \|u_\eta\|
\]

\[
\leq \frac{1}{2} \beta \|u_\eta\|^2 + c \eta,
\]

By using the Lemma of Gronwall, we have

\[
\|u_\eta(t)\|^2 \leq \|u_0\|^2 \exp(-\beta t) + \frac{1}{\beta} c \eta (1 - \exp(-\beta t)). \tag{3.5}
\]

Therefore, there exists a real constant \( c > 0 \), such that \( \|u_\eta\|^2 \leq c, \forall \eta \in (0, 1) \), and there also exists a \( t_1(R) > 0 \), such that if \( t \geq t_1(R) \), then \( \|u_0\|^2 \exp(-\beta t) < \eta \), i.e., \( \|u_\eta\|^2 \leq c \eta, \forall t \geq t_1 \).

Multiplying (3.1) with \( -\Delta u_\eta \) and integrating on \( R \) with respect \( x \)

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \Delta u_\eta\|^2 + \gamma \|\nabla \Delta u_\eta\|^2 + \|\Delta u_\eta\|^2 + \beta \|\nabla u_\eta\|^2
\]

\[
\leq \|f - f_\eta\| \|\Delta u_\eta\| + \frac{1}{2} \alpha \|\nabla u^2 (1 - \lambda_L(\eta))\| \|\Delta u_\eta\|
\]

\[
\leq \frac{\gamma}{2} \|\nabla \Delta u_\eta\|^2 + c \|f - f_\eta\|^2 + c \|\nabla u^2 (1 - \lambda_L)\|^2,
\]

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We have
\[
\frac{d}{dt} \| \nabla u \|^2 + \gamma \| \nabla \Delta u \|^2 + 2\beta \| \nabla u \|^2 \leq c\eta. \tag{3.6}
\]
By using the Lemma of Gronwall, we have
\[
\| \nabla u \|^2 \leq \| \nabla u_0 \|^2 \exp (-2\beta t) + \frac{1}{2\beta} c\eta (1 - \exp (-2\beta t)).
\]
there exists a real constant \( c > 0 \), such that
\[
\| \nabla u \|^2 \leq c, \quad \forall \eta \in (0, 1); \nonumber
\]
there also exists a \( t_2(R) \geq t_1(R) \)
\[
\| \nabla u_0 \|^2 \exp (-2\beta t) \leq R^2 \exp (-2\beta t) \leq \eta,
\]
i.e.
\[
\| \nabla u \|^2 \leq c\eta, \quad \forall t \geq t_2(R). \tag{3.7}
\]
Multiplying (3.1) with \( \Delta^2 u_\eta \) and integrating on \( R \) with respect \( x \), we have
\[
\frac{1}{2} \frac{d}{dt} \| \Delta u_\eta \|^2 + \gamma \| \Delta^2 u_\eta \|^2 + \| \nabla \Delta u_\eta \|^2 + \beta \| \Delta u_\eta \|^2
\]
\[
\leq \| f - f_\eta \| \| \Delta^2 u_\eta \|^2 + \frac{1}{2} \alpha \| \nabla u^2 (1 - \lambda_L) \| \| \Delta^2 u_\eta \|^2
\]
\[
\leq \frac{1}{2} \gamma \| \Delta^2 u_\eta \|^2 + c\eta.
\]
In a analogous way to above, there exits a \( t_3(R) \geq t_2(R) \), such that
\[
\| \Delta u_\eta \|^2 \leq c\eta, \quad \forall t \geq t_3(R), \forall \eta \in (0, 1). \tag{3.7}
\]
Multiplying (3.1) with \( -\Delta^3 u_\eta \) and integrating on \( R \) with respect \( x \), we have
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \Delta u_\eta \|^2 + \gamma \| \nabla \Delta^2 u_\eta \|^2 + \| \Delta^2 u_\eta \|^2 + \beta \| \nabla \Delta u_\eta \|^2
\]
\[
\leq \frac{1}{2} \gamma \| \nabla \Delta^2 u_\eta \|^2 + c \| f - f_\eta \|^2 + c \| \nabla u^2 (1 - \lambda_L) \|^2,
\]
We obtain
\[
\frac{d}{dt} \| \nabla \Delta u_\eta \|^2 + 2\beta \| \nabla \Delta u_\eta \|^2 \leq c\eta. \tag{3.8}
\]
Combing with (3.6)
\[
\gamma \int_t^{t+1} \| \nabla \Delta u_\eta \|^2 d\tau \leq c\eta + \sup_{\tau \in R^+} \| \nabla u_\eta(t) \|^2,
\]
By using Gronwall’s inequality, we get the result
\[
\| \nabla \Delta u_\eta \|^2 \leq c.
\]
**Lemma 3.2** Under the conditions of Lemma 2.5, there exist \( c_i(\eta) > 0, i = 1, 2, 3, \) such that if \( t \geq t^* \), then
\[
\| x w_\eta \|^2 \leq c_1(\eta), \| x \nabla w_\eta \|^2 \leq c_2(\eta), \| x \Delta w_\eta \|^2 \leq c_3(\eta).
\]
Proof: Multiplying (3.3) with $x^2 w_\eta$ and integrating on $R$ with respect $x$, we have
\[
(\gamma \Delta^2 w_\eta, x^2 w_\eta) = \gamma \|x \Delta w_\eta\|^2 + 4 \gamma (\Delta w_\eta, x \Delta w_\eta) - 2 \gamma \|\nabla w_\eta\|^2 ,
\]
\[
(\nabla \Delta w_\eta, x^2 w_\eta) = -3 (\Delta w_\eta, x w_\eta) .
\]
then
\[
\frac{1}{2} \frac{d}{dt} \|x w_\eta\|^2 + \gamma \|x \Delta w_\eta\|^2 + \|x \nabla w_\eta\|^2 + \beta \|x w_\eta\|^2 + 2 \gamma \|\nabla w_\eta\|^2 
\]
\[
\leq - 4 \gamma (\Delta w_\eta, x \nabla w_\eta) + 3 (\Delta w_\eta, x w_\eta) + (f_\eta, x^2 w_\eta) + \|w_\eta\|^2 - \frac{1}{2} \alpha (\nabla u^2 \lambda_L, x^2 w_\eta) 
\]
\[
\leq \frac{1}{2} \gamma \|x \Delta w_\eta\|^2 + \frac{1}{2} \beta \|x w_\eta\|^2 + c \|\nabla w_\eta\|^2 + c \|x f_\eta\|^2 + c \|x \nabla u^2 \lambda_L\|^2 ,
\]
If $t \geq t^*$, it follows that $u$ and $w_\eta$ are bounded in $H^3$. Furthermore, if $t \geq t^*$, $w_\eta$ is bounded in $H^3$. In fact, if $t \geq 0$, it follows that $w_\eta$ is bounded in $H^2$, combing with $f_\eta$ and $\nabla u^2 \lambda_L$ both have a compact support set, we obtain that $\|x f_\eta\|$ and $\|x \nabla u^2 \lambda_L\|$ are bounded. Thus we can prove
\[
\frac{d}{dt} \|x w_\eta\|^2 + \gamma \|x \Delta w_\eta\|^2 + \beta \|x w_\eta\|^2 \leq c (\eta) .
\] 
(3.10)

By using the Lemma of Gronwall, we have
\[
\|x w_\eta (t)\|^2 \leq c_1 (\eta), \forall t \geq 0.
\]

By applying $\Delta$ to the term on the both sides of (3.3)
\[
\Delta w_{\eta t} + \gamma \Delta^3 w_\eta + \nabla \Delta^2 w_\eta - \Delta^2 w_\eta + \beta \Delta w_\eta = \Delta f_\eta - \frac{1}{2} \alpha \Delta (\nabla u^2 \lambda_L) .
\] 
(3.11)

Multiplying (3.11) with $x^2 \Delta w_\eta$ and integrating on $R$ with respect $x$, we have
\[
(\gamma \Delta^3 w_\eta, x^2 \Delta w_\eta) = \gamma \|x \Delta w_\eta\|^2 + 4 \gamma (\Delta^2 w_\eta, x \nabla \Delta w_\eta) - 2 \gamma \|\nabla \Delta w_\eta\|^2 ,
\]
\[
(\Delta^2 w_\eta, x^2 w_\eta) = \|\Delta w_\eta\|^2 - \|\nabla \Delta w_\eta\|^2 ,
\]
\[
(\nabla \Delta^2 w_\eta, x^2 \Delta w_\eta) = -3 (\Delta w_\eta, x \Delta^2 w_\eta) ,
\]
\[
(\Delta f_\eta, x^2 \Delta w_\eta) = (f_\eta, x^2 \Delta^2 w_\eta) + 4 (f_\eta, x \nabla \Delta w_\eta) + 2 (f_\eta, \Delta w_\eta) ,
\]
\[
-\frac{\alpha}{2} (\nabla u^2 \lambda_L), x^2 \Delta w_\eta) = -\frac{\alpha}{2} (\nabla u^2 \lambda_L, x^2 \Delta^2 w_\eta) - \frac{\alpha}{2} (\nabla u^2 \lambda_L, 4x \nabla \Delta w_\eta) - \alpha (\nabla u^2 \lambda_L, \Delta w_\eta) ,
\]
we have
\[
\frac{1}{2} \frac{d}{dt} \|x \Delta w_\eta\|^2 + \gamma \|x \Delta^2 w_\eta\|^2 + \beta \|x \Delta w_\eta\|^2 + \|\Delta w_\eta\|^2 + 2 \gamma \|\nabla \Delta w_\eta\|^2 
\]
\[
\leq -4 \gamma (\Delta^2 w_\eta, x \nabla \Delta w_\eta) + 3 (\Delta w_\eta, x \Delta^2 w_\eta) + (f_\eta, x^2 \Delta^2 w_\eta) 
\]
\[
+ 4 (f_\eta, x \nabla \Delta w_\eta) + 2 (f_\eta, \Delta w_\eta) + \|\nabla \Delta w_\eta\|^2 - \frac{1}{2} \alpha (\nabla u^2 \lambda_L, x^2 \Delta^2 w_\eta) 
\]
\[
\leq \frac{1}{2} \gamma \|x \Delta^2 w_\eta\|^2 + c (\eta) ,
\]
therefore
\[
\frac{d}{dt} \| x \Delta w_\eta \|^2 + 2\beta \| x \Delta w_\eta \|^2 \leq c(\eta).
\] (3.12)

By using Gronwall’s inequality, we get the result
\[
\| x \Delta w_\eta \|^2 \leq c_3(\eta), \forall t \geq t^*,
\]
\[
\| x \nabla w_\eta \|^2 = \int x^2 (\nabla w_\eta)^2 dx = -2 \int x \nabla w_\eta w_\eta dx - \int x^2 \Delta w_\eta w_\eta dx \leq c_2(\eta).
\] (3.13)

The proof of Lemma 3.2 is complete.

4 The existence of the global attractor

If \( f(x) \in H^1(R^1), S(t) \) is the solution operator semigroups of Eq. (2.1)-(2.2) by the above proof, we know \( S(t) \) have a bounded absorbing set \( B \subset H^2(R^1) \) in \( H^2(R^1) \). In the following steps, using the Kuratowski \( \alpha \)-measure of non-compactness, we gain the compactness of \( \omega(B) \).

We define a set \( \mathbb{A} \) is \( \alpha \)-measure of non-compactness, if \( \mathbb{A} \) satisfies:
\[
\alpha(\mathbb{A}) = \inf \{ d | d(\mathbb{A}) < d \},
\]
where \( d(\mathbb{A}) \) is the diameter of globules which are a finite covering of \( \mathbb{A} \). In fact \( \alpha(\mathbb{A} \cup \mathbb{B}) \leq \alpha(\mathbb{A}) + \alpha(\mathbb{B}) \). Particularly, If \( \mathbb{A} \) is compact, so \( \alpha(\mathbb{A}) = 0 \). We can found the proof in [4, 5]

**Lemma 4.1** Given \( s, s_1 \in \mathbb{Z} \) and \( s > s_1 \), such that it is a compact imbedding form \( H^s(R^n) \cap H^{s_1}(R^n; (1 + x^2) dx) \) to \( H^{s_1}(R^n) \).

By using Lemma 3.2, 4.1, we obtain the compactness of \( S_{2\eta}(t) \) of Eq. (3.3)-(3.4) in \( H^2(R^1) \). Therefore, each bounded set \( \mathbb{B'} \) in \( H^2 \) satisfies
\[
\alpha(S_{2\eta}(t) \mathbb{B'}) = 0, \forall t \geq t^*.
\]

Thus we get the following Theorem

**Lemma 4.2** \( \mathbb{A} = \omega(\mathbb{B}) = \cap_{s \geq 0} \cup_{t \geq s} S(t) \mathbb{B} \) is the compact attractor of \( S(t) \) in \( H^2(R^1) \), where we consider the closure in \( H^2(R^1) \).

Proof: \( \forall \epsilon > 0 \), there exists a \( \eta > 0 \), \( t_0 > 0 \), such that
\[
\| S_{1\eta}(t) u_0 \| \leq \epsilon, \forall t \geq t_0, u_0 \in \mathbb{B},
\]

If \( t > t_0 \)
\[
\alpha(S(t) \mathbb{B}) \leq \alpha(S_{1\eta}(t) \mathbb{B}) + \alpha(S_{2\eta}(t) \mathbb{B}) = \alpha(S_{1\eta}(t) \mathbb{B}) \leq \epsilon,
\]
Thus we have \( \lim_{t \to \infty} \alpha(S(t) \mathbb{B}) = 0 \). i.e. \( S(t) \) is asymptotic smoothing.

5 Weak and strong limit as \( \gamma \to 0 \)

We now turn to the study of the behavior of the Couchy problem (2.1) as the dispersive parameter \( \gamma \) tends to zero. i.e.
\[
u t + \alpha u \nabla u + \nabla \Delta u - \Delta u + \beta u = f(t, x), \quad x \in R^1, t > 0, \quad u(x, 0) = u_0\] (5.1)

Where \( \alpha, \beta > 0 \).
**Theorem 5.1** Given $f \in L^\infty (R^+; L^2 (R^1))$, $u_0 \in H^2 (R^1)$, $u (x, 0) = u_0$, there exists a solution $u (x, t)$ in $L^\infty (R^+; H^2 (R^1))$, such that

$$u, \nabla u \in L^\infty (R^+; H^2 (R^1))$$

$$u_t + au \nabla u + \nabla \Delta u - \Delta u + \beta u = f (t, x)$$

$$u (x, 0) = u_0$$

In order to obtain Theorem 5.1, we proved the following Lemma 5.2-5.4

**Lemma 5.2** Given $u_0 \in L^2 (R^1)$, $0 < \gamma < 1$, $f \in L^\infty (R^+; L^2 (R^1))$, such that

$$\|u_\varepsilon\|_{L^\infty (R^+; L^2 (R^1))} \leq c$$

$$\sqrt{\gamma} \|\nabla u_\varepsilon\|_{L^2 (R^1)} \leq c$$

Where the constant $c$ is independent of $\varepsilon$.

Proof: Multiplying (2.1) with $u$ and integrating on $R$ with respect $x$, we have

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \gamma \|\Delta u\|^2 + \|\nabla u\|^2 + \beta \|u\|^2 \leq \|f\| \|u\| \leq \frac{1}{2} \beta \|u\|^2 + \frac{1}{2} \beta \|f\|^2.$$ 

We have

$$\frac{d}{dt} \|u\|^2 + 2\gamma \|\Delta u\|^2 \leq K.$$ 

Thus we complete the proof of (5.7)-(5.8).

**Lemma 5.3** Given $v (x) \in H^3 (R^1)$, such that

$$\|v\|_{L^4 (R^1)} \leq c \|v\|^{11/12}_{L^2 (R^1)} \left(\|v\|_{L^2 (R^1)} + \|\nabla v\|_{L^2 (R^1)}\right)^{1/12}$$

$$\left\|\frac{dv}{dx}\right\|_{L^4 (R^1)} \leq c \|v\|^{7/12}_{L^2 (R^1)} \left(\|v\|_{L^2 (R^1)} + \|\nabla v\|_{L^2 (R^1)}\right)^{5/12}$$

**Lemma 5.4** Given $u_0 (x) \in H^2 (R^1)$, $f \in L^\infty (R^+; L^2 (R^1))$, such that

$$\|\nabla u\|_{L^\infty (R^+; L^2 (R^1))} \leq c$$

$$\sqrt{\gamma} \|\nabla \Delta u\|_{L^2 (R^1)} \leq c$$

Where the constant $c$ is independent of $\varepsilon$.

Proof: Multiplying (2.1) with $-\Delta u$ and integrating on $R$ with respect $x$, we have

$$\frac{d}{dt} \|\nabla u\|^2 + \gamma \|\nabla \Delta u\|^2 + \|\Delta u\|^2 + \beta \|\nabla u\|^2 \leq \sup_t \|f\|^2 + c \|u\|^6 + c \|u\|^2$$

We have

$$\frac{d}{dt} \|\nabla u\|^2 + \gamma \|\nabla \Delta u\|^2 \leq K.$$ 

Thus we complete the proof of (5.7)-(5.8).

**Rewrite (2.1)**

$$u_t = -\gamma \Delta^2 u - au \nabla u - \nabla \Delta u + \Delta u - \beta u + f (x)$$

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Multiplying (5.13) with $\varphi$ and integrating on $R$ with respect $x$, we have

$$|(u_t, \varphi)| \leq \gamma \|\nabla u\|_{L^2} \|\varphi\|_{H^2} + \frac{1}{2} \|u\|_{L^\infty} \|\varphi\|_{H^2} + \|\nabla u\|_{L^2} \||\varphi|\|_{H^2} + \|u\|_{L^2} \||\varphi|\|_{H^2} + \beta \|u\|_{L^\infty} \|\varphi\|_{H^2} + \|f\|_{L^\infty} \|\varphi\|_{H^2}$$

where $\varphi \in L^2(R^+; H^2(R^1))$.

By the above proof, we have

$$\|u\|_{L^2}, \|\nabla u\|_{L^2}, \|\Delta u\|_{L^2}, \|\nabla \Delta u\|_{L^2} \leq c (c \neq 0)$$

Thus we obtain the uniformly bounded of $u_t$ in $L^2(R^+; H^2(R^1))$

Combing with (5.6),(5.7),(5.10),(5.11), we can select a subsequence $\{u_\varepsilon\}$ such that $u_\varepsilon$ weak limit to $u$ in $L^2(R^+; L^2(R^1))$; $\nabla u_\varepsilon$ weak limit to $\nabla u$ in $L^\infty(R^+; L^2(R^1))$; $\partial_u u_\varepsilon$ weak limit to $\partial_u u$ in $L^2(R^+; H^2(R^1))$. Furthermore, we have $u_\varepsilon$ strong limit to $u$ in $L^\infty(R^+; L^2(R^1))$; and $u \nabla u_\varepsilon$ weak limit to $u \nabla u$ in $L^\infty(R^+; L^2(R^1))$.

Therefore, we complete the proof. i.e. $\text{Eq. (2.1)} \rightarrow u_t + \alpha u \nabla u + \nabla \Delta u - \Delta u + \beta u = f$.

References


