Slow Manifolds of Lorenz-Haken System and its Application

Guoliang Cai ¹, Lixin Tian, Juanjuan Huang
Nonlinear Scientific Research Center, Faculty of Science, Jiangsu University
Zhenjiang, Jiangsu, 212013, P.R. China
(Received 30 September 2005, accepted 8 January 2006)

Abstract: In this paper, the slow manifolds of Lorenz-Haken system are discussed. By using three different methods, associated with the slow manifold equations of the L-H system are obtained. Firstly, the slow manifold equation of the nonlinear chaotic dynamic system is obtained by considering that the slow manifold is locally defined by a plane orthogonal to the tangent system’s left fast eigenvector. On the condition that $z^T_λ(\dot{X}) \cdot \dot{X} = 0$, the slow manifold equation of the L-H system is built. And secondly, another method consists of defining the slow manifold as the surface generated by the two slow eigenvectors associated with the two eigenvalues $\lambda_2(X)$ and $\lambda_3(X)$ of $J(X)$. Another slow manifold equation of the L-H system is obtained. Thirdly, by geometric singular perturbation theory, we give the new slow manifold equation that is concrete and terse of the L-H system. Finally, we apply our results to derive the slow manifold equations of some known classical chaotic systems, such as the Chua’s system, the Lorenz system, the Chen’s system and Lü’s system, and analyze the dynamical behavior of the S-FADS.

Keywords: Lorenz-Haken system; slow manifold; equation; analysis

1 Introduction

It is a popular problem for nonlinear chaotic dynamical system research in the nonlinear science field [1-3]. In the nonlinear chaotic dynamical system research, one method is that some chaotic systems can be seen as slow-fast systems to be qualitatively and quantitatively analyzed. It is known experimentally that the slow-fast autonomous dynamical systems (S-FADS) show a dichotomy of motion, which is alternatively slow and fast. This is confirmed theoretically by some recent studies in which they have brought to light that S-FADS (i.e. systems that are modeled by differential equations having a small parameter $\varepsilon$ multiplying one of their velocity components) have slow manifolds. Indeed, for these systems, they have shown that the Jacobian matrix possesses a fast eigenvalue (i.e. real, negative and dominant) and, under some conditions, the slow manifold is locally defined by a plane orthogonal to the tangent system’s left fast eigenvector, it is possible to compute the equation of the slow manifold [4]. In 2000, Ramdani et al. proposed another new definition of so-called S-FADS, i.e. they are systems having a fast eigenvalue in a large domain of the phase space. The systems can be seen as S-FADS as long as the influence of the nonlinear terms of the velocity is smaller than the fast component. In this case, the behavior is the same as the so-called linear tangent system and the slow manifold remains attractive in spite of the nonlinear part of the velocity. This condition is verified for systems having a fast eigenvalue as it is the case with the Lorenz system. The local slow manifold in the neighborhood of $X$ as the surface generated by the two slow eigenvectors associated with the two eigenvalues. In the case, they have given another method to compute the slow manifold equation by using the tangent system’s slow eigenvectors [5]. In this paper, we give another new method to derive

¹ Corresponding author. E-mail address: glcai@ujs.edu.cn Tel:+86-511-879 1467
the slow manifold equation of S-FADS by using geometric singular perturbation theory [6-8]. This method only needs to calculate the eigenvalues of degenerate fast sub-system and then make power expanding. The amount of calculation of our method is quit little and the analytic expression of the slow manifold is very clear. It is also easy to make qualitative analysis and numerical simulation.

The outline of this paper is organized as follows. The L-H chaotic system and its basic global dynamic behaviors are discussed in Section 2 and in Section 3. In Section 4, the first slow manifold equation of the L-H system is obtained. In Section 5, the second slow manifold equation of the L-H system is obtained. In Section 6, we put forward a new method to derive the third slow manifold equation of the L-H system (i.e. the first order expression of the slow manifold $M_\epsilon$ of the L-H system). In Section 7 and in section 8, we apply new method to the chua’s system, the Lorenz system, the Chen’s system and Lü’s system; the slow manifold equations of these systems are obtained. The qualitative behavior and the orbits of the S-FADS are analyzed in Section 9. In Section 10 contains the conclusion of this paper.

2 Lorenz-Haken system

In 1975, Haken gave a nonlinear optical slow-fast system, namely, the optical parametric oscillator model. Because the system model is similar with the Lorenz model, the system is called Lorenz-Haken model. The Lorenz-Haken laser system model can be written with the form of nonlinear differential equations as following

$$\begin{align*}
\dot{E} &= \kappa(P - E) \\
\dot{P} &= nE - P \\
\dot{n} &= \gamma(A - n - EP)
\end{align*}$$

(Lorenz – Haken) (1)

where $E$, $P$ and $n$ are respectively the real amplitude of the electromagnetic field in the laser cavity, the polarization of the cavity medium and the inversion of the population between the two levels of the transition due to the pumping.

The parameters $\kappa$ and $\gamma$ are given by $\kappa = \frac{\kappa_\perp}{\gamma_\perp}$, and $\gamma = \frac{\gamma_{11}}{\gamma_\perp}$, where $\kappa$, $\gamma_\perp$ and $\gamma_{11}$ are respectively the relaxation rates of the variables $E$, $P$, and $n$, and $A$ is the pump parameter.

If we replace $E$, $P$, $n$ with $x$, $y$, $z$, and replace $\kappa$, $\gamma$, $A$ with $\sigma$, $b$, $a$, the L-H system can be expressed by the nonlinear differential equations which is similar with the Lorenz system

$$\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= xz - y \\
\dot{z} &= b(a - z - xy)
\end{align*}$$

(L – H) (2)

3 The global dynamic behavior analysis of L-H system

We primarily analyze the global dynamic behavior of the L-H system, as follows:

First, we discuss the symmetry of the L-H system. For any arbitrary parameter $\sigma$, $b$, $a$, the L-H system is invariable under the transformation $(x, y, z) \rightarrow (-x, -y, z)$, which indicates that the L-H system is symmetric with respect to the $z$-axis.

Second, we discuss the dissipation of the system. Let $f = (f_1, f_2, f_3)^T = (\dot{x}, \dot{y}, \dot{z})^T$, so then we have

$$\text{div} f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -\sigma - 1 - b$$

If $-\sigma - 1 - b < 0$, i.e. $\sigma + b > -1$, then we have $\text{div} f < 0$. This is shown that the L-H system is the dissipative system. The volume of any attractor of the system must be absolute contract to zero.

Third, we discuss the equilibriums of the L-H system. Let

$$\begin{align*}
\sigma y - x &= 0, \\
xz - y &= 0, \\
b(a - z - xy) &= 0.
\end{align*}$$
When \( a < 1 \), the system has no equilibrium.

When \( a = 1 \), we get the only equilibrium \((0, 0, 1)\) of the system, and the corresponding characteristic equation is
\[
\lambda (\lambda + b) (\lambda + 1 + \sigma) = 0,
\]
and characteristic roots are \( \lambda_1 = -b < 0, \lambda_2 = 0, \lambda_3 = -(1 + \sigma) < 0 \), so the equilibrium \((0, 0, 1)\) is gradually convergence to stable.

When \( a > 1 \), we get another two equilibriums \((\pm \sqrt{a-1}, \pm \sqrt{a-1}, 1)\) of the system, and the corresponding characteristic equation is
\[
\lambda^3 + (b + 1 + \sigma) \lambda^2 + (ba + \sigma b) \lambda + 2b \sigma (a - 1) = 0,
\]
Because \( b + 1 + \sigma > 0, ba + \sigma b > 0, 2b \sigma (a-1) > 0 \), so characteristic roots are \( \lambda_1 \leq 0, \lambda_2 \leq 0, \lambda_3 \leq 0 \) or imaginary roots (Re(\( \lambda \)) \leq 0), so the two equilibriums are gradually inclined to be stable.

When parameter \( \sigma = 4, b = 0.4, a = 12 \), the fast eigenvalue existence condition is satisfied. In fact, the dynamics of this system are identical to those of the Lorenz model for these parameter values. So the L-H system is a slow-fast autonomous dynamical system. We derive the slow manifold equations of the L-H system by three deferent methods as following.

### 4 The first slow manifold equation of the L-H system

**Definition 1** On the attractive parts of the phase space (i.e. where \( J(X) \) have a fast eigenvalue) of the S-FADS, the slow manifold of the S-FADS is locally defined by a plane orthogonal to the tangent system’s left fast eigenvector [4].

**Theorem 1** On the attractive parts of the phase space of the L-H system, the first slow manifold equation of the L-H system (2) is:
\[
\begin{align*}
4 (y - x) \left( (\lambda_1 (x, y, z) + 1) (\lambda_1 (x, y, z) + 0.4) + 0.4 x^2 \right) \\
+ 4 (\lambda_1 (x, y, z) + 0.4) (xz - y) + 4x (4.8 - 0.4z - 0.4xy) = 0
\end{align*}
\]
(3)
where \( \lambda_1(x, y, z) \) is the fast eigenvalue of \( J(X) \).

**Proof:** When parameter \( \sigma = 4, b = 0.4, a = 12 \), the L-H system can be written as
\[
\begin{align*}
\dot{x} &= 4(y - x) \\
\dot{y} &= xz - y \\
\dot{z} &= 4.8 - 0.4z - 0.4xy
\end{align*}
\]
(4)

For a point \( X = (x, y, z)^T \) in the attractive parts of the phase space, the Jacobian matrix of the L-H system at the point \( X = (x, y, z)^T \)
\[
J(X) = J(x, y, z) = \begin{bmatrix}
-4 & 4 & 0 \\
4 & -1 & x \\
-0.4y & -0.4x & -0.4
\end{bmatrix}
\]
(5)

Let \( \lambda_1(X) = \lambda_1(x, y, z) \) be the fast eigenvalue and \( \lambda_2(X), \lambda_3(X) \) the two slow ones. The left fast eigenvector, i.e. the eigenvector of \( J^T(X) \) (the superscript “\(^T\)” denote “transpose”) is given by
\[
z_{\lambda_1}(x, y, z) = \begin{bmatrix} 1 \\ \frac{4(\lambda_1(x, y, z) + 0.4)}{4 \lambda_1(x, y, z) + 1)(\lambda_1(x, y, z) + 0.4) + 0.4x^2} \\ \frac{(\lambda_1(x, y, z) + 1)(\lambda_1(x, y, z) + 0.4) + 0.4x^2}{4x} \\ \end{bmatrix}
\]
(6)

On the attractive parts of the phase space (i.e. where \( J(X) \) have a fast eigenvalue), the equation of the slow manifold is given by
\[
z_{\lambda_1}^T(x, y, z) \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = 0
\]
(7)

If we replace \( z_{\lambda_1}(x, y, z) \) by its expression given by (6) and the velocities by Lorenz-Haken equation (4), we can obtain the slow manifold equation of the L-H system as (3).
5 The second slow manifold equation of the L-H system

**Definition 2** On the attractive parts of the phase space of the S-FADS, let \( z_{\lambda_2}(x, y, z) \) and \( z_{\lambda_3}(x, y, z) \) are the two slow eigenvectors associated with the two slow eigenvalues \( \lambda_2(X) \) and \( \lambda_3(X) \) of \( f(X) \). The local slow manifold in the neighborhood of \( X \) is generated by these two vectors [5].

**Theorem 2** On the attractive parts of the phase space of the L-H system, the second slow manifold equation of the L-H system (2) is:

\[
F_1(x, y, z)x + F_2(x, y, z)y + F_3(x, y, z)z + F_3(x, y, z)xy + F_4(x, y, z)xz + F_5(x, y, z) = 0, \tag{8}
\]

where

\[
F_1(x, y, z) = 4[g_2(x, y, z)g_3(x, y, z) - g_1(x, y, z)g_4(x, y, z)];
\]

\[
F_2(x, y, z) = 4[g_1(x, y, z)g_4(x, y, z) - g_2(x, y, z)g_3(x, y, z)] + g_4(x, y, z) - g_2(x, y, z);
\]

\[
F_3(x, y, z) = 0.4[g_1(x, y, z) - g_3(x, y, z)];
\]

\[
F_4(x, y, z) = g_2(x, y, z) - g_4(x, y, z);
\]

\[
F_5(x, y, z) = 4.8[g_3(x, y, z) - g_1(x, y, z)].
\]

**Proof:** For a point \( X = (x, y, z)^T \) in the attractive parts of the phase space, Let \( \lambda_2(X), \lambda_3(X) \) are the two slow eigenvalues of the L-H system at the point \( X = (x, y, z)^T \). It is easy to show that for \( k \in \{2, 3\} \), it is possible to write

\[
\begin{align*}
  z_{\lambda_k}(x, y, z) &= \begin{bmatrix}
1 \\
u_k(x, y, z) \\
v_k(x, y, z)
\end{bmatrix}
\end{align*}
\]

where

\[
u_k(x, y, z) = 1 + \frac{\lambda_k(x, y, z)}{4}
\]

\[
v_k(x, y, z) = \frac{1}{x}[(1 + \frac{\lambda_k(x, y, z)}{4})(1 + \lambda_k(x, y, z)) - z]
\]

The equation of the slow manifold can be derived from

\[
\begin{bmatrix}
  \dot{x} \\
  \dot{y} \\
  \dot{z}
\end{bmatrix} =
\begin{bmatrix}
1 & u_2(x, y, z) & v_2(x, y, z) \\
1 & u_3(x, y, z) & v_3(x, y, z)
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

which gives

\[
\begin{bmatrix}
4(y - x) & xz - y & 4.8 - 0.4z - 0.4xy \\
1 & u_2(x, y, z) & v_2(x, y, z) \\
1 & u_3(x, y, z) & v_3(x, y, z)
\end{bmatrix} = 0
\]

It is known that for some points \( X = (x, y, z)^T \), the two eigenvalues \( \lambda_2(X) \) and \( \lambda_3(X) \) are complex conjugate numbers and so are \( z_{\lambda_2}(x, y, z) \) and \( z_{\lambda_3}(x, y, z) \) (in fact their second and third components). So we have \( u_3(x, y, z) = u_2(x, y, z)^*, v_3(x, y, z) = v_2(x, y, z)^* \), where "*" denotes complex conjugate operation.

Now, the slow manifold equation must be real. So we have to take any linear combination of the slow eigenvectors which leads to a real determinant, for example

\[
\begin{align*}
  \bar{z}_{\lambda_2}(x, y, z) &= \frac{1}{2}(z_{\lambda_2}(x, y, z) + z_{\lambda_3}(x, y, z)) \\
  \bar{z}_{\lambda_3}(x, y, z) &= \frac{1}{2}((1 + i)z_{\lambda_2}(x, y, z) + (1 - i)z_{\lambda_3}(x, y, z))
\end{align*}
\]

so we have

\[
\begin{align*}
  \bar{z}_{\lambda_2}(x, y, z) &= \begin{bmatrix} 1 \\
Re[u_2(x, y, z)] \\
Re[v_2(x, y, z)]
\end{bmatrix}
\end{align*}
\]

**IJNS eamil for contribution:** editor@nonlinearscience.org.uk
where $x$ slow manifold

system. $M$ S-FADS, the slow manifold of the S-FADS is locally defined by a local invariable manifold perturbation theory.

6 The third slow manifold equation of L-H system equation of the L-H system as (8).

Let unique equation of the slow manifold as follows

Proof: On the attractive parts of the phase space of the L-H system, the first order expression of the determinant is

On the attractive parts of the phase space (i.e. where $J(X)$ have a fast eigenvalue) of the S-FADS, the slow manifold of the S-FADS is locally defined by a local invariable manifold $M_{e}$ of the system.

Theorem 3 On the attractive parts of the phase space of the L-H system, the first order expression of the slow manifold $M_{e}$ of the L-H system (2) is:

$$x = y + \varepsilon(y - yz)$$ (29)

Proof: Taking $\varepsilon=1/\sigma$, then we can treat the L-H system (2) as slow-fast autonomous system. Therefore we can precede the qualitative analysis to it and can obtain the slow manifold equation by using geometric singular perturbation method.

Taking $\varepsilon = 1/\sigma = 1/4$, the slow system of the L-H system is

$$\begin{cases} \varepsilon \dot{x} = y - x \\ \dot{y} = xz - y \\ \dot{z} = 0.4(12 - z - xy) \end{cases}$$ (30)

where $x$ is the fast variable, $y$ and $z$ are slow variables. The dualistic system of (30), namely fast system is

$$\begin{cases} x' = y - x \\ y' = \varepsilon(xz - y) \\ z' = \varepsilon(4.8 - 0.4z - 0.4xy) \end{cases}$$ (31)
where $\tau = t/\varepsilon$ is called the fast-time scale, $t$ is called the slow-time scale. As long as $\varepsilon \neq 0$, the system (31) is equivalent to the system (30).

In fast system, letting $\varepsilon \rightarrow 0$, we obtain a zero order approximate slow manifold $M_0$: $x = y$. Obviously, the dimension of $M_0$ is 2. On $M_0$, the Jacobin matrix of the system (31) is

$$J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which just contains $\text{Dim}(M_0)=2$ eigenvalues with zero real part (in fact, they are zero real roots), therefore $M_0$ is normal hyperbolic. Fenichel’s persistence theorem states that, provided some hypotheses are satisfied, the system (31) possesses a slow manifold that can be written as the graph of a function. The most critical hypotheses are satisfied in the systems (31), so by Fenichel’s theorem there exist a local invariable manifold $M_0$ for the system (31).

In general, one cannot directly calculate the expression of $M_0$. Following we investigate the leading order expression of $M_0$. From the invariable manifold theory by Fenichel, because of the $O(\varepsilon)$ closeness of $M_0$ and $M_0$, the relation of variables $x$ and $y$ in $M_0$ is $x = y + O(\varepsilon)$. We expand it in $\varepsilon$ series:

$$x = y + \varepsilon H(y, z) + O(\varepsilon^2) \quad (32)$$

Because $M_0$ is local invariable manifold of (30), so derivate (32) to get

$$\dot{x} = \dot{y} + \varepsilon \frac{\partial H}{\partial y} \dot{y} + \varepsilon \frac{\partial H}{\partial z} \dot{z} + O(\varepsilon^2) \quad (33)$$

and

$$\varepsilon \dot{\varepsilon} = \varepsilon \dot{y} + O(\varepsilon^2)$$

$$= \varepsilon \left[\left(y + \varepsilon H(y, z) + O(\varepsilon^2)\right) - y\right] + O(\varepsilon^2) \quad (34)$$

Plugging the formula (32) into the first term of (30), one has

$$\varepsilon \dot{\varepsilon} = y - \left[y + \varepsilon H(y, z) + O(\varepsilon^2)\right]$$

$$= - \left[\varepsilon H(y, z) + O(\varepsilon^2)\right] \quad (35)$$

Comparing the coefficients of the same power of $\varepsilon$ in (34) and (35), one gets $H(y, z) = y - yz$.

Thus the first order expression of the slow manifold $M_0$ of the L-H system (2) is (29).

$M_0$ presents many good properties, for example, $M_0$ is $O(\varepsilon)$ close to $M_0$, $M_0$ is local invariable. If $M_0$ is $C^r$ $(0 < r < \infty)$, it can be described as the graph of a smooth function. The slow manifold $M_0$ given by Fenichel’s theorem plays a central role in systems of the form (30). The dynamics on it is slow, while, due to its hyperbolic structure, nearby trajectories approach it exponentially. Also, the $O(\varepsilon)$ closeness of $M_0$ and $M_0$ is crucial for showing the existence of solutions to the three systems we present here.

Its graph is given in figure 1(a). Figure 1(b) shows the strange trajectories of the system (2).

From the above discussion, it is concluded that we can get the expression of slow manifold clearly and easily by geometric singular perturbation analysis and the acquired equation of the slow manifold (29) is concrete and brief rather than abstract as the one acquired by the two former nonstandard analysis is, furthermore the eigenvalue $\lambda_k(x, y, z)$ in the expressions (3) and (8) are also the abstract functions of $x, y, z$. So, it is more convenient for the equation (29) to be used for qualitative analysis and numerical simulation.

7 Application to the Chua’s system

In the following we discuss the slow manifold equation of the Chua’s system. The Chua’s system is an electric circuit system. Because of its extensive applied foreground, the Chua’s system has become a new focus in research of controlling nonlinear chaotic electric circuit and investigation of the nerve network.
in recent years. The Chua’s electric circuit system can emerge the state of the quantitative dynamics in any three order nonlinear system, among them including a subsection linear function of three parts with singular symmetry. The dynamics equation of the Chua’s system can be expressed as the system of nonlinear differential equations as follows

\[
\begin{align*}
\varepsilon \dot{x} &= y - x - f(x) \\
\dot{y} &= x - y + z \\
\dot{z} &= -\beta y
\end{align*}
\] (36)

where

\[f(x) = \begin{cases} 
  bx + a - b & x > 1 \\
  ax & -1 \leq x \leq 1 \\
  bx - a + b & x \leq -1
\end{cases}
\]

The Chua’s system (36) is a slow-fast autonomous system, where \(x\) is the fast variable, \(y\) and \(z\) are slow ones. When \(a = -8/7, b = -5/7, \beta = 100/7\), the Chua’s system has a chaotic attractor.

**Theorem 4.** On the attractive parts of the phase space of the Chua’s system, the first order expression of the slow manifold \(M_\varepsilon\) of the Chua’s system (36) is:

\[
M_\varepsilon : x = \begin{cases} 
\frac{y-a+b}{1+b} - \varepsilon \frac{1}{(1+b)^3} (b - a - by + (1 + b)z) & y \geq -(1 + a) \\
\frac{y-a-b}{1+b} - \varepsilon \frac{1}{(1+b)^3} (a - b - by + (1 + b)z) & y \leq 1 + a
\end{cases}
\] (37)

**Proof:** Given \(\varepsilon = 1/9, a = -8/7, b = -5/7, \beta = 100/7\), the fast system of the Chua’s system (36) is

\[
\begin{align*}
\dot{x}_f &= y - x - f(x) \\
\dot{y}_f &= \varepsilon (x - y - z) \\
\dot{z}_f &= -\varepsilon \beta y
\end{align*}
\] (38)

Where \(\tau = t/\varepsilon\) is a fast variable.

The equilibrium points of (36) are \(O(0,0,0), A(\frac{b-a}{1+b}, 0, \frac{a-b}{1+b}), B(\frac{a-b}{1+b}, 0, \frac{b-a}{1+b})\), when \(|x| > 1\), the Jacobian matrix is

\[
J_1 = \begin{pmatrix}
-1 + \frac{b}{\varepsilon} & 1 & 0 \\
1 & -1 & 1 \\
0 & -\beta & 0
\end{pmatrix}
\]

At given parameter value, we can get its eigenvalues \(\lambda_1 = -3.94, \lambda_{2,3} = 0.18 \pm 3.04i\), so \(A, B\) are unstable foci.

When \(|x| < 1\), the Jacobian matrix is

\[
J_1 = \begin{pmatrix}
\frac{-1 + \frac{b}{\varepsilon}}{\varepsilon} & \frac{1}{\varepsilon} & 0 \\
1 & -1 & 1 \\
1 & -\beta & 0
\end{pmatrix}
\]

Its eigenvalues are \(\lambda_1 = 2.21, \lambda_{2,3} = -0.965 \pm 2.711i\), so \(A, B\) still are unstable foci.
In fast system (38), letting \( \varepsilon \to 0 \), we can get the zero order approximate slow manifold \( M_0 \):

\[
x = \begin{cases} \frac{y-a+b}{1+b} & \quad y \geq -(1+a) \\ \frac{y+a-b}{1+b} & \quad y \leq 1+a \end{cases}
\]

The dimension of the zero order approximate slow manifold \( M_0 \) is 2.

When \( y \geq -(1+a) \) or \( y \leq 1+a \), the Jacobian matrix of the system (38) on the zero order approximate slow manifold \( M_0 \) is

\[
J = \begin{pmatrix} -1-b & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

Its eigenvalues are two zero roots and \(-1-b\), thus the zero order approximate slow manifold \( M_0 \) satisfies normal hyperbolic condition. We look for its first order approximately slow manifold \( M_\varepsilon \).

When \( y \geq -(1+a) \), let the equation of the slow manifold be

\[
x = \frac{y-a+b}{1+b} + \varepsilon H(y, z) + O(\varepsilon^2)
\]

Derive the above formula

\[
\dot{x} = \frac{1}{1+b} \dot{y} + \varepsilon \frac{\partial H}{\partial y} \dot{y} + \varepsilon \frac{\partial H}{\partial z} \dot{z} + O(\varepsilon^2)
\]

\[
= \frac{1}{1+b} \left( (x-y+z) + \varepsilon \frac{\partial H}{\partial y} \dot{y} + \varepsilon \frac{\partial H}{\partial z} \dot{z} + O(\varepsilon^2) \right)
\]

\[
\varepsilon \dot{x} = \frac{1}{1+b} \left( \frac{y-a+b}{1+b} - y + z \right) \varepsilon + O(\varepsilon^2)
\]

On the other hand,

\[
\varepsilon \dot{x} = y - x - f(x)
\]

\[
= y - x - bx - a + b
\]

\[
= -(1+b) H(y, z) \varepsilon + O(\varepsilon^2)
\]

\[
H(y, z) = -\frac{1}{(1+b)^2} \left( \frac{y-a+b}{1+b} - y + z \right)
\]

\[
= -\frac{1}{(1+b)^3} (b-a-by+(1+b)z)
\]

\[
\therefore M_\varepsilon : x = \frac{y-a+b}{1+b} - \varepsilon \frac{1}{(1+b)^3} (b-a-by+(1+b)z)
\]

When \( y \leq 1+a \), by similar calculating we can get

\[
M_\varepsilon : x = \frac{y+a-b}{1+b} - \varepsilon \frac{1}{(1+b)^3} (a-b-by+(1+b)z)
\]

So the slow manifold is divided into two parts and its equation is (37). The diagram is two disjunctive half planes which can be seen in figure 2.
In 1963, E. N. Lorenz discovered the Lorenz system which is the form of nonlinear ordinary differential equations as following

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= -xz + rx - y \\
\dot{z} &= xy - bz \\
\end{align*}
\]

(Taken \( \sigma = 10, b = 8/3, r = 28 \), the Lorenz system has a chaotic attractor [9].)

In 1999, Chen Guanrong, et al. discovered another chaotic attractor in studying anti-controlling chaos, and named it as Chen’s system. It also has the form of nonlinear ordinary differential equations

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= (r - \sigma)x - xz + ry \\
\dot{z} &= xy - bz \\
\end{align*}
\]

(Taken \( \sigma = 35, b = 3, r = 28 \), the Chen’s system has a chaotic attractor [10].)

In 2002, also in studying anti-controlling chaos, Lü Jinhua, et al. discovered a new chaotic system – Lü’s system

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= -xz + ry \\
\dot{z} &= xy - bz \\
\end{align*}
\]

(Taken \( \sigma = 36, b = 3, r = 20 \), Lü’s system has a chaotic attractor [11].

One can see that those three dynamical systems are all nonlinear ordinary differential equations systems with degree three. And all \( x \) derivative in those systems are multiplied by a number \( \sigma \). The only difference is the form of \( y \) derivative. According to a critical term \( a_{12}a_{21} \) that put forward by Vanecek and Celkovský in 1966, the above three systems belong to different kinds: The Lorenz system: \( a_{12}a_{21} > 0 \); The Chen system: \( a_{12}a_{21} < 0 \); The Lü’s system: \( a_{12}a_{21} = 0 \). This reflected the well correlation of these three systems. The Lü’s system connected the Lorenz system with the Chen system, and represented their transformation continuously. So they may have some common properties. It is known that all they have chaotic attractors. How about their invariant manifolds are? If we find their invariant manifolds, we can get an approximate knowledge of their dynamical behavior.

Using the same method in the section 6, i.e. to derive the slow manifold equation of the L-H system by using geometric singular perturbation theory, we can discuss the slow manifold of the Lorenz system, the Chen’s system and the Lü’s system.

**Theorem 5** On the attractive parts of the phase space of the Lorenz system (39), the first order expression of the slow manifold \( M_\varepsilon \) of the Lorenz system (39) is

\[
x = y + \varepsilon(yz - 27y).
\]
Theorem 6 On the attractive parts of the phase space of the Chen system (40), the first order expression of the slow manifold $M_\varepsilon$ of the Chen system (40) is

$$x = y + \varepsilon(yz - 21y).$$

Its graph is given in figure 4(a). Figure 4(b) shows the strange trajectories of the system (40).

Theorem 7 On the attractive parts of the phase space of the Lü’s system (41), the first order expression of the slow manifold $M_\varepsilon$ of the Lü’s system (41) is

$$x = y + \varepsilon(yz - 20y).$$

Its graph is given in figure 5(a). Figure 5(b) shows the strange trajectories of system (41).
9 The dynamical behavior analysis of the S-FADS

Geometric singular perturbation theory only requires the limiting slow system and the limiting fast system, which are both two-dimension systems and much simpler than their original systems. None of explicit information about their solution is required. We can make a brief image of the system’s dynamics. It is easy to prove that three equilibriums of the Lorenz system, \( O(0, 0, 0) \), \( A(\sqrt{br - 1}, \sqrt{br - 1}, r - 1) \), \( B(-\sqrt{br - 1}, -\sqrt{br - 1}, r - 1) \), all satisfy the equation (39), which means that they are all in the slow manifold \( M_\varepsilon \). Therefore we can use geometric singular perturbation theory to analyze the qualitative behavior and orbits of the system. Given initial data \((x(0), y(0), z(0))\), because the velocity of \( x, y, z \) is different, \( x \) is a fast variable, but \( y \) and \( z \) are slow variables, the fast movement takes places first, \( x \) is changing very fast, but \( y \) and \( z \) are remaining almost unchanged. Then \( x \) attains the half stability condition that is \( x \) reaches in the slow manifold. In the slow manifold, \( x, y \) and \( z \) all change slowly and the movement attains a certain equilibrium point \((O, A, B)\), for example \( O \), but can’t forever stay in the equilibrium point. Because \( x \) just attains half stability condition on slow manifold, it must lose stability and begin fast movement again. The fast movement ends very fast, the slow movement takes place again. Therefore the fast movement and the slow movement exercise alternately. The behavior of the system is most like this: it goes out from the point \( O \), and then comes into the point \( A \), then turns out from \( A \) into the point \( O \), then leaves the point \( O \) and head for point \( B \), turns out from point \( B \) again and comes into the point \( O \). A period begins after a period ends, going round and starting again, the back and forth is continuous. The circumvolution round the points \( A \) and \( B \) is on the plane approximately, however its type and numbers of turning are irregular, as a result it became the strange attractor like butterfly wings.

We analyze the relations and the difference of the slow manifold forms among three kinds of the Lorenz, Chen’s and Lü’s systems as following. Because the three systems are similar in many ways, the graphs of their slow manifolds also look like the same, which are all slightly folded planes. Because slow manifold are exponentially abstract, all trajectories starting in the abstractor tends to be the slightly folded planes, so one can see butterflies wings. Another interesting thing is that the coefficient of \( \varepsilon y \) in the expression of slow manifold is just as the difference between the first power term coefficient of variables \( x \) and \( y \) in second equation of the system, and the coefficient of \( \varepsilon y z \) is just as the opposite number of coefficient of second power term \( yz \).

10 Conclusion

The aim of this paper is to propose a qualitative approach to analyze the dynamics of L-H system. With three different methods, associated with the slow manifold equations of the L-H system is obtained. This study is based on the existence of a fast eigenvalue of the Jacobian matrix of the associated local tangent system, which allowed us to define the equation of a slow manifold. In general the Ramdani’s thought of calculating is clear. The only hard task is to develop a general program to realize numerical simulation which needs a lot of calculation. Additional, the eigenvalue \( \lambda_1(x, y, z) \), in the slow manifold equation, is abstract function of the variables \( x, y, z \), and it is not easy to make qualitative analysis of the system. The amount of calculation of our method is quite little in this paper, because we only need to calculate the eigenvalues of degenerate fast sub-system and then make power expanding. Because the degenerate system is a simple system, the analytic expression of the slow manifold \( M_\varepsilon \) is very clear. Thus it is easy to distinguish the relation between equilibrium and slow manifold, and it is also easy to make qualitative analysis of the orbits theoretically and numerical simulation.

Acknowledgement

This Research is supported by the National Natural Science Foundation of China (No.90210004) and the Department of Education Foundation of Jiangsu Province (No.03SJB790008).

IINS homepage: http://www.nonlinearscience.org.uk/
References


