Hopf Bifurcation Analysis of the Energy Resource Chaotic System

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Abstract: In this paper, the Hopf bifurcation of the energy resource chaotic system is studied by using an analytical method. It is shown that the energy resource chaotic system will display a subcritical Hopf bifurcation under certain conditions.

Keywords: energy resource chaotic system; Lyapunov coefficient; subcritical Hopf bifurcation

1 Introduction

Since Lorenz found the first classical chaotic attractor in 1963 [1], chaos, as a very interesting nonlinear phenomenon, has been intensively studied in the last decades [2-6]. It is found to be either useful or has great potential in many fields, such as in engineering, biology and economics.

In paper [7], we set up a new system was named the energy resource system which is a third order autonomous system. This new system displays very complex 2-scroll attractor. The value of Lyapunov exponents of this system is obtained as (0.068, 0.016, -0.016). This new attractor is different from the well-known Lorenz attractor, Rössler attractor, Chua’s attractor, Chen attractor, and so on. In paper [8], we further explored its dynamical behaviors based on two subsystems of the energy resource system and achieved chaotic synchronization for the energy resource system by applied the modified adaptive synchronization. In paper [9], chaos in energy resource chaotic system is controlled to equilibrium points or periodic orbits by using feedback control and adaptive control.

In this paper, Hopf bifurcation of the energy resource system is analyzed by using an analytical method. Analysis results show that this system will display Hopf bifurcation at two equilibria $O(0, 0, 0)$, $S_2(x_2, y_2, z_2)$ and Hopf bifurcation at equilibrium $O(0, 0, 0)$ is nondegenerate and subcritical. It is shown that the energy resource chaotic system has complex dynamics with some interesting characteristics.

2 Energy resource chaotic system

Energy resource chaotic system [7] is described by the following system of differential equations:

$$\begin{align*}
\frac{dx}{dt} &= a_1 x (1 - \frac{z}{M}) - a_2 (y + z) \\
\frac{dy}{dt} &= -b_1 y - b_2 z + b_3 x [N - (x - z)] \\
\frac{dz}{dt} &= c_1 z (c_2 x - c_3)
\end{align*}$$

(2.1)

where $x(t)$ the energy resource shortage in $A$ region, $y(t)$ the energy resource supply increment in $B$ region, $z(t)$ the energy resource import in $A$ region; $a_i$, $b_i$, $c_i$, $M$, $N$ are positive real constants. This system has

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a chaotic attractor shown in Fig.1, when $a_1 = 0.09, a_2 = 0.15, b_1 = 0.06, b_2 = 0.082, b_3 = 0.07, c_1 = 0.2, c_2 = 0.5, c_3 = 0.4, M = 1.8, N = 1$ and initial condition $[0.82, 0.29, 0.48]$. The value of Lyapunov exponents of this system is obtained as $(0.068, 0.016, -0.016)$. This system has three equilibria: $x, y$ vectors where $< p, q >$

$B$

$F$

3 Hopf bifurcation

According to [10], the first Lyapunov coefficient of the system $(2.1)$ at the origin point $O$ can be written as

$$l_1(0) = \frac{1}{2\omega} \text{Re}[< p, C(q, q, \bar{q}) > - 2 < p, B(q, A^{-1}B(q, \bar{q}) > + < p, B(\bar{q}, (2i\omega E - A)^{-1}B(q, q) >]$$


Figure 1: Energy resource attractor.
This formula is very convenient for analytical treatment of the Hopf bifurcation in n-dimensional systems with \( n \geq 2 \). When \( l_1(0) < 0 \), the Hopf bifurcation is nondegenerate and supercritical; when \( l_1(0) > 0 \), the Hopf bifurcation is nondegenerate and subcritical.

The Jacobian of system (2.1) at the point \( O (0, 0, 0) \) is given by

\[
\begin{bmatrix}
    a_1 & -a_2 & -a_2 \\
    b_3 N & -b_1 & -b_2 \\
    0 & 0 & -c_1 c_3
\end{bmatrix}
\]

For simplicity, we fix parameters:

\( a_2 = 0.15, \quad b_2 = 0.082, \quad b_3 = 0.07, \quad c_1 = 0.2, \quad c_2 = 0.5, \quad c_3 = 0.4, \quad M = 1.8, \quad N = 1. \)

By calculations, we can obtain the characteristic polynomal of Jacobian matrix of the system (2.1) at \( O (0, 0, 0) \) is

\[
\lambda^3 + (b_1 - a_1 + 0.08) \lambda^2 + (0.0105 - a_1 b_1 + 0.08 b_1 - 0.08 a_1) \lambda - 0.08 a_1 b_1 + 0.00084 = 0 \quad (3.7)
\]

When \( b_1 = b_h = a_1 \), Eq. (3.7) has pure imaginary roots \( \lambda_{1,2} = \pm i \omega = i \sqrt{0.0105 - a_1^2} \) (if \( a_1 < \sqrt{0.0105} = 0.1025 \)), and they satisfy

\[
\lambda'_h = \frac{\lambda^2 + (0.08 - a_1) \lambda - 0.08 a_1}{3 \lambda^2 + 2(b_1 - a_1 + 0.08) \lambda + 0.0105 - a_1 b_1 + 0.08 b_1 - 0.08 a_1},
\]

thus

\[
\alpha' (0) = \text{Re} \lambda'_h (b_h) = -\frac{(a_1 - 0.0715)(a_1 + 0.1915)}{4(0.0505 - a_1^2)}.
\]

When \( 0.0715 < a_1 < 0.1025 \), \( \alpha' (0) \neq 0 \).

Besides, the third root \( \lambda_3 = -0.08 < 0 \). Thus, a Hopf bifurcation takes place.

In order to analyze the bifurcation (i.e., to determine the direction of the limit cycle bifurcation), we will computer the first Lyapunov coefficient \( l_1 (0) \) of the restricted system on the center manifold at the critical parameter values.

When \( b_1 = b_h \), the Jacobian of system (2.1) at the point \( O (0, 0, 0) \) is

\[
A = \begin{bmatrix}
    a_1 & -0.15 & -0.15 \\
    0.07 & -a_1 & -0.082 \\
    0 & 0 & -0.08
\end{bmatrix}
\]

(3.8)

Next, we will calculate the corresponding vectors \( p, q \) of the matrix \( A \). By tedious calculations, we have

\[
q = \begin{pmatrix}
    1, \quad a_1 - i \sqrt{0.0105 - a_1^2} \quad 0
\end{pmatrix}^T
\]

(3.9)

and

\[
p = \frac{0.00525}{0.0105 - a_1^2 - a_1 i \sqrt{0.0105 - a_1^2}} \begin{pmatrix}
    1, \quad a_1 + i \sqrt{0.0105 - a_1^2} \quad 0
\end{pmatrix}^T
\]

(3.10)

which satisfy \( Aq = i \omega q \), \( A^T p = -i \omega p \) and \( < p, q > = 1 \).

For the energy resource chaotic system (2.1), the bilinear and trilinear functions are

\[
B \left( X, X' \right) = \begin{pmatrix}
    -\frac{2 a_1}{a_1^2} x x' \\
    -2 b_3 x x' + b_3 \left( x z' + z x' \right) \\
    c_1 c_2 \left( x z' + z x' \right)
\end{pmatrix} = \begin{pmatrix}
    -a_1^2 x x' \\
    -0.14 x x' + 0.07 \left( x z' + z x' \right) \\
    0.1 \left( x z' + z x' \right)
\end{pmatrix}
\]

(3.11)

and

\[
C \left( X, X', X'' \right) = \begin{pmatrix}
    0 \\
    0 \\
    0
\end{pmatrix}
\]

(3.12)

respectively, where \( X = (x, y, z)^T, X' = (x', y', z')^T \) and \( X'' = (x'', y'', z'')^T \).
By some tedious manipulations, from (3.8) - (3.12), we have

\[ B(q, q) = B(q, \bar{q}) = \begin{pmatrix} -\frac{a_1}{0.9} \\ -0.14 \\ 0 \end{pmatrix} \] (3.13)

and

\[ A^{-1} = \frac{1}{0.08(a_1^2 - 0.0105)} \begin{pmatrix} 0.08a_1 & -0.012 & 0.0123 - 0.15a_1 \\ 0.0056 & -0.08a_1 & 0.082a_1 - 0.0105 \\ 0 & 0 & 0.0105 - a_1^2 \end{pmatrix} \] (3.14)

Let \( s = A^{-1} B(q, \bar{q}) \), then from (3.13) and (3.14), we have

\[ B(q, s) = \begin{pmatrix} -\frac{a_1}{0.9} + 0.021 \\ -0.14 \\ 0 \end{pmatrix} \] (3.15)

therefore

\[ < p, B(q, s) > = \frac{0.00525 \left( -\frac{a_1}{0.9} + 0.021 \right)}{a_1^2 - 0.0105} \cdot \frac{-\frac{a_1}{0.9} + 2 \left( a_1 - i \sqrt{0.0105 - a_1^2} \right)}{0.0105 - a_1^2 + a_1 i \sqrt{0.0105 - a_1^2}} \] (3.16)

Again, let \( s' = \frac{1}{3 \sqrt{2} \sqrt{0.0105 - a_1^2 + 0.08 (a_1^2 - 0.0105)}} B \), we can obtain

\[ B = \begin{pmatrix} -\frac{a_1}{0.9} \left[ 4a_1^2 + 0.08a_1 - 0.042 + (0.16 + 2a_1) \sqrt{0.0105 - a_1^2} \right] +0.14 \left( 0.122 + 0.3 \sqrt{0.0105 - a_1^2} \right) \\ \left[ 0 \right] \\ \end{pmatrix} \]

\[ \Delta = -\frac{a_1}{0.9} \left[ 4a_1^2 + 0.08a_1 - 0.042 + (0.16 + 2a_1) \sqrt{0.0105 - a_1^2} \right] +0.14 \left( 0.122 + 0.3 \sqrt{0.0105 - a_1^2} \right) \]

then

\[ B(\bar{q}, s') = \begin{pmatrix} -\frac{a_1}{0.9} \Delta \\ -0.14 \Delta \\ 0 \end{pmatrix} \] (3.17)

\[ < p, B(\bar{q}, s') > = \frac{0.00525 \Delta}{3 \left( 2i \sqrt{0.0105 - a_1^2 + 0.08} \right) (a_1^2 - 0.0105)} \cdot \frac{-\frac{a_1}{0.9} + 2 \left( a_1 - i \sqrt{0.0105 - a_1^2} \right)}{0.0105 - a_1^2 + a_1 i \sqrt{0.0105 - a_1^2}} \] (3.18)

Consequently, from (3.16) and (3.18), we have

\[ l_1(0) = \frac{1}{2 \omega} Re \left[ -2 < p, B(q, s) > + < p, B(\bar{q}, s') > \right] = \frac{1.0288a_1 (a_1^2 + 0.2711) \left( 0.1235 - a_1^2 \right)}{(0.0105 - a_1^2)^{\frac{3}{2}} (0.0121 - a_1^2)} \] (3.19)
When \( 0.0715 < a_1 < 0.1025 (= \sqrt{0.0105}) \), the Lyapunov coefficient is clearly positive, so the Hopf bifurcation is nondegenerate and subcritical, i.e. a unique and unstable cycle bifurcates from the equilibrium via the Hopf bifurcation for \( b_1 > b_h = a_1 \).

In paper [7], we consider the Hopf bifurcation of system (2.1) at the point \( S_2 (x_2, y_2, z_2) \) and get the following conclusion: if \( b_2 = 0.1013 \), then the equilibrium point \( S_2 (x_2, y_2, z_2) \) is a Holf bifurcation point. Similarly to the case of the equilibrium point \( O (0, 0, 0) \), Hopf bifurcation at equilibrium point \( S_2 (x_2, y_2, z_2) \) can be further explored.

## 4 Conclusion

This paper investigates the Hopf bifurcation of the energy resource system by an analytical method. Yet, there are still many unknown dynamics of the chaotic system studied in this paper which deserve further studies in the near future.

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## References


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