On Solution of the Dullin-Gottwald-Holm Equation

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Abstract: In this paper, we study the inverse scattering problem for a new nonlinear dispersive shallow water wave equation, called Dullin-Gottwald-Holm equation. We used the inverse scattering transform method, first, to establish the inverse scattering equation of the DGH equation and a series of solving equations, and then, to represent the known one-soliton solution in a simple parametric form by using the scattering data, last, to present a few examples of the one-soliton solution profile.

Keywords: Keywords: DGH equation; one-soliton solution; parametric form; the inverse scattering method

1 Introduction

Dullin, Gottwald, Holm [1] discussed the following 1+1 quadratically nonlinear model which describes the dispersive unidirectional water waves with fluid velocity $u(x; t)$

$$m_t + 2\omega u_x + um_x + 2mu_x = -\gamma u_{xxx}, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}$$

(1.1)

where $m = u - \alpha^2 u_{xx}$ is a momentum variable, $\alpha$ is a constant.

Using the notation $m = u - \alpha^2 u_{xx}$, one can rewrite equation (1.1) into the following form

$$u_t - \alpha^2 u_{xx} + 2\omega u_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_xu_{xx} + uu_{xxx})$$

(1.2)

This equation contains with two separately integrable soliton models for water waves - Korteweg-de Vries (KdV) model and Camassa-Holm model. The former contains the linear dispersion while the latter contains the nonlinear/nonlocal dispersion. Hence, equation (1.2) is called a new shallow water wave equation with linear and nonlinear dispersions. This completely integrable model is of significance in physics.

When $\alpha^2 \to 0$, equation (1.2) becomes the KdV equation:

$$u_t + 2\omega u_x + 3uu_x = -\gamma u_{xxx}$$

(1.3)

While $\omega = 0$, there exists a smooth soliton solution. Instead, taking $\gamma \to 0$ in the equation (1.2), it turns out to be the Camassa-Holm equation:

$$u_t - \alpha^2 u_{xxx} + 2\omega u_x + 3uu_x = \alpha^3(2u_xu_{xx} + uu_{xxx})$$

(1.4)

Many researches have been carried on the above equations. In [1], equation (1.1) was derived by using asymptotic expansions directly in the Hamiltonian for Euler’s equations in the shallow water regime and thereby shown to be bi-Hamiltonian and has a Lax pair formulation, and then, it also identified how
the dispersion coefficients for the linearized water waves appear as parameters in the isospectral problem for this inverse scattering transform integrable shallow water wave equations. This new derivation and analysis in [1] attached additional physical meaning to equation 1.1. It was more important that in [1] it formulated that equation 1.1 was a very important integrable shallow water equation, which has soliton liking KdV equation and peakon liking Camassa-Holm equation. In [2], Tian, Gui, Liu denoted 1.2 for the Dullin-Gottwald-Holm DGHequation and solved the well-posedness problem, the isospectral problem and the scattering problem for the DGH equation. Meanwhile, the scattering data of the scattering problem for the equation was explicitly expressed. In [3] Constantin discussed the scattering and the inverse scattering problem of Camassa-Holm equation and converted the isospectral problem of the Camassa-Holm equation into the classical Schroedinger eigenvalues problem by the Liouville transformation. After solving the nonlinear second-order ordinary differential equation, the explicit solution can be constructed from the scattering data of the Schrodinger problem. Furthermore, in [4] Johnson represented the known solitary-wave solution in a simple parametric form and then obtained the general two- and three-soliton solutions. A number of examples and the profile of solutions were also presented in [4]. In [6], Tian and Yin introduced a fifth-order K(m,n,l) equation with nonlinear dispersion to obtain multi-compaction solutions by Adomain decomposition method. And in [7], Fan and Tian proved that solitary wave of mKdV-KS equation persisted when the perturbation parameter was suitably small. Tian,Yin [8] [9] considered the nonlinear generalized Camassa-Holm equation and derived some new compacton and floating compacton solutions by using four direct ansatzs. Tian and Xu in [10] discussed the traveling wave solutions and pair soliton solutions of (1.4), and introduced the definitions of concave or convex peaked soliton and smooth soliton solution. In [11], Tian and Song considered generalized Camassa-Holm equations and generalized weakly dissipative Camassa-Holm equations, derived some new exact peaked solitary wave solutions. And in [12], Ding and Tian studied the existence of global solution and got the existence of the global attractor on dissipative Camassa-Holm equation. In [15], Tian and Sun derived regular and singular solitonic structures of generalized Camassa-Holm models through Backlund transformation.

Based on the above works, we investigate the nonlinear dispersive DGH model (1.2) by the inverse scattering method. The inverse scattering equation and a series of solving equations are established by the scattering data in [1]. Then the exact solutions of the DGH equation are obtained.

2 Outline of the basic results

Without loss of generality, we may assume that \( \omega + \frac{\gamma}{2 \alpha^2} > 0 \). If \( \omega + \frac{\gamma}{2 \alpha^2} < 0 \) we can transform \( u(x, t) \to -u(-x, t) \).

First, according to the remark of Theorem 6.3.2 in [1], the Liouville transformation is

\[
\varphi(y) = [m(x) + \omega + \frac{\gamma}{2 \alpha^2}]^{1/4} \varphi(x), \quad y = \int_{-\infty}^{x} \sqrt{m(\xi) + \omega + \frac{\gamma}{2 \alpha^2}} d\xi,
\]

we may convert equation into the classical Sturm-Liouville problem

\[
\frac{d^2 \varphi}{dy^2} - (Q - \mu) \varphi, \varphi = \varphi(y; t)
\]

where \( \mu = -\frac{\alpha^2}{4 \omega \alpha^2 + 2 \gamma} - \eta \) with \( \eta = \frac{\lambda}{\alpha^2 - 2 \gamma} \)

\[
Q(y) = \frac{1}{4 \alpha^2 q(y)} + \frac{q_{yy}(y)}{4q(y)} - \frac{3 (q_y)^2}{16 q^2(y)} - \frac{\alpha^2}{4 \omega \alpha^2 + 2 \gamma}
\]

\[
q = m + \omega + \frac{\gamma}{2 \alpha^2} = u - \alpha^2 u_{xx} + \omega + \frac{\gamma}{2 \alpha^2}
\]

When the values of are fixed, we may obtain the solutions of the scattering problems by using 2.6. In fact, suppose that \( Q(y; 0) \)is given, we may have \( Q(y; t) \) by invoking the time evolution of the scattering data of Theorem 6.3.1 in [1].

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From equation 2.6 we have
\[ Q(y,t) = -2 \frac{d}{dy} K(y,y;t) \] (2.9)

Where \( K(y,x;t) \) is a solution of the Marchenko equation
\[ K(y,x;t) + F(y + x,t) + \int_{y}^{\infty} K(y,z;t) F(z + x,t) dz = 0 \] (2.10)

Where, for example, in the case of reflectionless potentials,
\[ F(x,t) = \sum_{n=1}^{N} \exp[-\sqrt{u_n}X - \frac{k_n}{\alpha^2} (\frac{1}{\eta} + 2\gamma)t + C_n] \] (2.11)

Let \( u_n = \frac{\alpha^2}{4\omega^2 + 2\gamma} + \eta, k_n = \frac{1}{4\alpha^2} + \frac{\eta(\omega + \frac{\gamma}{2\alpha^2})}{4\alpha^2}, C_n \) are arbitrary constants.

In order to solve the DGH equation, after giving \( Q \), we must obtain \( q \) from 2.7. It can be tested that the solution for \( q \) has been obtained in the form
\[ q = \frac{1}{\alpha^2} \left( \phi_1^2 - \frac{1}{4W^2} \phi_2^2 \right)^2 \] (2.12)

where \( \phi_1(y,t) \) and \( \phi_2(y,t) \) are two independent solutions of linear equation
\[ \frac{d^2 \phi}{dy^2} - \left[ Q(y,t) + \frac{\alpha^2}{4\omega^2 + 2\gamma} \right] \phi = 0 \]

So \( W \) is their Wronskian (non-zero and constant). This formulation, however, is not convenient for our purposes, so we need to make the following transformations:
\[ \phi_1'[\phi_1'' - (Q + \frac{\alpha^2}{4\omega^2 + 2\gamma})\phi_1] - \frac{1}{4W^2} \phi_2[\phi_2'' - (Q + \frac{\alpha^2}{4\omega^2 + 2\gamma})\phi_2] = 0 \] (2.13)

Let \( \phi = \phi_2^2 - \frac{1}{4W^2} \phi_2^2 \), so we have:
\[ \alpha^2 q = \phi^2 \] (2.14)

then 2.13 becomes
\[ \frac{1}{2} \phi'' - (Q + \frac{\alpha^2}{4\omega^2 + 2\gamma})\phi - (\phi_1^2 - \frac{1}{4W^2} \phi_1^2) = 0 \] (2.15)

On the other hand, we also have
\[ \phi_1'[\phi_1'' - (Q + \frac{\alpha^2}{4\omega^2 + 2\gamma})\phi_1] - \frac{1}{4W^2} \phi_2[\phi_2'' - (Q + \frac{\alpha^2}{4\omega^2 + 2\gamma})\phi_2] = 0 \]
which gives
\[ (\phi_1^2 - \frac{1}{4W^2} \phi_2^2)t - (Q + \frac{\alpha^2}{4\omega^2 + 2\gamma})\phi t = 0 \] (2.16)

and then 2.15 and 2.16 show
\[ \phi'' - 4(Q + \frac{\alpha^2}{4\omega^2 + 2\gamma})\phi t - 2Qt\phi = 0 \] (2.17)

where the above derivatives are all the derivatives with respect to \( y \). Hence, we can obtain \( \phi \) by 2.17, then from 2.14 we have
\[ q = \frac{1}{\alpha^2} \phi^2 \] (2.18)

The final stage in the construction of solution is to obtain \( u \) from
\[ u - \alpha^2 u_{xx} = q - \omega - \frac{\gamma}{\alpha^2} \] (2.19)

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However, it is more usefully expressed in terms of \( y \), where

\[
\frac{dy}{dx} = \sqrt{q} \tag{2.20}
\]

thus 2.19 is written as

\[
\alpha^2 q_{yy} + \frac{1}{2} \alpha^2 q_y q_y - u = \omega + \frac{\gamma}{2\alpha^2} - q \tag{2.21}
\]

The solution of this equation yields \( u = u(y; t) \). Coupled with the integration of 2.20, it provides a parametric representation of the solution.

From the preceding outline, we are now able to summarize the procedure. Thus we shall adopt for finding solutions of the DGH equation. First, we select \( F(X; t) \) by using equation 2.11, and then solve to find \( K(y, x; t) \) (equation 2.10), from which we obtain \( Q(y; t) \) (equation 2.9). \( Q(y; t) \) is used in 2.17 and is solved for \( \phi \). Now we can have \( q(y) \) from equation 2.18. Finally, from equation 2.20 and 2.21, we obtain \( x = x(y; t) \) and \( u = u(y; t) \) respectively, which are the parametric form of the solution of DGH equation, where the parameter is \( y \), and all this is at fixed \( t \).

### 3 One-soliton Solutions

From 2.11: \( F(x; t) = \sum_{n=1}^{N} \exp[-\sqrt{\alpha^2 \omega^2 + \eta^2} \cdot \frac{1}{2} \alpha^2 \frac{1}{\eta^2} + 2\gamma) t + C_n] \) Particularly, let \( N = 1 \) we can have

\[
F(x; t) = \exp[-\sqrt{\frac{\alpha^2}{4\omega^2 + \eta^2} + \frac{1}{2} \alpha^2 \frac{1}{\eta^2} + 2\gamma) t + 2C}]
\tag{3.22}
\]

where \( k = \frac{1}{\alpha^2} + \eta(\omega + \frac{\gamma}{\alpha^2}) \), \( C \) is arbitrary constant. First, from the Marchenko equation 2.10 we find directly that

\[
K(y, x; t) = -\frac{\exp[-\sqrt{\frac{\alpha^2}{4\omega^2 + \eta^2} + \eta(x + y) - \frac{k}{\alpha^2} \eta(\omega + \frac{\gamma}{\alpha^2}) t + 2C}]}{1 + \frac{1}{2} \sqrt{\frac{\alpha^2}{4\omega^2 + \eta^2} + \eta}} \exp[-2\sqrt{\frac{\alpha^2}{4\omega^2 + \eta^2} + \eta y - \frac{k}{\alpha^2} \eta(\omega + \frac{\gamma}{\alpha^2}) t + 2C}]
\]

therefore,

\[
K(y, y; t) = -\frac{\exp[-2\sqrt{\frac{\alpha^2}{4\omega^2 + \eta^2} + \eta y - \frac{k}{\alpha^2} \eta(\omega + \frac{\gamma}{\alpha^2}) t + 2C}]}{1 + \frac{1}{2} \sqrt{\frac{\alpha^2}{4\omega^2 + \eta^2} + \eta}} \exp[-2\sqrt{\frac{\alpha^2}{4\omega^2 + \eta^2} + \eta y - \frac{k}{\alpha^2} \eta(\omega + \frac{\gamma}{\alpha^2}) t + 2C}]
\]

from 2.9: \( Q(y; t) = -2 \frac{d}{dy} K(y, y; t) \), then we obtain

\[
Q(y; t) = -2(\frac{\alpha^2}{4\omega^2 + \eta^2} + \eta) \sec h^2[\sqrt{\frac{\alpha^2}{4\omega^2 + \eta^2} + \eta y + \frac{k}{2\alpha^2} \eta(\omega + \frac{\gamma}{\alpha^2}) t + C_1}]
\]

where \( C_1 = \frac{1}{2} \ln 2 \sqrt{\frac{\alpha^2}{4\omega^2 + \eta^2} + \eta} - C \) is arbitrary constant. Then, we will have \( \phi \) from 2.17, so that we can obtain \( q = \frac{1}{\alpha^2} \phi^2 \).

From this, we let

\[
\theta = \sqrt{\frac{\alpha^2}{4\omega^2 + \eta^2} + \eta y + \frac{k}{2\alpha^2} \eta(\omega + \frac{\gamma}{\alpha^2}) t + C_1} \tag{3.23}
\]

thus \( Q = -2(\frac{\alpha^2}{4\omega^2 + \eta^2} + \eta) \sec h^2 \theta \) then 2.17 becomes

\[
\frac{d^3 \phi}{d\theta^3} + 4[2 \sec h^2 \theta - \frac{1}{\eta(\omega^2 + \eta^2)}] \frac{d\phi}{d\theta} - 8 \sec h^2 \theta \tanh \theta \phi = 0 \tag{3.24}
\]

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After only a little investigation, it is clear that a solution of this equation is
\[ \phi = A \sec^2 \theta + B \tanh^2 \theta \]
from 3.24 we may have
\[ B/A = -\eta(4\omega\alpha^2 + 2\gamma)/\alpha^2 \]
then \( \phi \to \alpha^2 \omega + \frac{\gamma}{2\alpha^2} \). As \( |\theta| \to \infty \) that \( q = \frac{1}{\alpha^2} \phi^2 \to \omega + \frac{\gamma}{2\alpha^2} \), therefore we obtain
\[ \phi^2 = \frac{\alpha^4}{4\eta^2(4\omega\alpha^2 + 2\gamma)} [\sec h^2 \theta - \eta(4\omega\alpha^2 + 2\gamma)/\alpha^2 \tanh^2 \theta]^2 \] (3.25)

Next, we must solve 2.21 to find \( u \) this equation is most conveniently written in terms of \( \theta \)
\[ \alpha^2 (4\omega\alpha^2 + 2\gamma) + \eta)q \frac{d^2 u}{d\theta^2} + \frac{1}{2} \frac{dq}{d\theta} \frac{du}{d\theta} - u = \omega + \frac{\gamma}{2\alpha^2} - q \]
where \( q \) is obtained by 3.25.

A solution can be seeked in the form \( u = v(\frac{\theta}{\alpha}) \) and we easily find that there exists a solution for \( v(\frac{\theta}{\alpha}) \) that is proportional to \( \sec h^2 \theta \), so we may obtain
\[ u(\theta) = \frac{-\left(\frac{1}{\theta_0} + \frac{2\omega^2 + \gamma}{\alpha^2}\right) \sec h^2 \theta}{\sec h^2 \theta - \eta(4\omega\alpha^2 + 2\gamma)/\alpha^2 \tanh^2 \theta} \] (3.26)

Finally, the relation between \( x \) and \( y \) is needed in order to complete the parametric representation of the solution. From 2.20: \( \frac{dy}{dx} = \sqrt{q} \), we have \( \int \frac{dy}{\sqrt{q}} = \int dx \). Although it is obviously simple to use \( \theta \) as the parameter here. Thus the above equation with \( \sqrt{q} \) obtained from 3.25 becomes
\[ \int \frac{d\theta}{\sec h^2 \theta - \eta(4\omega\alpha^2 + 2\gamma)/\alpha^2 \tanh^2 \theta} = \frac{\alpha^2}{-2\eta(4\omega\alpha^2 + 2\gamma)} \sqrt{1 + \eta(4\omega\alpha^2 + 2\gamma)/\alpha^2} \int dx \]
, which can be integrated directly to give
\[ x = \frac{2\alpha}{\sqrt{\alpha^2 + \eta(4\omega\alpha^2 + 2\gamma)}} \theta + \ln \frac{\cosh(\theta - \theta_0)}{\cosh(\theta + \theta_0)} + C \] (3.27)
where \( \theta_0 = \arctan h \sqrt{1 + \eta(4\omega\alpha^2 + 2\gamma)/\alpha^2} \) \( C \) is regarded as a function of time.

Then we have the one-soliton solution of the DGH equation, represented by 3.26 and 3.27 with as the parameter.
In figure 1 we present a few examples of the one-soliton solution profile generated by 3.26 and 3.27. For this, we convert 3.26 into
\[ u(\theta) = \frac{-\frac{1}{\eta} \left[ 1 + \frac{\eta(4\omega^2+2\gamma)}{\alpha^2} \right]}{1 + \frac{\eta(4\omega^2+2\gamma)}{\alpha^2} - \frac{\eta(4\omega^2+2\gamma)}{\alpha^2} \cosh^2 \frac{\theta}{\alpha}} \]
and we also transform 3.27 into
\[ x - C = \frac{2}{\sqrt{1 + \frac{\eta(4\omega^2+2\gamma)}{\alpha^2}}} \theta + \ln \frac{\cosh(\theta - \theta_0)}{\cosh(\theta + \theta_0)} \]

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References