The Study of Solution of Dissipative Camassa-Holm Equation on Total Space

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Abstract. The dissipative Camassa-Holm equation is an important fluid equation which has close relation to turbulent phenomenon. Dissipative CH equation can be expressed as ODE through Galerkin method. Using prior estimate we get the existence of global solution for dissipative CH equation on total space about time t. Through norm estimate we investigate properties of solution of dissipative CH equation, then we find that the dissipative CH equation possesses global attractor under condition and the equation also possesses peaked solution. Stationary solution is studied, meanwhile the existence and uniqueness of the stationary solution belonging to absorbing set are obtained.

Keywords: dissipative Camassa-Holm Equation; prior estimate; global solution; peaked solution; stationary solution

1 Introduction and the existence of solution

Camassa-Holm equation was first introduced as an example with bi-Hamiltonian structure by B.Fuchssteniner and A.S.Fokas[1]. But in 1993 [2], it was re-introduced as shallow water equation and some newly special property—peaked and blow-up—in its solution are found. So, Camassa-Holm equation has caught a great deal of interest. In recent years, especially after 1998, a lot of papers about this equation appeared [3]-[14]. The conserved quantities and initial value problem of CH equation are investigated in reference [3]. Symmetries of CH equation are discussed in [4]. The soliton solution of CH equation is investigated with variation method in [5]. The double soliton solution, the definitions of concave, convex peaked soliton and smooth soliton solution of CH equation are discussed in reference [6] J.Vukadinovic studied the backwards behavior of solution periodic viscous Camassa—Holm equation in [7]. Tian and Gui investigate the relation between CH equation and DGH equation [15]. For all, these studies can be divided into three aspects basically. (a) the study of structure and property to this equation, such as, Hamiltonian structure, integrating and conservation and so on; (b) the study of solitons, especially peaked solution. (c) the study of the relation between CH equation and other water wave equations—KdV equation, Euler-α equation etc. C.Foias and his cooperators derived 3-D viscous CH equation to explain approximately turbulent phenomenon and make great achievement. Therefore, we study the existence and property of solution for CH equation with dissipation $\varepsilon (u_{xx} - u_{xxxx})$ on total space R. Foias and his cooperators study the existence and attractor of solution for dissipative CH equation on bounded domain. Nevertheless we study the problem on total space and the structure of attractor in this paper.

Here we are interested in the following equation

$$u_t - u_{xxt} + 3 uu_x - \varepsilon (u_{xx} - u_{xxxx}) = 2 u_x u_{xx} + uu_{xxx} \quad (1.1)$$

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Through using prior estimate and Galerkin method, we get the existence of global solution and the global attractor with any initial value in space of $v$. We also prove the stability of peaked solution and obtain the stationary solution for the dissipative CH equation.

Let $u(x, 0) = u_0(u, v) = \int_R u(x, v) d\tau, |u_2| = \sqrt{u^2 + u^2} = |u|_{L^2(R)}, D = \frac{\partial}{\partial x}$ and $D^k = \frac{\partial^k}{\partial x^k}$. We shall study the problem in Sobolev space $H_0^s(R), (s = 0, 1, 2, 3, 4)$ and the norm is given by $|u|_{H^s(R)} = 1/2$

\[
\left( \sum_{k \leq s} |D^k u|^2 \right). \hspace{1cm} \text{(1)}
\]

Multiplying (1) by $u$ and integrating respect to $x$ over $R$,

\[
\frac{d}{dt} |u|^2 + |D^2 u|^2 + 2 \varepsilon \left( |D^2 u|^2 + |D^3 u|^2 \right) = 0 \hspace{1cm} \text{(1.2)}
\]

Then there exist $K(t)$, such that $|D^2 u|^2 + |D^3 u|^2 \geq K(t)$, and we get

\[
\frac{d}{dt} |u|^2_{H^1} + 2 \varepsilon K(t) \leq 0 \hspace{1cm} \text{(1.3)}
\]

Due to Sobolev imbedding theorem,

\[
|u|_{L^\infty} \leq k_1 |u_0|_{H^1}, \hspace{1cm} t > 0 \hspace{1cm} \text{(1.4)}
\]

Where $k_1$ is a nonnegative number just depending on $R$.

Integrating (1.2) with respect to $t$ over $[t, t + \tau]$, where $t > 0$, is a nonnegative number, we get

\[
\int_t^{t+\tau} \left( |D^2 u(x, s)|^2 + |D^3 u(x, s)|^2 \right) ds \leq \frac{1}{\varepsilon} |u_0|^2_{H^1} \hspace{1cm} \text{(1.5)}
\]

Multiplying (1) by $u_{xxx}$ and integrating it over $R$. Using (refeq4) and Holder inequality and Young inequality, we get

\[
\frac{d}{dt} \left( |Du|^2 + |D^2 u|^2 \right) + \varepsilon \left( |D^2 u|^2 + |D^3 u|^2 \right) \leq \frac{9k_1^2}{\varepsilon} |u_0|^2_{H^1} \left( |Du|^2 + |D^2 u|^2 \right) \hspace{1cm} \text{(1.6)}
\]

By inequality (refeq5) and (refeq6), using uniform Gronwall inequality, we get

\[
|Du(x, t + \tau)|^2 + |D^2 u(x, t + \tau)|^2 \leq \frac{|u_0|^2_{H^1}}{\varepsilon \tau} \exp \left\{ \frac{9k_1^2}{\varepsilon} |u_0|^2_{H^1} \right\} = M_0 (\varepsilon, \tau, |u_0|_{H^1}, R) \hspace{1cm} \text{(1.7)}
\]

By (1.4) and (1.7), we get

\[
|u|^2_{H^2} \leq k_2 |u_0|^2_{H^1} \hspace{1cm} \text{(1.8)}
\]

Where $k_2 = k_2 (\varepsilon, \tau, |u_0|_{H^1}, R)$ is nonnegative.

From (1.8) and due to Sobolev embedding theorem we get

\[
|Du|_{L^\infty} \leq k_3 |u_0|_{H^1} \hspace{1cm} \text{(1.9)}
\]

Where $k_3 = k_3 (\varepsilon, \tau, |u_0|_{H^1}, R)$ is nonnegative.

Integrating (refeq6) with respect to $t$ on $[t, t + \tau], t \geq 0$,

\[
\varepsilon \int_t^{t+\tau} \left( |D^2 u(x, s)|^2 + |D^3 u(x, s)|^2 \right) ds \leq \frac{9k_1^2}{\varepsilon} |u_0|^4_{H^1} + M_0 (\varepsilon, \tau, |u_0|_{H^1}, R) \hspace{1cm} \text{(1.10)}
\]

Multiplying (1) by $u_{xxxx}$ and integrating it over $R$

\[
\frac{1}{2} \frac{d}{dt} \left( |D^2 u|^2 + |D^3 u|^2 \right) + \varepsilon (|D^2 u|^2 + |D^3 u|^2) \leq \frac{15}{2} |Du|_{L^\infty} (|D^2 u| + |D^3 u|^2) \hspace{1cm} \text{(1.11)}
\]

Obviously

\[
\frac{15}{2} \int_t^{t+\tau} |Du(x, s)|_{L^\infty} ds \leq \frac{15}{2} k_3^2 |u_0|_{H^1} \tau \hspace{1cm} \text{(1.12)}
\]
From (1.10), (1.11), (1.12) and using Gronwall inequality, we get
\[
|D^2 u(t + r)|^2 + |D^3 u(t + r)|^2 \leq M_1(\varepsilon, r, |u_0|_{H^1}, R)
\] (1.13)

Obviously, by (1.8) and (1.13), we have
\[
|u|_{H^1}^2 \leq k_4 |u_0|_{H^1}^2
\] (1.14)

Where \( k_4 = k_4(\varepsilon, r, |u_0|_{H^1}, R) \) is nonnegative, then by using Sobolev embedding theorem, we get
\[
|D^2 u|_{L^\infty} \leq k_5 |u_0|_{H^1}
\] (1.15)

Where \( k_5 = k_5(\varepsilon, r, |u_0|_{H^1}, R) \) is nonnegative.

Integrating (1.11) with respect to \( t \) on \([t, t + r]\), where \( t \geq 0 \), we get
\[
\varepsilon \int_t^{t+r} \left( |D^3 u(x, s)|^2 + |D^4 u(x, s)|^2 \right) ds
\leq \frac{15}{2^2} |Du|_{L^\infty} \left( \frac{9k_4^2}{2^2} |u_0|^4_{H^1} + M_0(\varepsilon, r, |u_0|_{H^1}, R) \right)
+ M_1(\varepsilon, r, |u_0|_{H^1}, R) = M_2(\varepsilon, r, |u_0|_{H^1}, R)
\] (1.16)

Multiplying (1) by \( D^5 u \) and integrating it over \( R \), we get
\[
\frac{1}{2} \frac{d}{dt} \left( |D^3 u|^2 + |D^4 u|^2 \right) + \varepsilon \left( |D^4 u|^2 + |D^5 u|^2 \right)
\leq (18 |Du|_{L^\infty} + 28 |D^2 u|_{L^\infty}) \left( |D^3 u|^2 + |D^4 u|^2 \right)
\] (1.17)

Through inequality (1.9) and (1.15), we have
\[
\int_t^{t+r} (18 |Du(x, s)|_{L^\infty} + 28 |D^2 u(x, s)|_{L^\infty}) ds \leq (18k_3 + 28k_5) |u_0|_{H^1} r
\] (1.18)

By (1.17) and using Gronwall inequality,
\[
|D^3 u|^2 + |D^4 u|^2 \leq \frac{1}{\varepsilon r} M_2(\varepsilon, r, |u_0|_{H^1}, R) \exp \{ (18k_3 + 28k_5) |u_0|_{H^1} \} = M_3(\varepsilon, r, |u_0|_{H^1}, R)
\] (1.19)

Furthermore, we get,
\[
|u|_{H^4} \leq k_6 |u_0|_{H^1}
\] (1.20)

Denote \( A = D^2 \). Multiplying (1) by \( A^{-1} u_t \) and integrating it over \( R \), we get
\[
(u, u_t) + (u_x, u_{xxt}) + \varepsilon |Du|^2 + \varepsilon |D^2 u|^2 = 0
\] (1.21)

From (1.21), we have
\[
|u_t|_{L^2} \leq M_4(\varepsilon, r, |u_0|_{H^1}, R)
\] (1.22)

Then through prior estimate, we obtain

**Theorem 1** If \( u_0 \in H^1_0(R) \), there exists a unique global solution (about \( t \)) of equation (1) with initial value \( u_0 \),
\[
u(x, t, u_0) \in C^1((0, \infty) , H^{s-1}(R)) \cap C((0, \infty), H^{s}(R)), s = 0, 1, 2, 3.
\]

**Proof:** Using Galerkin process, from equation (1) with its initial data, we get the ODE,
\[
\begin{align*}
u_{t}^{(m)} - u_{xxt}^{(m)} + 3u_{x}^{(m)}u_{xx}^{(m)} - \varepsilon (u_{xx}^{(m)} - u_{xxx}) &= 2u_{x}^{(m)}u_{xx}^{(m)} + u^{(m)}u_{xxx}^{(m)} \\
u_{0}^{(m)} &= u^{(m)}(x, 0)
\end{align*}
\] (1.23)

According to the ODE basic theory, ODE (1.21) has the unique solution \( u^{(m)} \) in \((0, T_m)\). From (1.4), (1.8), (1.14), (1.20), we get the \( u^{(m)} \) with initial value \( u_0 \in H^1_0(R) (t > 0) \) which is uniformly bounded.
in $H^1, H^2, H^3$ and $H^4$. By inequality (26), we can get $u_t$ is uniformly bounded in $L^2$. In other words, if $u_0 \in H^1_0(R)$, then there exist a subsequence $u^{(m)}$ of solution which converges to $u$ in $H^s(R)$ ($s = 0, 1, 2, 3$). By Lebesgue dominated convergence theorem, it is proved that $u$ is the global solution corresponding to initial value in $H^s(R)$.

Next, we discuss the uniqueness of global solution for equation (1.1)

Let $u, v$ be the globe solutions of equation (1) corresponding to data $u_0, v_0 \in H^1(R)$, so

$$\omega_t - \omega_{xxx} + 3u\omega_x + 3u_x\omega - 3\omega\omega_x - \varepsilon (\omega_{xx} - \omega_{xxxx}) = 2u_x\omega_{xx} + 2u_{xx}\omega_x + 2u_x\omega_{xx} + u\omega_{xxx} + u_{xxx}\omega - \omega_{xxxx}$$ \hspace{1cm} (1.24)

Multiplying (1.24) by $\omega$ and integrating it on $R$, using Holder inequality and Young inequality, we have

$$\frac{d}{dt} \left( |\omega|^2 + |D\omega|^2 \right) \leq 2k_6 \left( |\omega|^2 + |D\omega|^2 \right)$$ \hspace{1cm} (1.25)

Furthermore, we obtain $|\omega|^2 + |D\omega|^2 \leq \left( |\omega|^2 + |D\omega|^2 \right) \exp \{2k_6t\} = 0$. So, $|\omega|^2 = 0, |D\omega|^2 = 0$, we get $\omega \equiv 0$. Then, the theorem 1 is proved.

2 Global attractor

According to theorem 1, there exists an unique global solution of equation (1). We can define a semigroup of solution $S(t) : u_0 \in H^1_0 (R) \rightarrow u (\cdot, u_0) \in H^1_0 (R)$, so

Theorem 2 If $u_0 \in H^1_0 (R)$, then there exists an attractor about the solution of equation (1) in $H^1_0 (R)$.

Proof: By the prior estimate in section 1, there exists an absorbing set of semigroup $S(t)$ in $H^2_0 (R)$. By the theory of attractor about evolution equation, we only need to prove that $S(t)$ is a compact mapping.

Multiplying (1) by $t^2D^4u$ and integrating it over $R$, we have

$$\frac{1}{2} \frac{d}{dt} \left( |tD^2u|^2 + |tD^3u|^2 \right) + \varepsilon \left( |tD^3u|^2 + |tD^4u|^2 \right) \leq \left( \frac{15}{2} |Du|_\infty + \frac{1}{4} \right) \left( |tD^2u|^2 + |tD^3u|^2 \right) \left( |D^3u|^2 + |D^2u|^2 \right)$$ \hspace{1cm} (2.26)

Using uniform Gronwall inequality, we get

$$|D^3u| \leq \frac{E_1(\varepsilon, r, |u_0|_{H^1}), t}{t}$$ \hspace{1cm} (2.27)

So, $S(t)$ is equicontinuous. Due to the Ascoli-Arzela theorem, $S(t)$ is a completely continuous function. Then theorem 2 is proved.

3 Peaked Solution and Stationary Solution

Assume there exist traveling wave solution $\varphi (x - ct)$ of equation (1), where $c$ is wave speed, let $\zeta = x - ct$, the equation (1) becomes

$$-c\varphi + c\varphi'' + \frac{3}{2} \varphi^2 - \varepsilon (\varphi' - \varphi'') = \frac{1}{2} \varphi'^2 + \varphi\varphi''$$ \hspace{1cm} (3.28)

Obviously, $\varphi (\zeta) = ce^{-|\zeta|}$ is the solution of equation (3.33), and this solution is the peaked solution to CH equation. Next, we study the stationary solution of equation.

$$3uu_x - \varepsilon (u_{xx} - u_{xxxx}) = 2u_xu_{xx} + uu_{xxx}$$ \hspace{1cm} (3.29)

Integrating (3.29) over $(-\infty, x]$ for $\forall x \in R$, we get

$$\frac{3}{2} u^2 - \varepsilon (u_x - u_{xxx}) = uu_x + \frac{1}{2} u_x^2$$ \hspace{1cm} (3.30)
The differential equation (3.30) is equivalent to the equations as follows

\[
\begin{align*}
\frac{dx}{dt} &= y (u_x) \\
\frac{dy}{dt} &= z (u_{xx}) \\
\frac{dz}{dt} &= y - \frac{1}{2} y^2 + uz - \frac{3}{2} u^2
\end{align*}
\]  

(3.31)

According to the ODE basic theory, we easily prove that there exists a solution of equations (3.31), i.e. the solution of equation (3.30). Assume \( u, v \) be the solutions of equation (3.29), let \( w = u - v \), from (3.29), we have

\[
3wu_x + 3w_x u - 3w w_x - \varepsilon (w_{xx} - w_{xxxx}) = 2w_x u_{xx} + 2w_x w_{xx} - 2w_x w_x + w_{xxxx} + w_{xxx} - w_{xxxx} 
\]  

(3.32)

Multiplying (3.32) by \( w \) and integrating over \( R \)

\[
\int_R (3u_x - u_{xxx}) w^2 dx + \int_R (2\varepsilon + u_x) w_x^2 dx + \varepsilon \int_R w_{xx}^2 dx = 0 
\]  

(3.33)

Then we can obtain

**Theorem 3** If the stationary solution \( u(x) \) of dissipative Camassa-Holm Equation satisfy:

(i) \( 3u - u_{xx} \) is increasing; (ii) \( \varepsilon \geq -2\varepsilon \), then the stationary solution is unique.

**proof:** Under the condition of Theorem 3, through (3.33), we get \( w = 0, w_x = 0, w_{xx} = 0 \), so, the uniqueness is hold.

Through the above discussion, we know that stationary solution of dissipative Camassa-Holm Equation is unique. Naturally we can consider the relation between the global solution and stationary solution.

Assume \( u \) is a stationary solution satisfying conditions of Theorem 3, \( v \) is a global solution of equation (1), let \( w = u - v \), via (1.1) and (3.29), we get

\[
w_x - w_{xx} + 3wu_x + 3w_x u - 3ww_x - \varepsilon (w_{xx} - w_{xxxx}) = 2w_x u_{xx} + 2w_x w_{xx} - 2w_x w_x + w_{xxxx} + w_{xxx} - w_{xxxx} 
\]  

(3.34)

Multiplying (3.34) by \( w \) and integrating over \( R \)

\[
\frac{d}{dt} \left( |w|^2 + |Dw|^2 \right) + \int_R (3u_x - u_{xxx}) w^2 dx + \int_R (2\varepsilon + u_x) w_x^2 dx + \varepsilon \int_R w_{xx}^2 dx = 0 
\]  

(3.35)

so

\[
\frac{d}{dt} \left( |w|^2 + |Dw|^2 \right) \leq 0 
\]  

(3.36)

It is will divide into two cases:

(i) If \( \frac{d}{dt} \left( |w|^2 + |Dw|^2 \right) = 0 \), according to (3.35) we get

\[
\int_R (3u_x - u_{xxx}) w^2 dx + \int_R (2\varepsilon + u_x) w_x^2 dx + \varepsilon \int_R w_{xx}^2 dx = 0 
\]

So, \( w = 0, w_x = 0, w_{xx} = 0 \).

(ii) If \( \frac{d}{dt} \left( |w|^2 + |Dw|^2 \right) < 0 \), i.e.

\[
\int_R (3u_x - u_{xxx}) w^2 dx + \int_R (2\varepsilon + u_x) w_x^2 dx + \varepsilon \int_R w_{xx}^2 dx > 0 
\]

Assume \( K = \int_R (3u_x - u_{xxx}) w^2 dx + \int_R (2\varepsilon + u_x) w_x^2 dx + \varepsilon \int_R w_{xx}^2 dx \),

Obviously we only need to discuss the condition of \( K > 0 \), so we assume that \( K \) is connect with \( t \) (If \( K \to 0 \) with \( t \to 0 \), then after the enough large \( t \), we can get the same conclusion as follows through the above discussion. From (3.35) we get

\[
|w|^2_{H^2} \leq |w_0|^2_{H^1} e^{-Kt}, t > 0 
\]

(3.37)

By discussing, we get

**Proposition:** Under the conditions of Theorem 3, the stationary solution of equation (1) belongs to its absorbing set.
References


