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Comparison Results for AOR Iterative Method with a New Preconditioner

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Abstract: Linear system with M-matrices often appear in a wide variety of areas. In this work, we propose a new preconditioner for solving the system with nonsingular M-matrix. We show that our preconditioner increases the convergence rate of basic iterative methods. We also give a comparison between preconditioners with different parameters. Numerical results are also given.

Keywords: preconditioner; convergence; AOR iterative method
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1. Introduction

Consider the linear system

$$Ax = b, \quad A = (a_{ij}) \in \mathcal{R}^{n\times n}, \quad b \in \mathcal{R}^n \tag{1.1}$$

where A is a nonsingular matrix.

This system can arise from many practical problems, when we discrete corresponding differential equations[1, 2, 11, 13, 3].

For simplicity, in the following discussion, we assume that A has unit diagonal entries and consider usual splitting

$$A = I - L - U,$$

where $-L$ and $-U$ are strictly lower and strictly upper triangular parts of A, respectively.

The standard AOR(accelerated overrelaxation) iterative methods [6] is defined as

$$x^{(i+1)} = \mathcal{L}_{\gamma,\omega}x^{(i)} + (I-\gamma L)^{-1}\omega b, \quad i = 0, 1, 2, \cdots \tag{1.2}$$

with iteration matrix

$$\mathcal{L}_{\gamma,\omega} = (I-\gamma L)^{-1}[(1-\omega)I + (\omega-\gamma)L + \omega U], \tag{1.3}$$

where ω and γ are real parameters with $\omega \neq 0$.

It is well known that for certain values of the parameters ω and γ, we can obtain successive overrelaxation (SOR), Gauss-Seidel, JOR and Jacobi methods. To improve the convergence rate of the basic iterative method , several preconditioned iterative methods have been proposed [4, 5, 7, 8, 9, 10, 14, 15]. The main

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idea of these preconditioned iterative methods is to transform the original system into the preconditioned form

$$PAx = Pb, \tag{1.4}$$

where $P \in R^{n\times n}$ is nonsingular. We call the basic iterative methods corresponding to the preconditioned system the preconditioned iterative methods, such as the preconditioned Jacobi method, the preconditioned Gauss-Seidel method, etc.

In this paper, we introduce the preconditioner as follows

$$P(\alpha) = I + S(\alpha) = \begin{pmatrix} 1 & -\alpha_{1,2}a_{1,2} & \cdots & -\alpha_{1,n-1}a_{1,n-1} & -\alpha_{1,n}a_{1,n} \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & -\alpha_{n,2}a_{n,2} & \cdots & -\alpha_{n,n-1}a_{n,n-1} & 1 \end{pmatrix}$$

and consider the convergence of preconditioned AOR iterative method. Here, we denote

$$\alpha = [\alpha_{1,2}, \cdots, \alpha_{1,n}, \alpha_{n,2}, \cdots, \alpha_{n,n-1}] \in \mathcal{R}^{2n-3},$$

where $\alpha_{i,j} = 0$ if $a_{i,j} = 0, i = 1, n, j = 2, \cdots, n-1$.

For convenience, some notations, definitions and some results that will be used in the following parts are given below. A matrix A is called nonnegative, semi-positive and positive if each entry of A is nonnegative, nonnegative but at least a positive entry and positive, respectively. We denote them by $A \geq 0$, $A > 0$ and $A \gg 0$. Similarly, for n-dimensional vectors x, by identifying them with $n \times 1$ matrices, we can also define $x \geq 0$, $x > 0$ and $x \gg 0$. Additionally, we denote the spectral radius of A by $\rho(A)$.

Definition 1.1 *A matrix $A = (a_{ij})$ is called a Z-matrix if for any $i \neq j$, $a_{ij} \leq 0$; a nonsingular M-matrix if $A = sI - B$, $B \geq 0$ and $s > \rho(B)$.*

Definition 1.2 *Let A be a real matrix. The splitting $A = M - N$ is called*

(a) nonnegative if $M^{-1}N \geq 0$.

(b) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

(c) regular if $M^{-1} \geq 0$ and $N \geq 0$.

(d) M-splitting if M is a nonsingular M-matrix and $N \geq 0$.

Theorem 1.3 *[5] Let A be a nonnegative matrix.*

(a) If $\alpha x \leq Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \leq \rho(A)$.

(b) If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if

$$0 \neq \alpha x \leq Ax \leq \beta x$$

for some nonnegative vector x,then $\alpha \leq \rho(A) \leq \beta$, and x is a positive vector.

Lemma 1.4 *[10] Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is a nonsingular M-matrix.*

Theorem 1.5 *[12] Let A be a nonnegative $n \times n$ matrix,then*

(1) A has a nonnegative real eigenvalue equal to its spectral radius $\rho(A)$.

(2) To $\rho(A)$, there corresponds a nonzero eigenvector $x \geq 0$;

(3) $\rho(A)$ does not decrease when any entry of A is increased.

Theorem 1.6 *[12] If $A \geq 0$ is an $n \times n$ matrix, then the following are equivalent:*

(1) $\alpha > \rho(A)$

(2) $\alpha I - A$ is nonsingular, and $(\alpha I - A)^{-1} \geq 0$.

Theorem 1.7 *[1] Let A be a Z-matrix. Then the following statements are equivalent:*

(1) A is a nonsingular M-matrix;

(2) There is a positive vector x such that $Ax \gg 0$;

(3) There exists an inverse -positive matrix B and a nonsingular M-matrix C such that $A = BC$.

Lemma 1.8 *[7] Let $A_1, A_2 \in \mathcal{R}^{n\times n}$, and that $A_i = M_i - N_i, i = 1, 2$, are nonnegative splitting, that is $T_i = {M_i}^{-1}N_i \geq 0, i = 1, 2$. If the Perron eigenvector $z_2(\geq 0)$ of T_2 satisfies $T_1z_2 \leq T_2z_2$ then $\rho(T_1) \leq \rho(T_2)$.*

This paper is organized as follows. In Section 2 we will discuss the convergence and monotone of the preconditioned AOR iterative methods. Numerical results are then presented in Section 3.

2. Convergence and monotone of AOR iterative methods with the preconditioner $(I + S(\alpha))$

Applying $P(\alpha)$ on (1.1) we obtain the equivalent linear system

$$\tilde{A}(\alpha)x = \tilde{b}(\alpha) \tag{2.1}$$

with $\tilde{A}(\alpha) = (I + S(\alpha))A$ and $\tilde{b}(\alpha) = (I + S(\alpha))b$, where, if needed, we will write

$$\tilde{A}(\alpha) = \tilde{D}(\alpha) - \tilde{L}(\alpha) - \tilde{U}(\alpha), \tag{2.2}$$

with $\tilde{D}(\alpha)$ is diagonal and $\tilde{L}(\alpha)$ and $\tilde{U}(\alpha)$ strictly lower and strictly upper matrices.

The elements $\tilde{a}_{ij}(\alpha)$ of $\tilde{A}(\alpha)$ are given by the expressions:

$$\tilde{a}_{ij}(\alpha) = \begin{cases} a_{i,j} & 2 \leq i \leq n-1, \\ a_{1,j} - \sum\limits_{k=2}^{n} \alpha_{1,k}a_{1,k}a_{k,j} & i = 1, \\ a_{n,j} - \sum\limits_{k=2}^{n-1} \alpha_{n,k}a_{n,k}a_{k,j} & i = n. \end{cases} \tag{2.3}$$

In the following discussion, A is a Z-matrix. To preserve the nonpositivity of all the off-diagonal elements and the Z-matrix character of $\tilde{A}(\alpha)$, we assume that $[0, \cdots, 0] \leq \alpha \leq [1, \cdots, 1]$.

Then, if we define

$$\begin{aligned} D_\alpha &= diag(\sum_{k=2}^{n} \alpha_{1,k}a_{1,k}a_{k,1}, 0, \cdots, 0, \sum_{k=2}^{n-1} \alpha_{n,k}a_{n,k}a_{k,n}), \\ S(\alpha) &= S_1(\alpha) + S_2(\alpha), \\ S_1(\alpha)U &= E_1(\alpha) + F_1(\alpha) + G_1(\alpha), \\ S_2(\alpha)L &= E_2(\alpha) + F_2(\alpha) + G_2(\alpha), \end{aligned} \tag{2.4}$$

where $E_1(\alpha)$ and $E_2(\alpha)$ are diagonal matrices, $F_1(\alpha)$, $F_2(\alpha)$ and $S_1(\alpha)$ are strictly lower triangular, while $G_1(\alpha)$, $G_2(\alpha)$ and $S_2(\alpha)$ are strictly upper triangular. the three matrices on the right hand side of (2.2) are given by

$$\begin{aligned}\tilde{D}(\alpha) &= I - D_\alpha = I - E_1(\alpha) - E_2(\alpha),\\ \tilde{L}(\alpha) &= L - S_1(\alpha) + S_1(\alpha)L + F_1(\alpha) + F_2(\alpha), \qquad (2.5)\\ \tilde{U}(\alpha) &= U - S_2(\alpha) + S_2(\alpha)U + G_1(\alpha) + G_2(\alpha).\end{aligned}$$

The elements of $\tilde{L}(\alpha)$ and $\tilde{U}(\alpha)$ are non-negative.

In this section, we will show that AOR type iterative method for preconditioned linear system (2.1) is asymptotically faster than that for the original system (1.1).

The AOR method of (2.1) is defined as

$$x^{(i+1)} = \tilde{\mathcal{L}}_{\gamma,\omega}(\alpha)x^{(i)} + (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))^{-1}\omega b, \quad i = 0, 1, 2, \cdots \tag{2.6}$$

where

$$\tilde{\mathcal{L}}_{\gamma,\omega}(\alpha) = (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha)^{-1})[(1-\omega)\tilde{D}(\alpha) + (\omega-\gamma)\tilde{L}(\alpha) + \omega\tilde{U}(\alpha)]. \tag{2.7}$$

We first give a Lemma which is needed in the sequel.

Lemma 2.1 *Let* $A = (a_{i,j}) \in R^{n\times n}$ *be a nonsingular* M*-matrix and* $0 \le \gamma \le \omega \le 1$, $\omega \ne 0$. *Then, for* $0 \le \alpha_{i,j} \le 1$, $(I + S(\alpha))A$ *is also a nonsingular* M*-matrix.*

Proof. Since $A = (a_{i,j}) \in R^{n\times n}$ is a nonsingular M-matrix, by Lemma 1.7, there exists $x \gg 0$ such that $Ax \gg 0$. Then $(I + S(\alpha))Ax \gg 0$. From (2.3), it is easy to see that $(I + S(\alpha))A$ is a Z-matrix. The result is directly obtained from Theorem 1.7. □

Our main result in this section is as follows.

Theorem 2.2 *Let* $A = (a_{ij}) \in \mathcal{R}^{n\times n}$ *be a nonsingular* Z*-matrix and* $0 \le \gamma \le \omega \le 1$, $\omega \ne 0$.

(a) If $\rho(\mathcal{L}_{\gamma,\omega}) < 1$, *then for any* $[0, \cdots, 0] \le \alpha \le [1, \cdots, 1] \in \mathcal{R}^{2n-3}$, *we have*

$$\rho(\tilde{\mathcal{L}}_{\gamma,\omega}) \le \rho(\mathcal{L}_{\gamma,\omega}) < 1.$$

(b) If $\rho(\mathcal{L}_{\gamma,\omega}) > 1$, $\sum_{k=2}^{n} a_{1,k}a_{k,1} < 1$ *and* $\sum_{k=2}^{n-1} a_{n,k}a_{k,n} < 1$, *then for any* $[0, \cdots, 0] \le \alpha \le [1, \cdots, 1] \in \mathcal{R}^{2n-3}$, *we have*

$$\rho(\tilde{\mathcal{L}}_{\gamma,\omega}) \ge \rho(\mathcal{L}_{\gamma,\omega}) > 1.$$

Proof. For the needs of one of our main statements, we consider splittings:

$$A = M - N = \frac{1}{\omega}(I - \gamma L) - \frac{1}{\omega}[(1-\omega)I + (\omega-\gamma)L + \omega U], \tag{2.8}$$

and

$$\tilde{A}(\alpha) = E(\alpha) - F(\alpha) = \frac{1}{\omega}[\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha)] - \frac{1}{\omega}[(1-\omega)\tilde{D}(\alpha) + (\omega-\gamma)\tilde{L}(\alpha) + \omega\tilde{U}(\alpha)]. \tag{2.9}$$

Since A is a nonsingular Z-matrix, $0 \le \gamma \le \omega \le 1$ and $\omega \ne 0$, then L, U and N are nonnegative matrices. Thus the splitting (2.8) is an M-splitting.

For (a), if $\rho(\mathcal{L}_{\gamma,\omega}) < 1$. By Lemma 1.4, A is a nonsingular M-matrix. By Lemma 2.1, $\tilde{A}(\alpha)$ is also a nonsingular M-matrix. So $\tilde{D}(\alpha) \ge 0$, $\tilde{L}(\alpha) \ge 0$ and $\tilde{U}(\alpha) \ge 0$.

By virtue of (2.3), we know that $\tilde{D}(\alpha) = diag(1 - \sum_{k=2}^{n} \alpha_{1,k}a_{1,k}a_{k,1}, 1, \cdots, 1, 1 - \sum_{k=2}^{n-1} \alpha_{n,k}a_{n,k}a_{k,n})$. Thus, $1 - \sum_{k=2}^{n} \alpha_{1,k}a_{1,k}a_{k,1} > 0$ and $1 - \sum_{k=2}^{n-1} \alpha_{n,k}a_{n,k}a_{k,n} > 0$ hold for any $[0, \cdots, 0] \le \alpha \le [1, \cdots, 1]$. This implies that $\tilde{D}(\alpha)$ is an invertible positive matrix. On the other hand, since $\gamma \ge 0$ and $\tilde{L}(\alpha)$ is nonnegative, it is obvious that the diagonal elements of $E(\alpha)$ are positive and the off-diagonal elements of $E(\alpha)$ are non-positive, i.e., $E(\alpha)$ is an L-matrix. Since $\gamma D^{-1}(\alpha)L(\alpha)$ is a strictly lower triangular matrix, $\rho(\gamma D^{-1}(\alpha)L(\alpha)) = 0 < 1$ and $\gamma D^{-1}(\alpha)L(\alpha) \ge 0$. By Theorem 1.6, we have $(I - \gamma D^{-1}(\alpha)L(\alpha))^{-1} \ge 0$. Then

$$E^{-1}(\alpha) = (I - \gamma D^{-1}(\alpha)L(\alpha))^{-1}D^{-1}(\alpha) \ge 0,$$

which means $E(\alpha)$ is an nonsingular M-matrix.

From (2.3), it is easy to see $\tilde{U}(\alpha) \ge 0$ for $[0, \cdots, 0] \le \alpha \le [1, \cdots, 1]$ and therefore $F(\alpha) \ge 0$. This implies $\tilde{A}(\alpha) = E(\alpha) - F(\alpha)$ is also an M-splitting.

For (b), since $[0, \cdots, 0] \le \alpha \le [1, \cdots, 1]$, $\sum_{k=2}^{n} a_{1,k}a_{k,1} < 1$ and $\sum_{k=2}^{n-1} a_{n,k}a_{k,n} < 1$, from above analysis, we can conclude that $A(\alpha) = E(\alpha) - F(\alpha)$ is an M-splitting.

Now, according to Theorem 1.5, there exists a Perron vector $x \ge 0$ such that

$$\mathcal{L}_{\gamma,\omega}x = \rho(\mathcal{L}_{\gamma,\omega})x,$$

where we denote $\rho(\mathcal{L}_{\gamma,\omega})$ by λ. From the expression of $\mathcal{L}_{\gamma,\omega}$, we obtain the following equality

$$[(1-\omega)I + (\omega - \gamma)L + \omega U]x = \lambda(I - \gamma L)x,$$

which is equivalent to

$$[(1 - \omega - \lambda)I + (\omega - \gamma + \lambda\gamma)L + \omega U]x = 0. \tag{2.10}$$

Premultiplying (2.10) with $S(\alpha)$ and $S_2(\alpha)$, we obtain

$$[(1 - \omega - \lambda)S(\alpha) + (\omega - \gamma + \lambda\gamma)S(\alpha)L + \omega S(\alpha)U]x = 0 \tag{2.11}$$

and

$$[(1 - \omega - \lambda)S_2(\alpha) + (\omega - \gamma + \lambda\gamma)S_2(\alpha)L + \omega S_2(\alpha)U]x = 0. \tag{2.12}$$

From (2.11) and (2.12), we can obtain

$$\omega[-S(\alpha) + S(\alpha)L + S(\alpha)U]x = [(\lambda - 1)S(\alpha) + \gamma(1 - \lambda)S(\alpha)L]x \tag{2.13}$$

and

$$S_2(\alpha)Lx = \frac{[(-1 + \omega + \lambda)S_2(\alpha) - \omega S_2(\alpha)U]x}{\omega - \gamma + \lambda\gamma}. \tag{2.14}$$

Then, we have

$$\begin{aligned}
& \tilde{\mathcal{L}}_{\gamma,\omega}x - \lambda x \\
= \;& (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))^{-1}[(1-\omega)\tilde{D}(\alpha) + (\omega-\gamma)\tilde{L}(\alpha) + \omega\tilde{U}(\alpha) - \lambda(\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))]x \\
= \;& (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))^{-1}[(1-\omega-\lambda)\tilde{D}(\alpha) + (\omega-\gamma+\lambda\gamma)\tilde{L}(\alpha) + \omega\tilde{U}(\alpha)]x \\
= \;& (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))^{-1}[(1-\omega-\lambda)(I - E_1(\alpha) - E_2(\alpha)) + (\omega-\gamma+\lambda\gamma)(L - S_1(\alpha) + S_1(\alpha)L \\
& + F_1(\alpha) + F_2(\alpha)) + \omega(U - S_2(\alpha) + S_2(\alpha)U + G_1(\alpha) + G_2(\alpha))]x \\
= \;& (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))^{-1}\{[(1-\omega-\lambda)I + (\omega-\gamma+\lambda\gamma)L + \omega U] + \\
& [-(1-\omega-\lambda)(E_1(\alpha) + E_2(\alpha)) + (\omega-\gamma+\lambda\gamma)(-S_1(\alpha) + S_1(\alpha)L \\
& + F_1(\alpha) + F_2(\alpha)) + \omega(-S_2(\alpha) + S_2(\alpha)U + G_1(\alpha) + G_2(\alpha))]\}x \\
= \;& (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))^{-1}[-(1-\omega-\lambda)(E_1(\alpha) + E_2(\alpha)) + (\omega-\gamma+\lambda\gamma)(-S_1(\alpha) + S_1(\alpha)L + \\
& F_1(\alpha) + F_2(\alpha)) + \omega(-S_2(\alpha) + S_2(\alpha)U + G_1(\alpha) + G_2(\alpha))]\}x \\
= \;& (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))^{-1}[(\lambda-1)(E_1(\alpha) + E_2(\alpha)) + \gamma(\lambda-1)(-S_1(\alpha) + S_1(\alpha)L + F_1(\alpha) + F_2(\alpha)) \\
& + \omega(S(\alpha)L + S(\alpha)U - S(\alpha))]x \\
= \;& (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))^{-1}[(\lambda-1)(E_1(\alpha) + E_2(\alpha)) + \gamma(\lambda-1)(-S_1(\alpha) + S_1(\alpha)L + F_1(\alpha) + F_2(\alpha)) \\
& + (\lambda-1)S(\alpha) + \gamma(1-\lambda)S(\alpha)L]x \\
= \;& (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))^{-1}[(\lambda-1)(E_1(\alpha) + E_2(\alpha)) + \gamma(\lambda-1)(F_1(\alpha) + F_2(\alpha)) \\
& + (1-\gamma)(\lambda-1)S_1(\alpha) + (\lambda-1)S_2(\alpha) + \gamma(1-\lambda)S_2(\alpha)L]x \\
= \;& (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))^{-1}[(\lambda-1)(E_1(\alpha) + E_2(\alpha)) + \gamma(\lambda-1)(F_1(\alpha) + F_2(\alpha)) \\
& + (1-\gamma)(\lambda-1)S_1(\alpha) + (\lambda-1)S_2(\alpha) + \gamma(1-\lambda)\frac{[(-1+\omega+\lambda)S_2(\alpha) - \omega S_2(\alpha)U]}{\omega-\gamma+\lambda\gamma}]x \\
= \;& (\tilde{D}(\alpha) - \gamma\tilde{L}(\alpha))^{-1}[(\lambda-1)(E_1(\alpha) + E_2(\alpha)) + \gamma(\lambda-1)(F_1(\alpha) + F_2(\alpha)) \\
& + (1-\gamma)(\lambda-1)S_1(\alpha) + \frac{\omega(\lambda-1)[(1-\gamma)S_2(\alpha) + \gamma S_2(\alpha)U]}{\omega-\gamma+\lambda\gamma}]x
\end{aligned}$$

If $\lambda < 1$, then $\tilde{\mathcal{L}}_{\gamma,\omega}x - \lambda x \leq 0$, i.e., $\tilde{\mathcal{L}}_{\gamma,\omega}x \leq \lambda x$. By Lemma 1.8, we have

$$\rho(\tilde{\mathcal{L}}_{\gamma,\omega}) \leq \rho(\mathcal{L}_{\gamma,\omega}) < 1.$$

Similarly, if $\lambda > 1$, then

$$\tilde{\mathcal{L}}_{\gamma,\omega}x - \lambda x \geq 0,$$

i.e., $\tilde{\mathcal{L}}_{\gamma,\omega}x \geq \lambda x$. By Theorem 1.3, we have

$$\rho(\tilde{\mathcal{L}}_{\gamma,\omega}) \geq \rho(\mathcal{L}_{\gamma,\omega}) > 1.$$

The proof is completed. □

If we choose $\alpha_{n,j} = 0, j = 2, \cdots, n-1$ or $\alpha_{1,j} = 0, j = 2, \cdots, n$, we obtain preconditioner $I + S_1(\alpha)$ or $I + S_2(\alpha)$, where $S_1(\alpha)$ and $S_2(\alpha)$ are strict lower triangular part and strict upper triangular part of $S(\alpha)$, respectively.

Corollary 2.3 *Let* $A = (a_{ij}) \in R^{n\times n}$ *be a nonsingular Z-matrix and* $0 \le \gamma \le \omega \le 1$, $\omega \ne 0$. *If* $\rho(\mathcal{L}_{\gamma,\omega}) < 1$. *then for either preconditioner* $I + S_1(\alpha)$ *or preconditioner* $I + S_2(\alpha)$, *we have*

$$\rho(\tilde{\mathcal{L}}_{\gamma,\omega}) \le \rho(\mathcal{L}_{\gamma,\omega}) < 1.$$

Remark 2.4 *By choose special parameters, similar results about SOR, JOR, Gauss-Seidel and Jacobi method can be copied words by words from above Theorem. For simplicity, we omit them here.*

We can prove that the spectral radii of the $\tilde{\mathcal{L}}_{\gamma,\omega}(\alpha)$ are nonincreasing function of $\alpha_{i,j} \in [0,1]$. First, we give our results for preconditioner $I + S_1(\alpha)$ and $I + S_2(\alpha)$.

Theorem 2.5 *Under the assumptions that* A *is a nonsingular* M*-matrix,* $0 \le \gamma \le \omega < 1$ *and* $\omega \ne 0$, *let*

$$\alpha = [\alpha_{1,2}, \alpha_{1,3}, \cdots, \alpha_{1,n}] \in \mathcal{R}^{n-1} \quad \textit{and} \quad \tilde{\alpha} = [\alpha'_{1,2}, \alpha'_{1,3}, \cdots, \alpha'_{1,n}] \in \mathcal{R}^{n-1}.$$

where both $\alpha_{1,j}$ *and* $\alpha'_{1,j}$ *are zeros if* $a_{1,j} = 0$ *and* $[0, \cdots, 0] \le \alpha \le \tilde{\alpha} \le [1, \cdots, 1]$.

Then, for preconditioner $I + S_1(\alpha)$, *we have*

$$\rho(\tilde{\mathcal{L}}_{\gamma,\omega}(\tilde{\alpha})) \le \rho(\tilde{\mathcal{L}}_{\gamma,\omega}(\alpha)) \le \rho(\mathcal{L}_{\gamma,\omega}) < 1. \tag{2.15}$$

Proof. Note that the AOR iteration matrices associated with any $A = D - L - U$ (D invertible diagonal, L and U strictly lower and upper triangular), are the same with those associated with $D^{-1}A = I - D^{-1}L - D^{-1}U$. Next, by virtue of Theorem 2.2 and Corollary 2.3, we know that $\rho(\tilde{\mathcal{L}}_{\gamma,\omega}(\alpha)) \le \rho(\mathcal{L}_{\gamma,\omega}) < 1$, which implies the AOR iterative method associated a preconditioned $\tilde{A}$ is no worse than the corresponding one of the unpreconditioned matrix A. Since $\tilde{D}^{-1}\tilde{A}$ has the same iteration matrices with $\tilde{A}$, its elements, denoted by the same symbols as those of $\tilde{A}$, are

$$\begin{aligned} &\tilde{a}_{i,i} = 1, \quad i = 1, \cdots, n, \quad \tilde{a}_{i,j} = a_{i,j}, \quad i \ne 1, \quad j \ne i, \\ &\tilde{a}_{1,j} = \frac{a_{1,j} - \sum\limits_{k=2}^{n} \alpha_{1,k} a_{1,k} a_{k,j}}{1 - \sum\limits_{k=2}^{n} \alpha_{1,k} a_{1,k} a_{k,1}}, \quad j = 2, \cdots, n. \end{aligned} \tag{2.16}$$

Consider the vector $\beta \in \mathcal{R}^{n-1}$ whose components are defined by

$$\begin{cases} \beta_{1,j} = 0, & \text{if} \quad \sum\limits_{k=2}^{n} \alpha_{1,k} a_{1,k} a_{k,j} - a_{1,j} = 0, \quad j = 2, \cdots, n \\ \beta_{1,j} = \dfrac{(\alpha_{1,j} - \alpha'_{1,j}) a_{1,j}}{\sum\limits_{k=2}^{n} \alpha_{1,k} a_{1,k} a_{k,j} - a_{1,j}} & \text{if} \quad \sum\limits_{k=2}^{n} \alpha_{1,k} a_{1,k} a_{k,j} - a_{1,j} \ne 0, \quad j = 2, \cdots, n. \end{cases} \tag{2.17}$$

Apply $\tilde{D}^{-1}\tilde{A}$ the preconditioner $I + S_1(\beta)$, The AOR iterative methods associated with the new preconditioned matrix $\tilde{\tilde{A}}(\beta) = (I + S_1(\beta))\tilde{D}^{-1}\tilde{A}$ will be no worse than the ones corresponding to $\tilde{D}^{-1}\tilde{A}$. The elements $\tilde{\tilde{a}}_{i,j}$ of the matrix $\tilde{\tilde{D}}^{-1}\tilde{\tilde{A}}(\beta)$ will be given by the same expression as those in (2.16) where the $a_{i,j}$'s will be replaced by $\tilde{a}_{i,j}$'s and the $\alpha_{i,j}$'s by $\beta_{i,j}$'s. The $\tilde{\tilde{a}}_{i,j}$'s are given by

$$\begin{aligned} &\tilde{\tilde{a}}_{i,i} = 1, \quad i = 1, \cdots, n, \quad \tilde{\tilde{a}}_{i,j} = \tilde{a}_{i,j}, \quad i \ne 1, \quad j \ne i, \\ &\tilde{\tilde{a}}_{1,j} = \frac{\tilde{a}_{1,j} - \sum\limits_{k=2}^{n} \beta_{1,k} \tilde{a}_{1,k} \tilde{a}_{k,j}}{1 - \sum\limits_{k=2}^{n} \beta_{1,k} \tilde{a}_{1,k} \tilde{a}_{k,1}}, \quad j = 2, \cdots, n. \end{aligned} \tag{2.18}$$

Substituting in (2.18) the $\tilde{a}_{i,j}$'s and the $\beta_{i,j}$'s, after some algebra, we end up with:

$$\begin{aligned}&\tilde{\tilde{a}}_{i,i}=1,\quad i=1,\cdots,n,\quad \tilde{\tilde{a}}_{i,j}=a_{i,j},\quad i\neq 1,\quad j\neq i,\\&\tilde{\tilde{a}}_{1,j}=\frac{a_{1,j}-\sum\limits_{k=2}^{n}\alpha^{'}_{1,k}a_{1,k}a_{k,j}}{1-\sum\limits_{k=2}^{n}\alpha^{'}_{1,k}a_{1,k}a_{k,1}},\quad j=2,\cdots,n.\end{aligned}\tag{2.19}$$

In fact, (2.17) is equivalent to

$$(\alpha_{1,j}+\beta_{1,j}-\alpha^{'}_{1,j})a_{1,j}=\beta_{1,j}\sum_{k=2}^{n}\alpha_{1,k}a_{1,k}a_{k,j},\quad j=2,\cdots,n,$$

which can be written as

$$\begin{aligned}&\left(\begin{array}{ccc}(\alpha_{1,2}+\beta_{1,2})a_{1,2}, & \cdots, & (\alpha_{1,n}+\beta_{1,n})a_{1,n}\end{array}\right)\\&-\left(\begin{array}{ccc}\alpha_{1,2}a_{1,2}, & \cdots, & \alpha_{1,n}a_{1,n}\end{array}\right)\begin{pmatrix}a_{2,2} & a_{2,3} & \cdots & a_{2,n}\\ a_{3,2} & a_{3,3} & \cdots & a_{3,n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n,2} & a_{n,3} & \cdots & a_{n,n}\end{pmatrix}\begin{pmatrix}\beta_{1,2} & \cdots & 0\\ \cdots & \cdots & \cdots\\ 0 & \cdots & \beta_{1,n}\end{pmatrix}\\&=\left(\begin{array}{ccc}a_{1,2}, & \cdots, & a_{1,n}\end{array}\right)\begin{pmatrix}\alpha^{'}_{1,2} & \cdots & 0\\ \cdots & \cdots & \cdots\\ 0 & \cdots & \alpha^{'}_{1,n}\end{pmatrix}.\end{aligned}\tag{2.20}$$

Note that

$$\begin{aligned}&\tilde{a}_{1,j}\\=\;&\frac{a_{1,j}-\sum\limits_{k=2}^{n}\alpha_{1,k}a_{1,k}a_{k,j}}{1-\sum\limits_{k=2}^{n}\alpha_{1,k}a_{1,k}a_{k,1}}\\=\;&\frac{a_{1,j}-\quad\alpha_{1,2}a_{1,2},\quad\cdots,\quad\alpha_{1,n}a_{1,n}\quad\begin{pmatrix}a_{2,j}\\ \cdots\\ a_{n,j}\end{pmatrix}}{1-\quad\alpha_{1,2}a_{1,2},\quad\cdots,\quad\alpha_{1,n}a_{1,n}\quad\begin{pmatrix}a_{2,1}\\ \cdots\\ a_{n,1}\end{pmatrix}},\end{aligned}$$

and

$$\begin{aligned}&\tilde{\tilde{a}}_{1,j}\\=\;&\frac{\tilde{a}_{1,j}-\sum\limits_{k=2}^{n}\beta_{1,k}\tilde{a}_{1,k}\tilde{a}_{k,j}}{1-\sum\limits_{k=2}^{n}\beta_{1,k}\tilde{a}_{1,k}\tilde{a}_{k,1}}\\=\;&\frac{\tilde{a}_{1,j}-\quad\tilde{a}_{1,2},\quad\cdots,\quad\tilde{a}_{1,n}\quad\begin{pmatrix}\beta_{1,2}a_{2,j}\\ \cdots\\ \beta_{1,n}a_{n,j}\end{pmatrix}}{1-\quad\tilde{a}_{1,2},\quad\cdots,\quad\tilde{a}_{1,n}\quad\begin{pmatrix}\beta_{1,2}a_{2,1}\\ \cdots\\ \beta_{1,n}a_{n,1}\end{pmatrix}}\\=\;&\frac{a_{1,j}-\quad(\alpha_{1,2}+\beta_{1,2})a_{1,2},\quad\cdots,\quad(\alpha_{1,n}+\beta_{1,n})a_{1,n}\quad -T\begin{pmatrix}a_{2,j}\\ \cdots\\ a_{n,j}\end{pmatrix}}{1-\quad(\alpha_{1,2}+\beta_{1,2})a_{1,2},\quad\cdots,\quad(\alpha_{1,n}+\beta_{1,n})a_{1,n}\quad -T\begin{pmatrix}a_{2,1}\\ \cdots\\ a_{n,1}\end{pmatrix}}\end{aligned}$$

where

$$T \;=\; \begin{pmatrix} \alpha_{1,2}a_{1,2}, & \cdots, & \alpha_{1,n}a_{1,n} \end{pmatrix} \begin{pmatrix} a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} \beta_{1,2} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \beta_{1,n} \end{pmatrix}.$$

So, to realize the equalities

$$\begin{aligned} &\tilde{\tilde{a}}_{1,j} \\ =\; &\frac{a_{1,j}-\sum\limits_{k=2}^{n}\alpha'_{1,k}a_{1,k}a_{k,j}}{1-\sum\limits_{k=2}^{n}\alpha'_{1,k}a_{1,k}a_{k,1}} \\ =\; &\frac{a_{1,j}- \quad \alpha'_{1,2}a_{1,2}, \quad \cdots, \quad \alpha'_{1,n}a_{1,n} \quad \begin{pmatrix} a_{2,j} \\ \cdots \\ a_{n,j} \end{pmatrix}}{1- \quad \alpha'_{1,2}a_{1,2} \quad \cdots \quad \alpha'_{1,n}a_{1,n} \quad \begin{pmatrix} a_{2,1} \\ \cdots \\ a_{n,1} \end{pmatrix}}, \end{aligned}$$

we only need (2.20). The proof is completed. □

Theorem 2.6 *Under the assumptions that A is a nonsingular M-matrix, $0 \le \gamma \le \omega < 1$ and $\omega \neq 0$, let*

$$\alpha = [\alpha_{n,2}, \alpha_{n,3}, \cdots, \alpha_{n,n-1}] \in \mathcal{R}^{n-2}, \quad \textit{and} \quad \tilde{\alpha} = [\alpha'_{n,2}, \alpha'_{n,3}, \cdots, \alpha'_{n,n-1}] \in \mathcal{R}^{n-2},$$

where both $\alpha_{n,j}$ and $\alpha'_{n,j}$ are zeros if $a_{n,j} = 0$ and $[0, \cdots, 0] \le \alpha \le \tilde{\alpha} \le [1, \cdots, 1]$.

Then, for preconditioner $I + S_2(\alpha)$, we have

$$\rho(\tilde{\mathcal{L}}_{\gamma,\omega}(\tilde{\alpha})) \le \rho(\tilde{\mathcal{L}}_{\gamma,\omega}(\alpha)) \le \rho(\mathcal{L}_{\gamma,\omega}) < 1. \tag{2.21}$$

Proof. Similar to the Proof of Theorem 2.5 and according to Theorem 2.2 and Corollary 2.3, we know that $\rho(\tilde{\mathcal{L}}_{\gamma,\omega}(\alpha)) \le \rho(\mathcal{L}_{\gamma,\omega}) < 1$, which implies the AOR iterative method associated a preconditioned $\tilde{A}(\alpha)$ is no worse than the corresponding one of the unpreconditioned matrix A. Since $\tilde{D}^{-1}\tilde{A}$ has the same iteration matrices with $\tilde{A}$, its elements, denoted by the same symbols as those of $\tilde{A}$, are

$$\begin{aligned} &\tilde{a}_{i,i} = 1, \quad i = 1, \cdots, n, \quad \tilde{a}_{i,j} = a_{i,j}, \quad i \neq n, \quad j \neq i, \\ &\tilde{a}_{n,j} = \frac{a_{n,j}-\sum\limits_{k=2}^{n-1}\alpha_{n,k}a_{n,k}a_{k,j}}{1-\sum\limits_{k=2}^{n-1}\alpha_{n,k}a_{n,k}a_{k,n}}, \quad j = 1, \cdots, n-1. \end{aligned} \tag{2.22}$$

Consider the vector $\beta \in \mathcal{R}^{n-2}$ whose components are defined by

$$\begin{cases} \beta_{n,j} = 0, & \text{if } \sum\limits_{k=2}^{n-1}\alpha_{n,k}a_{n,k}a_{k,j} - a_{n,j} = 0, \quad j = 2, \cdots, n-1 \\ \beta_{n,j} = \dfrac{(\alpha_{n,j}-\alpha'_{n,j})a_{n,j}}{\sum\limits_{k=2}^{n-1}\alpha_{n,k}a_{n,k}a_{k,j}-a_{n,j}} & \text{if } \sum\limits_{k=2}^{n-1}\alpha_{n,k}a_{n,k}a_{k,j} - a_{n,j} \neq 0, \quad j = 2, \cdots, n-1. \end{cases} \tag{2.23}$$

Apply $\tilde{D}^{-1}\tilde{A}$ the preconditioner $I + S_2(\beta)$, the AOR iterative methods associated with the new preconditioned matrix $\tilde{\tilde{A}}(\beta) = (I + S_2(\beta))\tilde{D}^{-1}\tilde{A}$ will be no worse than the ones corresponding to $\tilde{D}^{-1}\tilde{A}$. The elements $\tilde{\tilde{a}}_{i,j}$ of the matrix $\tilde{\tilde{D}}^{-1}\tilde{\tilde{A}}(\beta)$ will be given by the same expression as those in (2.22) where the $a_{i,j}$'s will be replaced by $\tilde{a}_{i,j}$'s and the $\alpha_{i,j}$'s by $\beta_{i,j}$'s. The $\tilde{\tilde{a}}_{i,j}$'s are given by

$$\begin{aligned} &\tilde{\tilde{a}}_{i,i} = 1, \quad i = 1, \cdots, n, \quad \tilde{\tilde{a}}_{i,j} = \tilde{a}_{i,j}, \quad i \neq n, \quad j \neq i, \\ &\tilde{\tilde{a}}_{n,j} = \frac{\tilde{a}_{n,j} - \sum\limits_{k=2}^{n-1} \beta_{n,k}\tilde{a}_{n,k}\tilde{a}_{k,j}}{1 - \sum\limits_{k=2}^{n-1} \beta_{n,k}\tilde{a}_{n,k}\tilde{a}_{k,n}}, \quad j = 1, \cdots, n-1. \end{aligned} \tag{2.24}$$

Substituting in (2.24) the $\tilde{a}_{i,j}$'s and the $\beta_{i,j}$'s, after some algebra, we end up with:

$$\begin{aligned} &\tilde{\tilde{a}}_{i,i} = 1, \quad i = 1, \cdots, n, \quad \tilde{\tilde{a}}_{i,j} = a_{i,j}, \quad i \neq n, \quad j \neq i, \\ &\tilde{\tilde{a}}_{n,j} = \frac{a_{n,j} - \sum\limits_{k=2}^{n-1} \alpha'_{n,k}a_{n,k}a_{k,j}}{1 - \sum\limits_{k=2}^{n-1} \alpha'_{n,k}a_{n,k}a_{k,n}}, \quad j = 1, \cdots, n-1. \end{aligned} \tag{2.25}$$

In fact, (2.23) is equivalent to

$$(\alpha_{n,j} + \beta_{n,j} - \alpha'_{n,j})a_{n,j} = \beta_{n,j}\sum_{k=2}^{n-1}\alpha_{n,k}a_{n,k}a_{k,j}, \quad j = 2, \cdots, n-1,$$

which can be written as

$$\begin{aligned} &\left((\alpha_{n,2}+\beta_{n,2})a_{n,2}, \quad \cdots, \quad (\alpha_{n,n-1}+\beta_{n,n-1})a_{n,n-1} \right) \\ &- \left(\alpha_{n,2}a_{n,2}, \quad \cdots, \quad \alpha_{n,n-1}a_{n,n-1} \right) \begin{pmatrix} a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} \\ a_{3,2} & a_{3,3} & \cdots & a_{3,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{pmatrix} \begin{pmatrix} \beta_{n,2} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \beta_{n,n-1} \end{pmatrix} \\ &= \left(a_{n,2}, \quad \cdots, \quad a_{n,n-1} \right) \begin{pmatrix} \alpha'_{n,2} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \alpha'_{n,n-1} \end{pmatrix}. \end{aligned} \tag{2.26}$$

Note that

$$\begin{aligned} &\tilde{\tilde{a}}_{n,j} \\ = \; &\frac{a_{n,j} - \sum\limits_{k=2}^{n-1}\alpha_{n,k}a_{n,k}a_{k,j}}{1 - \sum\limits_{k=2}^{n-1}\alpha_{n,k}a_{n,k}a_{k,n}} \\ = \; &\frac{a_{n,j} - \quad \alpha_{n,2}a_{n,2}, \quad \cdots, \quad \alpha_{n,n-1}a_{n,n-1} \quad \begin{pmatrix} a_{2,j} \\ \cdots \\ a_{n-1,j} \end{pmatrix}}{1 - \quad \alpha_{n,2}a_{n,2}, \quad \cdots, \quad \alpha_{n,n-1}a_{n,n-1} \quad \begin{pmatrix} a_{2,n} \\ \cdots \\ a_{n-1,n} \end{pmatrix}}, \end{aligned}$$

and

$$\begin{aligned}
&\tilde{\tilde{a}}_{n,j}\\
= \;& \frac{\tilde{a}_{n,j}-\sum\limits_{k=2}^{n-1}\beta_{n,k}\tilde{a}_{n,k}\tilde{a}_{k,j}}{1-\sum\limits_{k=2}^{n-1}\beta_{n,k}\tilde{a}_{n,k}\tilde{a}_{k,n}}\\
= \;& \frac{\tilde{a}_{n,j}- \quad \tilde{a}_{n,2}, \quad \cdots, \quad \tilde{a}_{n,n-1} \quad \begin{pmatrix} \beta_{n,2}a_{2,j} \\ \cdots \\ \beta_{n,n-1}a_{n-1,j} \end{pmatrix}}{1- \quad \tilde{a}_{n,2}, \quad \cdots, \quad \tilde{a}_{n,n-1} \quad \begin{pmatrix} \beta_{n,2}a_{2,n} \\ \cdots \\ \beta_{n,n-1}a_{n-1,n} \end{pmatrix}}\\
= \;& \frac{a_{n,j}- \quad (\alpha_{n,2}+\beta_{n,2})a_{n,2}, \quad \cdots, \quad (\alpha_{n,n-1}+\beta_{n,n-1})a_{n,n-1} \quad -T \begin{pmatrix} a_{2,j} \\ \cdots \\ a_{n-1,j} \end{pmatrix}}{1- \quad (\alpha_{n,2}+\beta_{n,2})a_{n,2}, \quad \cdots, \quad (\alpha_{n,n-1}+\beta_{n,n-1})a_{n,n-1} \quad -T \begin{pmatrix} a_{2,n} \\ \cdots \\ a_{n-1,n} \end{pmatrix}}
\end{aligned}$$

where

$$T = \begin{pmatrix} \alpha_{n,2}a_{n,2}, & \cdots, & \alpha_{n,n-1}a_{n,n-1} \end{pmatrix} \begin{pmatrix} a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} \\ a_{3,2} & a_{3,3} & \cdots & a_{3,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{pmatrix} \begin{pmatrix} \beta_{n,2} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \beta_{n,n-1} \end{pmatrix}.$$

So, to realize the equalities

$$\begin{aligned}
&\tilde{\tilde{a}}_{n,j}\\
= \;& \frac{a_{n,j}-\sum\limits_{k=2}^{n-1}\alpha'_{n,k}a_{n,k}a_{k,j}}{1-\sum\limits_{k=2}^{n-1}\alpha'_{n,k}a_{n,k}a_{k,n}}\\
= \;& \frac{a_{n,j}- \quad \alpha'_{n,2}a_{n,2}, \quad \cdots, \quad \alpha'_{n,n-1}a_{n,n-1} \quad \begin{pmatrix} a_{2,j} \\ \cdots \\ a_{n-1,j} \end{pmatrix}}{1- \quad \alpha'_{n,2}a_{n,2}, \quad \cdots, \quad \alpha'_{n,n-1}a_{n,n-1} \quad \begin{pmatrix} a_{2,n} \\ \cdots \\ a_{n-1,n} \end{pmatrix}},
\end{aligned}$$

we only need (**??**). The proof is completed. □

Now, the following result can be derived directly.

Theorem 2.7 *Under the assumptions that A is a nonsingular M-matrix, $0 \le \gamma \le \omega < 1$ and $\omega \ne 0$, let*

$$\begin{aligned}
\alpha &= [\alpha_{1,2}, \cdots, \alpha_{1,n-1}, \alpha_{1,n}, \alpha_{n,2}, \alpha_{n,3}, \cdots, \alpha_{n,n-1}] \in \mathcal{R}^{2n-3},\\
\tilde{\alpha} &= [\alpha'_{1,2}, \cdots, \alpha'_{1,n-1}, \alpha_{1,n}, \alpha'_{n,2}, \alpha'_{n,3}, \cdots, \alpha'_{n,n-1}] \in \mathcal{R}^{2n-3},
\end{aligned}$$

where for $i = 1, n$, both $\alpha_{i,j}$ and $\alpha'_{i,j}$ are zeros if $a_{i,j} = 0$ and $[0, \cdots, 0] \le \alpha \le \tilde{\alpha} \le [1, \cdots, 1]$.

Then, for preconditioner $I + S(\alpha)$, if $a_{1,n} = 0$ or $\alpha_{1,n} = 0$ we have

$$\rho(\tilde{\mathcal{L}}_{\gamma,\omega}(\tilde{\alpha})) \le \rho(\tilde{\mathcal{L}}_{\gamma,\omega}(\alpha)) \le \rho(\mathcal{L}_{\gamma,\omega}) < 1. \tag{2.27}$$

Proof. The key observation is that

$$\begin{aligned}
&(I + S(\alpha))\\
= \;& (I + S_2(\alpha_n))(I + S_1(\alpha_1))\\
= \;& (I + S_1(\alpha_1))(I + S_2(\alpha_n))
\end{aligned}$$

where

$$\alpha_1 = [\alpha_{1,2}, \cdots, \alpha_{1,n-1}, 0] \in \mathcal{R}^{n-1}, \quad \alpha_n = [\alpha_{n,2}, \cdots, \alpha_{n,n-1}] \in \mathcal{R}^{n-2}.$$

According to Theorem 2.5 and 2.6, for

$$\tilde{\alpha}_1 = [\alpha'_{1,2}, \cdots, \alpha'_{1,n-1}, 0] \in \mathcal{R}^{n-1}, \quad \tilde{\alpha}_n = [\alpha'_{n,2}, \cdots, \alpha'_{n,n-1}] \in \mathcal{R}^{n-2},$$

we have that the AOR iterative methods for preconditioner $(I + S_1(\tilde{\alpha}_1))(I + S_2(\alpha_n))$ is no worse than the ones for preconditioner $(I + S_1(\alpha_1))(I + S_2(\alpha_n))$, and the ones associated with preconditioner $(I + S_2(\tilde{\alpha}_n))(I + S_1(\tilde{\alpha}_1))$ is no worse than ones associated with preconditioner $(I + S_n(\alpha_n))(I + S_1(\tilde{\alpha}_1))$. In summery, it holds that

$$\rho(\tilde{\mathcal{L}}_{\gamma,\omega}(\tilde{\alpha})) \le \rho(\tilde{\mathcal{L}}_{\gamma,\omega}(\alpha)) \le \rho(\mathcal{L}_{\gamma,\omega}) < 1.$$

□

3. Numerical example

Consider $n \times n$ coefficient matrix

$$A = \begin{pmatrix} 1 & q & r & s & q & \cdots \\ s & 1 & q & r & \ddots & q \\ r & s & \ddots & \ddots & \ddots & s \\ q & \ddots & \ddots & 1 & q & r \\ s & \ddots & r & s & 1 & q \\ \cdots & s & q & r & s & 1 \end{pmatrix} \tag{3.1}$$

where $q = -\frac{1}{n}$, $r = -\frac{1}{n+1}$, $s = -\frac{1}{n+2}$. Obviously, A is a M-matrix.

In the experiment, the initial approximation of $x^{(0)}$ is taken as a zero vector, and the right hand side vector is chosen so that $e = [1, 1, \cdots, 1]^T$ is the solution of the considered system. Here $\|x^{(k)} - e\|_2 \le 10^{-6}$ is used as the stopping criterion.

In the following table, we report the spectral radius of the corresponding iteration matrix, the CPU time for the iterative part and the number of iterations. We denote spectral radius by ρ, CPU time by CPU, iterative number by IT, basic iterative AOR method by AOR, preconditioned AOR method proposed in our paper by PAOR. For preconditioned AOR method we choose all the parameters as a constant β except $\alpha_{1,n} = 0$.

From the table, we can see that the preconditioned AOR iterative methods are superior to the basic AOR iterative methods. The table have also shown the preconditioned AOR iterative method associated with $\beta = 0.9$ is better than the one associated with $\beta = 0.8$. So, all the numerical results have illustrated our theoretical analysis.

Finally, natural problems are how to choose the optimal parameter α, and how to analyze the AOR iterative method for $\omega > 1$ or $\gamma > 1$. Further research is required.

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Table 1: Numerical Results ($n = 100$)

ω	γ	AOR			PAOR $\beta = 0.8$			PAOR $\beta = 0.9$		
		IT	CPU	ρ	IT	CPU	ρ	IT	CPU	ρ
0.6	0.6	952	1.609	0.983205	937	1.531	0.982939	920	1.5	0.98263
0.6	1	682	1.125	0.976626	672	1.109	0.976258	660	1.094	0.975831
0.8	0.6	712	1.172	0.977606	701	1.14	0.977252	688	1.125	0.97684
0.8	0.8	611	1.016	0.973952	602	0.985	0.973541	591	0.954	0.973064
1	0.6	568	0.938	0.972008	559	0.906	0.971565	549	0.89	0.97105
1	0.8	488	0.828	0.96744	480	0.797	0.966927	471	0.781	0.966329
1	1	406	0.688	0.961043	400	0.672	0.96043	393	0.656	0.959718

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