

A Computational Method for Positive Solutions of the Semilinear Elliptic Boundary Value Problems

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Abstract. In this paper we study the existence of positive solutions for problem of the type $-\Delta u = \lambda|u|^{p-1}u$ for $x \in \Omega$, together with Dirichlet boundary condition.

By a numerical method we will show that for any positive constant λ , there exists a positive solution for this problem

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1 Introduction

Let Ω be a bounded domain in \mathbf{R}^N with smooth boundary $\partial\Omega$ and consider the semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda f(u) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where $f(t) = |t|^{p-1}t$ for $t \in \mathbf{R}$ and $p < \frac{N+2}{N-2}$ and λ is a positive parameter.

The existence of possibly solution of (1) has been extensively investigated even for the case there exists a weight function $a(x)$ beside λ , see for example [1,2,4,8] for a survey.

There are several methods for solving problems such as (1). In particular two main approaches are the use of critical point theory like the mountain pass lemma and fixed point argument [3,4,6].

When $f(u) = u^q$, $0 < q < 1$ sub and super solutions easily provide the existence of unique positive solution of (1) for all $\lambda > 0$, see for example [2].

In this paper we present a variational tool, such as constraining method that is quite convenient when f is superlinear, say, $f(t) = |t|^{p-1}t$ for $1 < p < \frac{N+2}{N-2}$. Indeed one can show that when p equal critical Sobolev exponent $\frac{N+2}{N-2}$ the problem becomes delicate because the lack of compactness.

We don't want to show how this problem has a solution but we compare existence result obtained by variational method with the existence result obtained by numerical method, that is "finite difference approach".

This paper is organized as follows:

Section 2 contains a theoretical argument that leads us to find what we are looking for in numerical discussion, and why we certainly know such solution exists.

In section 3 we briefly summarize the numerical method used in this work and all of numerical data are presented in this section.

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2 Minimal nonnegative solution

In this section we define the action functional whose critical points are the solutions we seek, and state the main existence results from [7].

Throughout this paper we work in the Sobolev space $W = W_0^{1,2}(\Omega)$ equipped with the Energy norm

$$\|u\|_W^2 = \int_{\Omega} |\nabla u(x)|^2 dx.$$

Henceforth we assume integrals are over Ω .

Observe that the equation (1) is the Euler-Lagrange equation of the functional

$$I_{\lambda}(u(x)) = \frac{1}{2} \int |\nabla u(x)|^2 dx - \frac{\lambda}{p+1} \int |u(x)|^{p+1} dx.$$

By using the regularity argument for elliptic boundary value problems [5], u is a solution to (1) if and only if u is a critical point of (1).

It is apparent that the action functional $I_{\lambda}(u)$ is neither bounded from below nor from above on W (because of $p > 1$).

Hence we can consider that part of $I_{\lambda}(u)$ which is bounded below, that is, $J(u) = \frac{1}{2} \int |\nabla u|^2 dx$ and look for a constrained minimization for this functional on the Hilbert space W , restricted to the set $S = \{u \in W; \lambda \int |u|^{p+1} dx = 1\}$.

It is easy to see that $J(u)$ is semi lower continuous i.e. for every $u_n \rightarrow u_0$ in W , $J(u) \leq \liminf J(u_n)$ and is coercive, i.e., if $\|u_n\| \rightarrow \infty$ then $J(u_n) \rightarrow \infty$.

Moreover by the Sobolev embedding theorem the injection $W \hookrightarrow L^p(\Omega)$ is completely continuous for $p < 2^*$, $N \geq 3$ and hence S is weakly closed in W , (i.e. if $u_n \rightarrow u_0$ in S then $u \in S$).

Now we can apply [7, Theorem 1.2] and find a $u_1 \in S$ such that J attains its infimum at u_1 .

We notice since $J(u) = J(|u|)$ we may assume u_1 is nonnegative.

We will show that the conditional critical point of the problem

$$\text{Crit}\{J(u), \Psi_{\lambda}(u) = \lambda \int |u|^{p+1} dx - 1 = 0\}$$

is a critical point of $I_{\lambda}(u)$ and hence a weak solution of the main problem.

At first note that $\Psi'_{\lambda}(u_1)v = (p+1)\lambda \int |u_1|^{p-1} u_1 v dx$ and so

$$\Psi'_{\lambda}(u_1)u_1 = (p+1)\lambda \int |u_1|^{p+1} dx = p+1 \neq 0.$$

Now by using Lagrange multiplier theorem we find a constant $\mu \in \mathbf{R}$ such that $J'(u_1) = \mu \Psi'_{\lambda}(u_1)$, i.e.,

$$\int \nabla u_1 \nabla v = (p+1)\mu \lambda \int |u_1|^{p-1} u_1 v dx, \quad \forall v \in W. \quad (*)$$

Let $v = u_1$ then

$$\int |\nabla u_1|^2 = (p+1)\mu \lambda \int |u_1|^{p+1} = (p+1)\mu.$$

We claim that $\mu > 0$. Indeed if we suppose otherwise we will have $\nabla u_1 = 0$ a.e. $x \in \Omega$ and so for a constant C ; $u_1 \equiv C$. From Dirichlet boundary condition $u_1 \equiv 0$, that is a contradiction with $u_1 \in S$.

So we can define $w = \mu^{\frac{1}{1-p}} u_1 \in W$, by substitution $\mu^{\frac{1}{1-p}} w$ instead of u_1 in the (*) we arrive at

$$\mu^{\frac{1}{1-p}} \int \nabla w \nabla v = (p+1)\mu \mu^{\frac{p}{1-p}} \lambda \int |w|^{p-1} w v dx, \quad \forall v \in W.$$

i.e.

$$\int \nabla w \nabla v = (p+1)\lambda \int |w|^{p-1} w v dx, \quad \forall v \in W,$$

that shows w is a weak solution of (1).

By using the regularity argument and maximum principle in [5] w is a classical positive solution, and so we have proved the following Theorem:

Theorem 1. For any positive λ there exists a positive solution $w \in C^2(\Omega) \cap C^0(\bar{\Omega})$ for problem (1).

Remark 1. The same result is valid in the case $N = 1, 2$ for any $p > 1$.([4])

Remark 2. It is well-known by using bifurcation theory that the norm of positive solutions obtained by Theorem 1 tend to infinity when $\lambda \rightarrow 0$ and tend to zero when $\lambda \rightarrow \infty$. We verify this fact in the following numerical results.

3 Numerical results

In this section we want to find a solution for the equation (1) by using numerical method.

Our method in this section is *finite difference method*, that is one of discretization. We introduce briefly this method as follows:

The purpose of discretization methods is reduce of continuous system to a simple discrete system that is equivalent with it. The essential idea is substitution the continuous region Ω with a grid of discrete points of this area.

Instead of finding a solution over all Ω we find just solution that on points of this grid is exact solution. Mean value, integrals, derivatives and other operators can obtain with different techniques for example finite difference technique, Variational method and iterative method.

In following subsection we introduce fundamental concepts of *finite difference method* and set up the framework for later subsection that involves data.

3.1 Finite difference method

Partial derivatives can approximated by finite differences in some manner. All of these approximations have "truncation error". We look for an approximation that its truncation error is ignorable.

consider the boundary value problem in 2 dimension. (In higher dimension all of formulas and results are similar by using suitable modification.)

Without loss of generality we can suppose our region is in the form $[a, b] \times [c, d]$ and divide this region into $n \times m$ section and hence we have a grid with nm points.

Let $h = \frac{b-a}{n}$ and $l = \frac{d-c}{m}$, $x_i = a + ih$ and $y_j = c + jl$.

We seek an array $\mathbf{u} = \{\mathbf{u}_{ij} ; 1 \leq i \leq n, 1 \leq j \leq m\}$ where $\mathbf{u}_{ij} = u(x_i, y_j)$, i.e., is compatible with exact solution u on points of grid $\Omega \subset \Omega$.

By using Taylor expansion about (x, y) we have

$$u(x+h, y) = u(x, y) + h \frac{\partial u}{\partial x}(x, y) + \frac{h^2}{2!} \frac{\partial^2 u}{\partial x^2}(x, y) + \dots$$

After dividing this formula by h and ignoring the higher degree of h we arrive at

$$\frac{\partial u}{\partial x}(x, y) \approx \frac{1}{h} [u(x+h, y) - u(x, y)],$$

that is forward difference approximation for partial derivative.

We can write last formula in this form

$$\frac{\partial u}{\partial x}|_{i,j} = \frac{1}{h} [u_{i+1,j} - u_{i,j}]. \quad (I)$$

Backward difference can be obtained by using the same approach, i.e.,

$$\frac{\partial u}{\partial x}|_{i,j} = \frac{1}{h} [u_{i,j} - u_{i-1,j}]. \quad (II)$$

Putting together (I) and (II) we arrive at the central difference that is the best approximation for partial derivative:

$$\frac{\partial u}{\partial x}|_{i,j} = \frac{1}{2h}[u_{i+1,j} - u_{i-1,j}].$$

The same procedure as above yields

$$\begin{cases} \frac{\partial u}{\partial y}|_{i,j} &= \frac{1}{2l}[u_{i,j+1} - u_{i,j-1}] \\ \frac{\partial^2 u}{\partial x^2}|_{i,j} &= \frac{1}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] \\ \frac{\partial^2 u}{\partial x \partial y}|_{i,j} &= \frac{1}{4h^2}[u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i+1,j+1}]. \quad (h = l) \end{cases} \quad (3.2)$$

By using these approximation in a partial differential equation we get a system of equations that after solving we can obtain a numerical solution for main equation that is compatible with exact solution on points of grid $\Omega \subset \Omega$. (generally this system has $(n - 1)^N$ equations and variations, where n is the number of division and N is dimension of Ω .)

3.2 A numerical solution for (1)

To solve the problem (1) we have to consider $N \geq 3$ and $p < \frac{N+2}{N-2}$.

For example $N = 3$ and $p = 2$.

We want to apply the method presented in this paper and solve the problem

$$\begin{cases} -(u_{xx} + u_{yy} + u_{zz}) = \lambda[u(\xi)]^2 & \xi \in \Omega \\ u(\xi) = 0 & \xi \in \partial\Omega, \end{cases} \quad (3.3)$$

where $\Omega = [0, 1] \times [0, 1] \times [0, 1]$ is a cuboid.

For simplicity we can suppose all of its lateral sides divide to 4 section i.e. $h = l = s = \frac{1}{4}$.

In this case achieved system has this form

$$-\frac{1}{h^2}[u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j+1,k} + u_{i,j-1,k} + u_{i,j,k+1} + u_{i,j,k-1} - 6u_{i,j,k}] = \lambda(u_{i,j,k})^2, \quad (4)$$

for $1 \leq i, j, k \leq 3$ (since Dirichlet boundary condition yields

$$u_{0,j,k} = u_{i,0,k} = u_{i,j,0} = 0 \quad \forall 1 \leq i, j, k \leq 3).$$

The array of solution $\{u_{i,j,k}; 1 \leq i, j, k \leq 3\}$ of (4) is a numerical solution for (3) that is compatible with exact solution on $\Omega \subset \Omega$.

Some numerical results are presented below that certify our guess of the behavior of the branch of solution.

λ	u		
10^{-6}	$u_{111}=21011.9$	$u_{112}=21836.2$	$u_{113}=10116.6$
	$u_{121}=41598.9$	$u_{122}=45921.1$	$u_{123}=17668.6$
	$u_{131}=20994.4$	$u_{132}=21893.3$	$u_{133}=10233.7$
	$u_{211}=55984.2$	$u_{212}=47134.6$	$u_{213}=19740.3$
	$u_{221}=135700.$	$u_{222}=97035.8$	$u_{223}=35370.7$
	$u_{231}=55863.8$	$u_{232}=47358.8$	$u_{233}=20355.4$
	$u_{311}=83496.8$	$u_{312}=55596.4$	$u_{313}=20175.7$
	$u_{321}=277885.$	$u_{322}=134857.$	$u_{323}=39547.1$
	$u_{331}=83154.4$	$u_{332}=56188.9$	$u_{333}=23192.2$

λ	u		
10^{-2}	$u_{111}=16.1011$	$u_{112}=21.6398$	$u_{113}=20.1692$
	$u_{121}=21.029$	$u_{122}=12.1453$	$u_{123}=9.9809$
	$u_{131}=20.0214$	$u_{132}=10.2116$	$u_{133}=8.84293$
	$u_{211}=18.1598$	$u_{212}=20.3552$	$u_{213}=18.8908$
	$u_{221}=21.2471$	$u_{222}=9.7653$	$u_{223}=8.63669$
	$u_{231}=10.6218$	$u_{232}=9.49621$	$u_{233}=16.6283$
	$u_{311}=22.2679$	$u_{312}=20.6373$	$u_{313}=21.3654$
	$u_{321}=19.9223$	$u_{322}=16.0407$	$u_{323}=19.8503$
	$u_{331}=10.4565$	$u_{332}=20.5419$	$u_{333}=23.7996$

λ	u		
5	$u_{111}=0.0108887$	$u_{112}=0.00344111$	$u_{113}=0.0136107$
	$u_{121}=0.00344111$	$u_{122}=0.0143538$	$u_{123}=0.0557282$
	$u_{131}=0.0136107$	$u_{132}=0.0557282$	$u_{133}=0.0583006$
	$u_{211}=0.0649857$	$u_{212}=0.0246028$	$u_{213}=0.0104586$
	$u_{221}=0.0246028$	$u_{222}=0.0307928$	$u_{223}=0.0205658$
	$u_{231}=0.0104586$	$u_{232}=0.0205658$	$u_{233}=0.0272532$
	$u_{311}=0.0323823$	$u_{312}=0.0268549$	$u_{313}=0.0148198$
	$u_{321}=0.0268549$	$u_{322}=0.0389693$	$u_{323}=0.0354365$
	$u_{331}=0.0148198$	$u_{332}=0.0354365$	$u_{333}=0.0576495$

λ	u		
1000	$u_{111}=0.000175334$	$u_{112}=0.0000587831$	$u_{113}=0.000171755$
	$u_{121}=0.000283776$	$u_{122}=0.0000252565$	$u_{123}=0.000272404$
	$u_{131}=0.00027304$	$u_{132}=0.000282462$	$u_{133} = 8.3722 * 10^{-6}$
	$u_{211}=0.000266744$	$u_{212}=0.0000848869$	$u_{213}=0.000228161$
	$u_{221}=0.0000314731$	$u_{222}=0.000184164$	$u_{223}=0.000332404$
	$u_{231}=0.000260695$	$u_{232}=0.000159176$	$u_{233}=0.000346859$
	$u_{311}=0.000326214$	$u_{312}=0.0000552452$	$u_{313}=0.0000216326$
	$u_{321}=0.000178712$	$u_{322}=0.000365419$	$u_{323} = 9.87329 * 10^{-6}$
	$u_{331}=0.00010257$	$u_{332}=0.0000446018$	$u_{333}=0.000314295$

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