

Global Conservative Solutions of the Generalized Shallow Water Wave Equation

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Abstract: In this paper, the existence of global conservative solutions of the Cauchy problem for the generalized Dullin-Gottwald-Holm equation is studied. It is transformed into an ODE system in a Banach space. By using a contraction map and some Sobolev's inequalities, the existence of the short-time solutions is obtained. Particularly the global conservative solutions with respect to the initial data are obtained. The continuity of the solution semigroup is proved.

Keywords: global conservative solutions; Lipschitz; continuous

1 Introduction

In [1], Camassa and Holm derived a new completely integrable dispersive wave equation for water waves by Hamiltonian method, namely Camassa-Holm equation:

$$u_t + 2\omega u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \quad (1.1)$$

As we know Eq. (1.1) is integrable for all ω , and it has peakon solutions of the form $ce^{-|x-ct|}$ (c is the speed). More and more researches have been carried on Camassa-Holm equation. Lixin Tian et al [2,3] discussed the traveling wave solutions and double solitary wave solutions, and introduced the definitions of concave, convex peaked solitary solutions and smooth solitary solutions. Lixin Tian, Guilong Gui, Boling Guo [4] studied the limit behavior of the solutions to a class of nonlinear dispersive wave equations.

Lixin Tian et al [5] also considered generalized Camassa-Holm equation or the modified Camassa-Holm equation

$$u_t + ku_x - u_{xxt} + au^2u_x = 2u_x u_{xx} + uu_{xxx}$$

and derived some new exact peaked solitary wave solutions.

Dullin, Gottwald and Holm [6] derived an integrable shallow water equation that combines the linear dispersion, namely DGH equation

$$u_t + 2\omega u_x - \alpha^2 u_{xxt} + 3uu_x + \gamma u_{xxx} = \alpha^2 (2u_x u_{xx} + uu_{xxx}). \quad (1.2)$$

Lixin Tian et al [7] discussed the well-posedness problem and the scattering problem for the DGH equation, and also obtained peakon solutions of the form $ce^{-|x-ct|}$. When $\alpha^2 \rightarrow 0$, it becomes the well-known KdV equation

$$u_t + 2\omega u_x + 3uu_x + \gamma u_{xxx} = 0. \quad (1.3)$$

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The KdV equation is completely integral and its solitary waves are solitons (see [8,9]). The Cauchy problem of Eq.(1.3) has been the subject of a number of studies, and a satisfactory local or global existence theory is now in hand in [10]. Lixin Tian, Xiuming Li [10] discussed the modified Degasperis-Procesi (MDP) equation. In [11-13] it is shown that it has the global existence of strong solutions and weak solutions, and the precise blow-up scenario and a blow-up result. In [14] they proved the optimal control of the viscosity Degasperis-Procesi equation in $H^2(\mathbb{R})$, Lixin Tian etc.[15] also discussed the initial value problem of DGH equation as

$$\begin{cases} u_t + 2\omega u_x - \alpha^2 u_{xxt} + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}) \\ u(0, x) = u_0(x) \end{cases} \quad (1.4)$$

They proved that Eq.(1.4) has a global solution and the solitary waves of Eq.(1.4) are stable. It is also shown that the solutions of DGH equation convergence to the solution of corresponding KdV equation (as α^2 tends to zero).

Jiuli Yin etc.[16] also discussed the simplified MDGH equation

$$u_t + 2\omega u_x - u_{xxt} + au^2 u_x + \gamma u_{xxx} = 2u_x u_{xx} + uu_{xxx}. \quad (1.5)$$

It is painleve integrable and also has some conservation laws. They used the truncated painleve expansion to obtain an auto-Backlund transformation and smooth solitary wave solutions, particularly obtained the double solitary wave solution with peakon and double singular solitary wave solution. And they also proved that the smooth solitary waves to this rod equation are orbital stable.

For the nonlinear partial differential equation

$$u_t - u_{xxt} + f(u)_x - f(u)_{xxx} + \left(g(u) + \frac{1}{2} f''(u)(u_x)^2 \right)_x = 0,$$

when $f(u) = \frac{1}{2}\gamma u^2$, $g(u) = \frac{3-\gamma}{2}u^2$, we add a linear dispersive term $-\gamma u_{xxx}$ to the equation and get the DGH equation

$$u_t - u_{xxt} + 3uu_x - \gamma(2u_x u_{xx} + uu_{xxx} + u_{xxx}) = 0.$$

Furthermore, for $f(u) = \frac{1}{2}\gamma u^2$, we find the generalized DGH wave equation

$$u_t - u_{xxt} + \frac{1}{2}g(u)_x - \gamma(2u_x u_{xx} + uu_{xxx} + u_{xxx}) = 0. \quad (1.6)$$

This is the equation which we will consider in this paper. In this paper, when $g(u) = \frac{3}{2}u^2$, Eq.(1.6) becomes Eq.(1.4). When $g(u) = \frac{a}{3}u^3$, Eq.(1.6) becomes Eq.(1.5). Here we take a different approach, based on recent techniques developed for Camassa-Holm equation, see [17-20]. From [21,22], Eq.(1.6) can be rewritten as

$$u_t + \gamma uu_x + P_x = 0, P - P_{xx} = \frac{1}{2}(g(u) - \gamma u^2 + \gamma u_x^2 - 2\gamma u_{xx}) \quad (1.7)$$

It is advantageous to rewrite the equation as

$$u_t + f(u)_x + P_x = 0, \quad (1.8a)$$

$$P - P_{xx} = g(u) + \frac{1}{2}f''(u)(u_x^2 - 2u_{xx}), \quad (1.8b)$$

where we assume

$$f \in W_{loc}^{2,\infty}(\mathbb{R}), f''(u) \neq 0, u \in \mathbb{R}; g \in W_{loc}^{1,\infty}(\mathbb{R}), g(0) \neq 0. \quad (1.9)$$

2 Existence of solutions

2.1 Transport equation for the energy density and reformulation in terms of Lagrangian variables

In (1.8 b), P can be written in explicit form:

$$P(t, x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-z|} \left(g \circ u + \frac{1}{2} f'' \circ u (u_x^2 - 2u_{xx}) \right) (t, z) dz \quad (2.1)$$

After differentiating (1.8a) with respect to x and using (1.8b), that

$$u_{xt} + \frac{1}{2}f''(u)u_x^2 + f'(u)u_{xx} + P - g(u) + f''(u)u_{xx} = 0. \tag{2.2}$$

Multiply (1.8a) by u , (2.2) by u_x , add the two to find the following equation

$$(u^2 + u_x^2)_t + (f'(u)(u^2 + u_x^2))_x = -2(Pu)_x + (2g(u) + f''(u)u^2 - 2f''(u)u_{xx})u_x. \tag{2.3}$$

Define

$$G(v) = \int_0^v (2g(z) + f''(z)z^2 - 2f''(z)z_{xx})dz. \tag{2.4}$$

Then (2.3) can be rewritten as

$$(u^2 + u_x^2)_t + (f'(u)(u^2 + u_x^2))_x = (G(u) - 2Pu)_x, \tag{2.5}$$

which is transport equation for the energy density $u^2 + u_x^2$.

Let us introduce the characteristics $y(t, \xi)$ defined as the solutions

$$y_t(t, \xi) = f'(u(t, y(t, \xi))) \tag{2.6}$$

with $y(0, \xi)$ given. Equation (2.5) gives us information about the evolution of the amount of energy contained between two characteristics. Indeed, given ξ_1, ξ_2 in \mathbb{R} , let $H(t) = \int_{y(t, \xi_1)}^{y(t, \xi_2)} (u^2 + u_x^2)dx$ be the energy contained between the two characteristic curves $y(t, \xi_1), y(t, \xi_2)$. Then, we have

$$\frac{dH}{dt} = [y_t(t, \xi)(u^2 + u_x^2) \circ y(t, \xi)]_{\xi_1}^{\xi_2} + \int_{y(t, \xi_1)}^{y(t, \xi_2)} (u^2 + u_x^2)_t dx. \tag{2.7}$$

We use (2.5) and integrate by parts. The first term on the right-hand side of (2.7) was canceled by using (2.6) and we get

$$\frac{dH}{dt} = [(G(u) - 2Pu) \circ y]_{\xi_1}^{\xi_2}. \tag{2.8}$$

We now derive a system equivalent to (1.7). The calculations will be justified later. Let y still denote the characteristics. We introduce two other variables, the Lagrangian velocity U and cumulative energy distribution H defined by

$$U(t, \xi) = u(t, y(t, \xi)), \tag{2.9}$$

$$H(t, \xi) = \int_{-\infty}^{y(t, \xi)} (u^2 + u_x^2)dx. \tag{2.10}$$

From the definition of the characteristics, it follows from (1.8a) that

$$U_t(t, \xi) = u_t(t, y) + y_t(t, \xi)u_x(t, y) = (u_t + f'(u)u_x) \circ y(t, \xi) = -P_x \circ y(t, \xi). \tag{2.11}$$

After the change of variable $z = y(t, \eta)$, and $H_\xi = (u^2 + u_x^2) \circ yy_\xi$,

$$P_x \circ y(t, \xi) = -\frac{1}{2} \int_{\mathbb{R}} \text{sgn}(y(\xi) - y(\eta)) e^{-|y(\xi) - y(\eta)|} \times \left((g(U) - \frac{1}{2}f''(U)U^2) y_\xi + \frac{1}{2}f''(U)(H_\xi - 2U_{\xi\xi}) \right) (\eta) d\eta. \tag{2.12}$$

If we take this for granted, then $P_x \circ y$ is equivalent to Q , where

$$Q(t, \xi) = -\frac{1}{2} \int_{\mathbb{R}} \text{sgn}(\xi - \eta) \exp(-\text{sgn}(\xi - \eta)(y(\xi) - y(\eta))) \times \left((g(U) - \frac{1}{2}f''(U)U^2) y_\xi + \frac{1}{2}f''(U)(H_\xi - 2U_{\xi\xi}) \right) (\eta) d\eta. \tag{2.13}$$

Slightly abusing the notation, we write

$$P(t, \xi) = \frac{1}{2} \int_{\mathbb{R}} \exp(-\text{sgn}(\xi - \eta)(y(\xi) - y(\eta))) \times \left((g(U) - \frac{1}{2}f''(U)U^2) y_\xi + \frac{1}{2}f''(U)(H_\xi - 2U_{\xi\xi}) \right) (\eta) d\eta. \tag{2.14}$$

$P_x \circ y$ and $P \circ y$ can be replaced by equivalent expressions given by (2.13) and (??) which only depend on our new variables U, H , and y . We introduce yet another variable $\zeta(t, \xi)$, simply defined as $\zeta(t, \xi) = y(t, \xi) - \xi$.

It will turn out that $\zeta \in L^\infty(\mathbb{R})$. We have now derived a new system of equations, formally equivalent to (1.7). Equations (2.6), (2.8) and (2.11) give us

$$\begin{cases} y_t = f'(u) \\ U_t = -Q \\ H_t = G(U) - 2PU \end{cases} \quad (2.15)$$

Detailed analysis will reveal that the system (2.15) of ordinary differential equations for $(\zeta, U, H): [0, T] \rightarrow E$ is well-posed, where E is a Banach space to be defined in the next section. We have

$$Q_\xi = -\frac{1}{2}f''(U)(H_\xi - 2U_{\xi\xi}) + \left(P + \frac{1}{2}f''(U)U^2 - g(U) \right) y_\xi$$

and

$$P_\xi = Qy_\xi.$$

Hence, differentiating (2.15) yields

$$\begin{cases} \zeta_{\xi t} = f''(U)U_\xi, (y_{\xi t} = f''(U)U_\xi) \\ U_{\xi t} = \frac{1}{2}f''(U)H_\xi - \left(P + \frac{1}{2}f''(U)U^2 - g(U) \right) y_\xi - f''(U)U_{\xi\xi} \\ H_{\xi t} = -2QUy_\xi + (2g(U) + f''(U)U^2 - 2P)U_\xi - (2f''(U)U_\xi/y_\xi)U_{\xi\xi} \end{cases} \quad (2.16)$$

Because we add the term $-\gamma u_{xxx}$, the regularity of the solutions has increased .

2.2 Existence and uniqueness of solutions in Lagrangian variables

In this section, we focus our attention on the system of Eqs. (2.15) and prove, by a contraction argument, that it admits a unique solution. Let V be the Banach space defined by $V = \{f \in C_b(\mathbb{R}) | f_\xi \in L^2(\mathbb{R})\}$, where $C_b(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and the norm of V is given by $\|f\|_V = \|f\|_{L^\infty(\mathbb{R})} + \|f_\xi\|_{L^2(\mathbb{R})}$. Of course $H^1(\mathbb{R}), H^2(\mathbb{R}) \subset V$, but the converse is not true as V contains functions that do not vanish at infinity. We will employ the Banach space E which is defined by $E = V \times H^2(\mathbb{R}) \times V$ to carry out the contraction map argument. For any $X = (\zeta, U, H) \in E$, the norm on E is given by $\|X\|_E = \|\zeta\|_V + \|U\|_{H^2(\mathbb{R})} + \|H\|_V$.

Lemma 2.1 *Let $B_M = \{X \in E | \|X\|_E \leq M\}$, given $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{H})$ in E , there exists a time T depending only on $\|X\|_E$ such that the system (2.15) admits a unique solution in $C^2([0, T], E)$ with initial data \bar{X} .*

Lemma 2.2 *([23]) $y_\xi H_\xi = y_\xi^2 U^2 + U_\xi^2$ almost everywhere.*

If $\|\bar{\zeta}_\xi\| + \|\bar{U}_{\xi\xi}\| + \|\bar{H}_\xi\| < \infty$, we can prove the the solution exists in $[0, T]$ for any time $t \in [0, T]$ (see [20, Lemma 2.4]).

Definition 2.1 $\bar{\xi} = \int_{-\infty}^{\bar{y}(\xi)} (\bar{u}^2 + \bar{u}_x^2) dx + \bar{y}(\xi)$, $\bar{U} = \bar{u} \circ \bar{y}$, $\bar{H} = \int_{-\infty}^{\bar{y}(\xi)} (\bar{u}^2 + \bar{u}_x^2) dx$, $y(\xi) = \zeta(\xi) + \xi$, $\varepsilon = \left\{ (f, g) | (f, g) \in W_{loc}^{2,\infty}(\mathbb{R}) \times W_{loc}^{1,\infty}(\mathbb{R}) \right\}$.

It is easy to prove $H_\xi \geq 0$, H_ξ is an increasing function with respect to ξ . We have $\lim_{\xi \rightarrow \pm\infty} U(t, \xi) = 0$ and $H(t, \xi) = H(0, \xi) + \int_0^t [G(U) - 2PU](\tau, \xi) d\tau$. Hence we can prove $H(t, \pm\infty) = H(0, \pm\infty)$, so $\lim_{\xi \rightarrow \pm\infty} H(t, \xi)$ exists and is independent of time. Let's define $\sup_{t \in [0, T]} \|H(t, \cdot)\|_{L^\infty(\mathbb{R})} = \|\bar{H}\|_{L^\infty(\mathbb{R})} = h$ (see [24]).

Lemma 2.3 *([23]) Given $\bar{u} \in H^2(\mathbb{R})$, the initial date $(\bar{y}, \bar{U}, \bar{H})$ belongs to Ω .*

$$\Omega = \{(y, U, H) | (y, U, H) \in E \cap (W^{1,\infty}(\mathbb{R}) \times W^{2,\infty}(\mathbb{R}) \times W^{1,\infty}(\mathbb{R}))\}$$

Theorem 2.1 *(2.15) admits a unique global solution $X(t) = (y(t), U(t), H(t))$ in $C^1(\mathbb{R}_+, E)$ with initial date $\bar{X} = (\bar{y}, \bar{U}, \bar{H})$.*

Proof. We want to prove $\sup_{t \in [0, T]} \|X(t)\|_E < \infty$,

$$U^2(\xi) = 2 \int_{-\infty}^{\xi} U(\eta)U_{\xi}(\eta)d\eta = 2 \int_{\{\eta \leq \xi | y_{\xi}(\eta) > 0\}} U(\eta)U_{\xi}(\eta)d\eta$$

$$|U(\xi)U_{\xi}(\xi)| = \left| \sqrt{y_{\xi}}U_{\xi}(\xi) \frac{U_{\xi}(\xi)}{\sqrt{y_{\xi}}} \right| \leq \frac{1}{2} \left(U(\xi)^2 y_{\xi}(\xi) + \frac{U_{\xi}^2(\xi)}{y_{\xi}(\xi)} \right) = \frac{1}{2} H_{\xi}(\xi),$$

We get $U^2(\xi) \leq H(\xi)$, $U(t, \xi) \in I := [-\sqrt{h}, \sqrt{h}]$ for all the $t \in [0, T]$, $\xi \in \mathbb{R}$,

$$\sup_{t \in [0, T]} \|U(t, \cdot)\|_{L^{\infty}(\mathbb{R})} < \infty.$$

We get $k = \|f\|_{W^{2, \infty}(I)} + \|g\|_{W^{1, \infty}(I)}$ (see [25]), $|\zeta(t, \xi)| \leq |\zeta(0, \xi)| + \kappa T$, so $\sup_{t \in [0, T]} \|\zeta(t, \cdot)\|_{L^{\infty}(\mathbb{R})}$ is

bounded.

From (2.13) we know $Q(t, \xi) = Q_a + Q_b$,

$$|Q_a(t, \xi)| \leq C_1 \int_{\mathbb{R}} e^{-|y(\xi) - y(\eta)|} y_{\xi}(\eta) d\eta = 2C_1(C_1 \text{ dependson } \kappa, h).$$

$$|Q_b(t, \xi)| \leq \frac{\kappa}{4} \int_{\mathbb{R}} e^{-|y(\xi) - y(\eta)|} [H_{\xi}(\eta) - 2U_{\xi\xi}(\eta)] d\eta$$

$$\leq \left(\frac{\kappa h}{4} + \frac{\kappa \sqrt{h}}{2} \right) \int_{\mathbb{R}} e^{-|y(\xi) - y(\eta)|} y_{\xi}(\eta) d\eta = \frac{\kappa h}{2} + \kappa \sqrt{h}.$$

So $Q(P)$ is bounded by a constant that only depends on κ, h .

Let $h(\xi) = \chi_{\{\xi > 0\}}(\xi)e^{-\xi}$, and A be the map defined by $A : v \rightarrow h * v$. Then use the way in [23] to define $Q = Q_1 + Q_2$, and

$$Q_1(X)(\xi) = -\frac{e^{-\zeta(\xi)}}{2} A \circ R(\zeta, U, H)(\xi), Q_2(X)(\xi) = \frac{e^{-\zeta(\xi)}}{2} A \circ R(\zeta, U, H)(\xi).$$

Let $C_2 = \sup_{t \in [0, T]} \left\{ \|U(t, \cdot)\|_{L^{\infty}(\mathbb{R})} + \|H(t, \cdot)\|_{L^{\infty}(\mathbb{R})} + \|\zeta(t, \cdot)\|_{L^{\infty}(\mathbb{R})} + \|P(t, \cdot)\|_{L^{\infty}(\mathbb{R})} + \|Q(t, \cdot)\|_{L^{\infty}(\mathbb{R})} \right\}$

(C_2 is finite and only depends on $\|\bar{X}\|_E, T, \kappa$),

$$\|A \circ R\|_{L^2(\mathbb{R})} \leq C \left(\|U(t, \cdot)\|_{L^2(\mathbb{R})} + \|U_{\xi\xi}(t, \cdot)\|_{L^2(\mathbb{R})} + \|\zeta_{\xi}(t, \cdot)\|_{L^2(\mathbb{R})} + \|H_{\xi}(t, \cdot)\|_{L^2(\mathbb{R})} \right)$$

(C only depends on C_2, h and k).

The same bounds hold for Q, P ,

$$Z(t) = \|U(t, \cdot)\|_{L^2(\mathbb{R})} + \|U_{\xi\xi}(t, \cdot)\|_{L^2(\mathbb{R})} + \|\zeta_{\xi}(t, \cdot)\|_{L^2(\mathbb{R})} + \|H_{\xi}(t, \cdot)\|_{L^2(\mathbb{R})}.$$

From all the above we get $Z(t) \leq Z(0) + C \int_0^t Z(\tau) d\tau$, by Grownwall's Lemma, $\sup_{t \in [0, T]} Z(t) < \infty$, so the solutions exist globally in time. ■

Theorem 2.2 The map $S : \Omega \times \varepsilon \times \mathbb{R}_+ \rightarrow \Omega$ defined as $S_t(\bar{X}, f, g) = X(t)$ is a continuous semigroup with respect to all the variables on any bounded set of ε, Ω means the topology induced by the E -norm.

Proof. Now we will prove the continuity of the semigroup. we will prove sequential continuity.

Let ε' be a bounded set of ε . Let $\bar{X}_n = (\bar{y}_n, \bar{U}_n, \bar{H}_n) \in \Omega, (f_n, g_n) \in \varepsilon'$ be sequences that converge to $\bar{X} = (\bar{y}, \bar{U}, \bar{H}) \in \Omega, (f, g) \in \varepsilon'$.

Define

$$X_n(t) = S_t(\bar{X}_n, f_n, g_n), X(t) = S_t(\bar{X}, f, g), U_n(t, \xi) \in I := [-\sqrt{M}, \sqrt{M}].$$

Hence we have $I_n := \left[-\|\bar{H}\|_{L^{\infty}(\mathbb{R})}^{1/2}, \|\bar{H}\|_{L^{\infty}(\mathbb{R})}^{1/2} \right] \subset I, k_n = \|f_n\|_{W^{2, \infty}(I_n)} + \|g_n\|_{W^{1, \infty}(I_n)} \leq k$, for all $n \geq 1$.

$$\sup_{t \in [0, T]} \|X_n(t, \cdot)\|_E \leq C \left(\|\bar{X}_n\|_E, T, \kappa_n \right) \leq C(M, T, \kappa) = C'.$$

So $\sup_{t \in [0, T]} \|X_n(t, \cdot)\|_E$ is bounded uniformly with respect to n , we have

$$\|X_n(t) - X(t)\|_E \leq \|\bar{X}_n - \bar{X}\|_E + \int_0^t \|F(X_n, f_n, g_n) - F(X, f, g)\|_E(s) ds.$$

We fix the time $t \in [0, T]$ for the moment. We have

$$\begin{aligned} & \|F(X_n, f_n, g_n) - F(X, f, g)\|_E \leq \\ & \|F(X_n, f_n, g_n) - F(X_n, f, g)\|_E + \|F(X_n, f, g) - F(X, f, g)\|_E. \end{aligned} \quad (2.17)$$

For $X \rightarrow F(X, f, g)$ is Lipschitz on any bounded set of E , we get

$$\|F(X_n, f, g) - F(X, f, g)\|_E \leq L \|X_n - X\|_E. \quad (2.18)$$

Next we will prove $\|F(X_n, f_n, g_n) - F(X_n, f, g)\|_E$ is bounded.

Let $G_n = \int_0^{U_n} (2g_n(z) + f_n''(z)(z^2 - 2z_{xx})) dz$, $G_n' = \int_0^{U_n} (2g(z) + f''(z)(z^2 - 2z_{xx})) dz$.

Using the same notation to others, we get Q_n, Q_n', R_n, R_n' .

Let $\delta_n^1 = \|g_n(U) - g(U)\|_{L^2(\mathbb{R})}$. We know

$$\begin{aligned} \|g_n(U_n) - g(U)\|_{L^2(\mathbb{R})} & \leq \|g_n(U_n) - g_n(U)\|_{L^2(\mathbb{R})} + \|g_n(U) - g(U)\|_{L^2(\mathbb{R})} \\ & + \|g(U_n) - g(U)\|_{L^2(\mathbb{R})} \leq 2\kappa \|U_n - U\|_{L^2(\mathbb{R})} + \delta_n^1. \end{aligned}$$

For $g_n \rightarrow g$ in $L^\infty(I)$, and $\|g_n(u) - g(u)\|_{L^2(\mathbb{R})} \leq 2k \|U\|_{L^2(\mathbb{R})}$, by Lebesgue dominated convergence theorem, we get $\lim_{n \rightarrow \infty} \delta_n^1 = 0$.

$$\begin{aligned} \|G_n - G_n'\|_V & = \|G_n - G_n'\|_{L^\infty(\mathbb{R})} + \|G_n - G_n'\|_{L^2(\mathbb{R})} \\ & \leq \sqrt{M} \left(2 \|g_n(U_n) - g(U_n)\|_{L^\infty(\mathbb{R})} + (M + 2\sqrt{M}) \|f_n''(U_n) - f''(U_n)\|_{L^\infty(\mathbb{R})} \right) \\ & + 2C' \|g_n(U_n) - g(U_n)\|_{L^2(\mathbb{R})} + C' \|f_n''(U_n) - f''(U_n)\|_{L^2(\mathbb{R})} \leq C\delta_n^2 \end{aligned} \quad (2.19)$$

($\delta_n^2 = \|f_n - f\|_{W^{2,\infty}(I)} + \|g_n - g\|_{W^{1,\infty}(I)}$, $n \rightarrow \infty$, $\delta_n^2 \rightarrow 0$).

The same way applies to R_n, R_n' , we get

$$\|R_n - R_n'\|_V \leq C(\delta_n^1 + \delta_n^2 + \|U_n - U\|_{L^2(\mathbb{R})}).$$

Let $\delta_n = \delta_n^1 + \delta_n^2$, we get $\|Q_n - Q_n'\|_V \leq C(\delta_n + \|U_n - U\|_{L^2(\mathbb{R})})$, $\|P_n - P_n'\|_V \leq C(\delta_n + \|U_n - U\|_{L^2(\mathbb{R})})$. We get

$$\|F(X_n, f_n, g_n) - F(X_n, f, g)\|_E \leq C(\delta_n + \|U_n - U\|_{L^2(\mathbb{R})}). \quad (2.20)$$

From (2.17-2.20), we know

$$\|X_n(t) - X(t)\|_E \leq \|\bar{X}_n - \bar{X}\|_E + CT\delta_n + (L + C) \int_0^t \|X_n - X\|_E(s) ds.$$

By Gronwall's Lemma, $\|X_n(t) - X(t)\|_E \leq (\|\bar{X}_n - \bar{X}\|_E + CT\delta_n) e^{(L+C)T}$. So $X_n \rightarrow X$ uniformly in $[0, T]$, hence S_t is a continuous semigroup on ε . ■

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