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Similarity Flow Solutions of a Non-Newtonian Power-law Fluid

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Abstract: In this paper we present a mathematical analysis for a steady-state laminar boundary layer flow, governed by the Ostwald-de Wael power-law model of an incompressible non-Newtonian fluid past a semi-infinite power-law stretched flat plate with uniform free stream velocity. A generalization of the usual Blasius similarity transformation is used to find similarity solutions [1]. Under appropriate assumptions, partial differential equations are transformed into an autonomous third-order nonlinear degenerate ordinary differential equation with boundary conditions. Using a shooting method, we establish the existence of an infinite number of global unbounded solutions. The asymptotic behavior is also discussed. Some properties of those solutions depend on the viscosity power-law index.

Key words: Boundary-layer, Power-law fluid, Multiple solutions, Similarity transformation

1 Introduction

In view of their wide applications in different industrial processes, and also by the interesting mathematical features presented their equations, boundary-layer flows of non-Newtonian fluids have motivated researchers in many branches of engineering in recent years. The most frequently used model in non-Newtonian fluid mechanics is the Ostwald-de Wael model (with a power-law rheology [2–6]), which the relationship between the shear stress and the strain rate is given as follows

$$\tau_{xy} = k|u_y|^{n-1}u_y \tag{1}$$

for $n = 1$ the fluid is called Newtonian with dynamic coefficient of viscosity k. For $n > 1$ the behavior of the fluid is dilatant or shear-thickening and for $0 < n < 1$ the behavior is shear-thinning, in these cases the fluid is non-Newtonian and k is the fluid consistency. In this work we shall restrict our study to the dilatant fluids, then throughout all the paper, the exponent n will be taken in the range $(1, \infty)$. The problem of laminar flows of power-law non-Newtonian fluids have been studied by several authors. For the sake of brevity, we mention here some examples, Acrivos et al.[7] and Pakdemirli [8] derived the boundary layer equations of power-fluids, Mansutti and Rajagobal [9] investigated the boundary layer flow of dilatant fluids. Adopting the Crocco variable formulation, Nachman and Talliafero [10] established existence and uniqueness of similarity solution for a mass transfer problem. Filipussi et al. [11] obtained similarity solutions and their properties using a phase-plane formalism. Recently numerical solutions have been given by Ece and Büyük in [12] for the steady laminar free convection over a heated flat plate.

More recently Guedda [13] studied the free convection problem of a Newtonian fluid, he showed the existence of an infinite number of solution and studied their asymptotic behavior. In this work we aim to extend the analysis of [13] to the non-Newtonian case, we are interested also in the effect of the power-law index on the existence and the asymptotic behavior of solutions.

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The remainder of this work is organized as follows, in the next section, we introduce the mathematical formulation of the problem, section 3 deals with some preliminary tools which will be useful in section 4 and 5 to prove the main results. Finally, we give some concluding remarks in section 6.

2 Similarity procedure

The problem is geometrically defined by a semi-infinite power-law stretched rigid plate, over which flows a non-Newtonian fluid obeying to (1). The main hypotheses for the mathematical formulation of this problem are given by:

- Two-dimensional, incompressible and steady-state laminar flow,
- Physical properties are taken as constants,
- Body force, external gradients pressure and viscous dissipation are neglected.

Under these assumptions, and referred to a Cartesian system of coordinates Oxy, where $y = 0$ is the plate, the x-axis is directed upwards to the plate and the y-axis is normal to it, the continuity and momentum equations can be simplified, within the range of validity of the Boussinesq approximation [7], to the following equations

$$\begin{cases} uu_x + vu_y = \nu(|u_y|^{n-1}u_y)_y, \\ u_x + v_y = 0, \end{cases} \tag{2}$$

The functions u and v are the velocity components in the $x-$ and $y-$ directions respectively.
The boundary conditions accompanied equation (2) are given by

$$u(x,0) = U_w(x), \quad v(x,0) = V_w(x), \quad u(x,y) \to 0 \quad \text{as } y \to \infty. \tag{3}$$

The functions : $U_w(x) = u_w x^m$ is called the stretching velocity and $u_w > 0$, the exponent m is negative, and $V_w(x) = v_w x^{\frac{m(2n-1)-n}{n+1}}$ is the suction/injection velocity where $v_w > 0$ for suction and $v_w < 0$ for injection.

From the incompressibility of the fluid we introduce the dimensionless stream function $\psi = \psi(x,y)$ satisfying $(u = \psi_y,\ v = -\psi_x)$.

Hence equations (2) are reduced to the single equation

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = \nu(|\psi_{yy}|^{n-1}\psi_{yy})_y. \tag{4}$$

The boundary conditions (3) are transformed into

$$\psi_y(x,0) = u_w x^m, \quad \psi_x(x,0) = -v_w x^{\frac{m(2n-1)-n}{n+1}}, \quad \psi_y(x,y) \to 0 \text{ as } y \to \infty. \tag{5}$$

Since the broad goal of this paper is to obtain similarity solutions to (4),(5) we introduce the following similarity transformations

$$\psi(x,y) := Ax^\alpha f(t), \quad t := B\frac{y}{x^\beta}. \tag{6}$$

where A, B, α and β are real numbers, f is the transformed dimensionless stream function and t is the similarity variable. In terms of (6), equation (4) can satisfy the ordinary differential equation of the shape function : f

$$(|f''|^{n-1}f'')' + \alpha f f'' = (\alpha - \beta)f'^2, \tag{7}$$

where the primes denote differentiation with respect to t, if and only if the following

$$\alpha(2-n) + \beta(2n-1) = 1, \text{ and } \quad \alpha - \beta = m,$$

holds, and the parameters A, B and ν satisfy

$$\nu A^{n-2}B^{2n-1} = 1, \quad \text{and} \quad m = \alpha - \beta. \tag{8}$$

It follows that

$$\alpha = \frac{1+m(2n-1)}{n+1}, \quad \beta = \frac{1+m(n-2)}{n+1}, \quad \text{and} \quad p = \frac{m(2n-1)-n}{n+1}.$$

Consequently, we have

$$\psi(x,y) := \nu^{\frac{1}{n+1}} x^{\frac{1+m(2n-1)}{n+1}} f(t), \quad t := \nu^{-\frac{1}{n+1}} y x^{-\frac{1+m(n-2)}{n+1}}. \tag{9}$$

The corresponding boundary conditions (5) are expressed as

$$f(0) = \frac{-v_w}{A\alpha}, \quad f'(0) = \frac{u_w}{A}, \quad f'(\infty) = \lim_{t\to\infty} f'(t) = 0. \tag{10}$$

In the remainder, we deal with the following problem

$$\begin{cases} (|f''|^{n-1}f'')' + \alpha f f'' - m f'^2 = 0, \\ f(0) = a, \quad f'(0) = b, \quad f'(\infty) = \lim_{t\to\infty} f'(t) = 0. \end{cases} \tag{11}$$

For Newtonian fluids $(n = 1)$, problem (11) reads

$$\begin{cases} f''' + \alpha f f'' - m f'^2 = 0, \\ f(0) = a, \quad f'(0) = b, \quad f'(\infty) = 0. \end{cases} \tag{12}$$

We notice that this problem arises from two different contexts in fluid mechanics when looking for similarity solutions. First, in natural convection along a vertical heated flat plate, embedded in a saturated porous medium, where the temperature is a power function with the exponent m, for more details, we refer the reader to [13, 14, 16] and the references therein. Equation (12) appears also in the study of the boundary layer flow, of a Newtonian fluid, adjacent to a stretching surface with a power-law velocity (see [15, 17]). In [14], the authors proved that (12) with $a = b = 0$, has a solution (which is bounded) if $m \geq -\frac{1}{3}$ and this solution is unique for $0 \leq m \leq \frac{1}{3}$. In [15] the author gives a complete study about existence and nonexistence of solutions to (12), where $b = 1$.

Recently some new results have been obtained in [13]. The author considered the problem (12), where $m \in (-\alpha, 0)$. He showed that, under some assumptions, problem (12) has an infinite number of unbounded solutions and these solutions satisfy $f(t) \sim t^{\frac{\alpha}{\alpha-m}}$, as t goes to infinity.

Based on the results of [13], the interest in this work will be in existence and asymptotic behavior of solutions of problem (11).

3 Preliminary results

As it is announced above, the existence of solutions will be established by a shooting method. We replace the boundary condition at infinity by $f''(0) = d$, where $d \neq 0$. Therefore, we consider the initial value problem

$$\begin{cases} (|f''|^{n-1}f'')' + \alpha f f'' - m f'^2 = 0, \\ f(0) = a, \quad f'(0) = b, \quad f''(0) = d. \end{cases} \tag{13}$$

we shall see that for appropriate d problem (13) has a global unbounded solution and this solution satisfies the boundary condition at infinity.

Remark 1 We notice that for $n \neq 1$, equation $(11)_1$ can be degenerate or singular at the point t_0 where $f''(t_0) = 0$. The existence the t_0 is done for $d > 0$. We shall see also that f''' is not bounded at t_0 (the solution f is then not classical). By a solution to (11) we will mean a function $f \in C^2(0,\infty)$ such that $|f''|^{n-1}f'' \in C^1(0,\infty)$, $f'(\infty) = 0$ and $f''(\infty) = 0$. Note also that any solution is classical on any interval where the second derivative does not change the sign.

Consider now the initial value problem (13) with $n > 1, a, d \in \mathbb{R}, b \geq 0$ and $m \in (-\alpha, 0)$.

By the classical theory of ordinary differential equations the above problem has local (maximal) solutions on some interval $(0, T_d), T_d \leq \infty$ and they are uniquely determined by d $(d \neq 0)$. Let us denote this such solution by f_d. Integrating $(13)_1$ to het the following identity

$$|f_d''|^{n-1} f_d''(t) + \alpha f_d'(t) f_d(t) = |d|^{n-1} d + \alpha ab + (m + \alpha) \int_0^t f_d'(s)^2 ds, \quad \forall\, t < T_d. \tag{14}$$

which will be used later for proving some results.

A solution f_d of (11), is of class C^2 on $[0, T_d)$, and satisfies $|f_d''|^{n-1} f_d'' \in C^1([0, T_d))$. We shall investigate whether f_d admits an entire extension. First, we give the following result characterizing the existence time T_d.

Proposition 2 *Let f_d be the local solution to (13), if T_d is finite then the functions f_d, f_d' and f_d'' are unbounded as t approaches T_d from below.*

Proof. Similar to [15, 18]. ■

Let us note also that if we require a classical solution of (13) (ie. $f \in C^3(0, \infty)$), it is possible that f ceases to exist at some $T < \infty$ and such that f, f' and f'' remain bounded on $[0, T)$. More precisely we have the following result.

Proposition 3 *Let f_d be the local solution to (13) where $n > 1$ ans $d \neq 0$. Assume that there exists $t_0 \in (0, T_d)$ such that $f_d''(t_0) = 0$. Then $d > 0$, $f_d'' < 0$ on (t_0, T_d) and f_d''' is unbounded on $(0, t_0)$.*

Proof. Assume first that $d < 0$. Therefore $f_d'' < 0$ on $[0, \varepsilon), \varepsilon$ small, and the following equation

$$n|f_d''|^{n-1} f_d'' + \alpha f f'' - m f'^2 = 0, \tag{15}$$

holds on $(0, \varepsilon)$. Hence

$$(f_d'' e^F)' = \frac{m}{n} e^F |f_d''|^{1-n} f_d'^2, \quad \text{on} \quad (0, \varepsilon), \tag{16}$$

where

$$F(t) = \frac{\alpha}{n} \int_0^t f_d |f_d''|^{1-n}(s) ds.$$

Consequently, the function $t \to f_d'' e^F(t)$ is decreasing, and then $f_d''(t)$ remains negative for all $t \in [0, T_d)$. A contradiction. Then $d > 0$. Actually, we have $f_d'' > 0, f_d' > b$ on $(0, t_0)$ and equation (15) holds on $(0, t_0)$. Assume now that f_d''' is bounded on $(0, t_0)$. Thanks to equation $(11)_1$ we deduce that $f_d'(t_0) = 0$ this is contradiction with $f'(0) > b$. ■

From the above we can see, in particular, that $f_d'' < 0$ on $(0, T_d)$ for any $d < 0$. Then $f_d \in C^\infty([0, T_d))$. While for the case $d > 0$ the solution f_d is not classical.

Proposition 4 *Let f_d be the local solution to (13) for $d \neq 0$ and $n > 1$. If $T_d < \infty$ then $\lim_{t \to T_d} f_d(t) = -\infty$.*

Proof. First we show that $\sup_{[0, T_d)} |f_d(t)| = \infty$. Suppose not and $f_d''(t_0) = 0$ holds, for some $t_0 \in (0, T_d)$. From (14) we get

$$-(-f_d'')^n(t) + \alpha f_d'(t) f_d(t) = \alpha f_d'(t_0) f_d(t_0) + (m + \alpha) \int_{t_0}^t f_d'(s)^2 ds, \quad \forall\, t_0 < t < T_d.$$

Hence

$$\frac{\alpha}{2} f_d^2(t) - \alpha f_d'(t_0) f_d(t_0)(t - t_0) - \frac{\alpha}{2} f_d^2(t_0) = (m + \alpha) \int_{t_0}^t \int_{t_0}^\tau f_d'^2(s) ds d\tau + \int_{t_0}^t (-f_d'')^n(s) ds.$$

Since the right-hand side of the above is monotonic increasing with respect to t, the function f_d has a finite limit as $t \to T_d$. Consequently the function $(-f_d'')^n$ is integrable on (t_0, T_d). Since $n > 1$ we deduce that f_d'' is also integrable on (t_0, T_d). Therefore the function f_d' is bounded. Next we use (14) to deduce that f_d''

is also bounded. A contradiction with Proposition 2.
It remains to prove that the hypothesis $f_d'' > 0$ on $(0, T_d)$ leads also to a contradiction. Actually, in such situation, we know that f_d is classical and satisfies (15), which yields to

$$(f_d'')^{n-2} f_d''' \leq -\frac{\alpha}{n} f_d,$$

and then

$$(f_d'')^{n-2} f_d''' \leq \frac{\alpha}{n} \sup_{[0,T_d)} |f_d(t)|.$$

Therefore f_d'' and f_d' are bounded. A contradiction.Because f_d is monotonic on (τ, T_d), for some $0 < \tau < T_d$, we deduce that $|f_d(t)|$ goes to infinity as $t \to T_d$. Finally, to show that $f_d(t) \to -\infty$ as $t \to T_d$ we assume on the contrary that $\lim_{t \to T_d} f_d(t) = \infty$. Hence the functions f_d and f_d' are positive on (τ, T_d). Moreover, using $(11)_1$, we can deduce from the Energy-function defined by

$$E(t) = \frac{n}{n+1} |f_d''(t)|^{n+1} - \frac{m}{3} {f_d'}^3, \tag{17}$$

and satisfies $E'(t) = -\alpha f_d {f_d''}^2 \leq 0$. That f_d'' and f_d' are bounded. Hence f_d is also bounded and this is a contradiction with Proposition.2. We conclude that if T_d is finite the function f_d goes to minus infinity as $t \to \infty$. ■

4 Existence of solutions

In this section we shall obtain a sufficient condition on d such that the local solution f_d of (13) is global and satisfies the condition $f_d'(\infty) = 0$. We show that, for each d satisfying $|d|^{n-1}d > -\alpha ab,$ f_d exists on the entire positive axis $\mathbb{R}^+$ and satisfies $f_d'(\infty) = 0$. We begin by a simple observation that: if $m + \alpha > 0$ and $|d|^{n-1}d > -\alpha ab$, (14) yields the important fact that f_d cannot have a local maximum. Thus we prove the following result.

Theorem 5 *Let $a \in \mathbb{R}, b \geq 0$ and $m \in (-\alpha, 0)$. For any d such that $|d|^{n-1}d > -\alpha ab$, there exists a unique global solution f_d, to (13), which goes to infinity with t, and its first and second derivative tend to 0 as t approaches infinity.*

For our analysis, we need to distinguish two cases for the parameter $a = f_d(0)$; namely $a \geq 0$ and $a < 0$. First we prove the following lemma.

Lemma 6 *If $a \geq 0$ and $|d|^{n-1}d > -\alpha ab$ the functions f_d' and f_d are positive on $(0, T_d)$ and $T_d = \infty$; that is f_d is global. Moreover f_d' and f_d'' are bounded.*

Proof. Because $|d|^{n-1}d + \alpha ab > 0$, the first assertion of the lemma is immediate from (14). To demonstrate that $T_d = \infty$ it suffices to show that f_d remains bounded on any bounded interval $[0, T]$. Let us consider the Lyapunov function E for f_d defined by (17). Since

$$E'(t) = -\alpha f_d {f_d''}^2 \leq 0,$$

thanks to $(11)_1$, it is seen that

$$\frac{n}{n+1} |f_d''(t)|^{n+1} - \frac{m}{3} f_d'(t)^3 \leq \frac{n}{n+1} |d|^{n+1} - \frac{m}{3} b^3, \quad \forall t < T_d.$$

this in turn implies that f_d'', f_d' and then f_d are bounded on $[0, T]$. ■

Lemma 7 *If $a \geq 0$ and $|d|^{n-1}d > -\alpha ab$, $f_d(t)$ tends to infinity with t, f_d' and f_d'' tend to zero as $t \to \infty$.*

Proof. Since f'_d is monotonic on (t_1, ∞), t_1, large enough, and bounded there exists a $l \geq 0$ such that

$$\lim_{t\to\infty} f'_d(t) = l.$$

This implies the existence of a sequence (t_n) tending to infinity with n satisfying $\lim_{n\to+\infty} f''_d(t_n) = 0$ and then $\lim_{t\to\infty} f''_d(t) = 0$, with the help of the energy function E.
Now we assume that f_d is bounded, therefore $l = 0$. Subsequently

$$|d|^{n-1}d + \alpha ab + (m+\alpha)\int_0^\infty f'_d(t)^2 dt = 0.$$

This is impossible. Therefore f_d is unbounded and then $\lim_{t\to+\infty} f_d(t) = \infty$. It remains to prove that $l = 0$. Assume on the contrary that $l > 0$. Together with (14) we get

$$|f''_d|^{n-1} f''_d(t) = -\alpha l^2 t + (m+\alpha) l^2 t + o(t),$$

$$|f''_d|^{n-1} f''_d(t) = m l^2 t + o(t),$$

as t approaches infinity, that This is only possible if $m = 0$. Consequently $l = 0$. ■

Next we consider the case $a < 0$. The first simple consequence is that $f_d(t) < 0$ and $f'_d(t) > 0$ for small $t > 0$. Since f_d cannot have a local maximum, we have two possibilities

- Either $f_d(t)$ vanishes at a some point and remains positive after this point.
- Or $f_d(t) < 0$ for all $t > 0$.

Hence the proof of Theorem 5 is completed by the following lemma.

Lemma 8 *Assume $a < 0$ and $|d|^{n-1}d > -\alpha ab$. Then f_d has exactly one zero, goes to ∞ with t, and the functions f'_d, f''_d converge to 0 as $t \to \infty$.*

Proof. Assume that the first assertion holds. Since f'_d is positive we deduce that f_d is bounded and then is global. On the other hand, using (14) one sees that $f''_d > 0$. Therefore we get $\lim_{t\to\infty} f_d(t) \in (a, 0]$ and $\lim_{t\to\infty} f'_d(t) = 0$, since f'_d is monotonic. This is absurd since f'_d is positive and increasing function. Hence f_d has exactly one zero, say t_0. To finish the Proof of Lemma 8 and therewith that of Theorem 5 we note that the new function

$$h(t) = f_d(t + t_0)$$

satisfies equation $(11)_1$ and

$$h(0) \geq 0, \quad h''(0) > -\alpha h(0)h'(0).$$

Therefore we use Lemmas 6 and 7 to conclude. ■
In the next result we complete our analysis on the existence of global solutions by the case $b < 0$.

Theorem 9 *Let $b < 0, a > 0$ and $m \in (-\alpha, 0)$. For any $d > 0$ satisfying*

$$ad^n - \frac{1}{2}b^2 d^{n-1} + \alpha a^2 b > 0. \tag{18}$$

The unique local solution, f_d to (13) is global unbounded and satisfies $\lim_{t\to\infty} f'_d(t) = \lim_{t\to\infty} f''_d(t) = 0$.

Proof. Since $a, d > 0$ and $b < 0$, there exists a real $t_0 > 0$ such that f_d is positive, decreasing and convex on $(0, t_0)$. Define

$$T = \sup\left\{t : f_d(s) > 0, f'_d(s) < 0, f''_d(s) > 0, \text{ for all } s \in (0,t)\right\}.$$

The real number T is larger than t_0 and may be infinite.
Assume that $T = \infty$. Then the function f_d has a finite limit at infinity and $f'_d(t)$ (and f''_d) go to zero as $t \to \infty$. Since the function

$$H = f_d|f''_d|^{n-1}f''_d - \frac{1}{2}f'^2_d|f''_d|^{n-1} + \alpha f_d^2 f'_d,$$

satisfies

$$H' = f(f_d')^2\left[m + 2\alpha + \frac{\alpha(n-1)}{2n}\right] - \frac{m(n-1)}{2n}(f_d')^4(f_d'')^{-1},$$

thanks to $(11)_1$, we deduce that H is increasing on $(0,\infty)$. Hence for $t > 0$ we have

$$H(0) < \lim_{t\to\infty} H(t) = 0,$$

which yields to

$$ad^n - \frac{1}{2}b^2d^{n-1} + \alpha a^2 b < 0.$$

A contradiction. Therefore T is infinite. Next, we assume that $f_d(T) = 0$ or $f_d''(T) = 0$. Arguing as above we deduce $H(0) < 0$ and then we get a contradiction. In conclusion if condition (18) holds the function f_d has a local positive minimum at some $t_1 > 0$. We use Theorem 5 to deduce that the new function $h(t) = f_d(t + t_1)$ is global, unbounded and satisfies $h'(\infty) = h''(\infty) = 0$. The proof is finished. ■

Remark 10 We notice that we can extend the results of Theorem 5 to the case $(-2\alpha, -\alpha)$ by using the function H defined above, as in the work by Guedda [13] for the Newtonian case.

5 Asymptotic behavior

In this section we shall derive the asymptotic behavior of any possible global unbounded solution to (11) for $m \in (-2\alpha, 0)$. First we give the following result

Lemma 11 *Let f be a positive solution to (11) for $m \in (-2\alpha, 0)$. Then f' goes to zero at infinity and f'' is negative.*

Proof. Since f is monotonic on $[t_0,\infty)$, t_0 large enough, we get the positivity of f' and f on (t_0,∞). In addition we use the Lyapunov function to get the boundedness of f' and f''. Arguing as in the previous section we get that $f' \to 0$ and $f'' < 0$ for large t. ■

Proposition 12 *Assume that $n > 1$ and $m \in (-2\alpha, 0)$. Let f be a positive solution to (11). Then*

$$\lim_{t\to\infty} f_d(t)f_d''(t) = \lim_{t\to\infty}\left(|f''|^{n-1}f''\right)'(t) = 0.$$

Proof. Thanks to lemma (11) we have $f'(t) > 0, f''(t) < 0$ for all $t > t_0$, t_0 large enough and f' and f'' tend to 0 as $t \to \infty$. Then equation $(11)_1$ can be written as

$$f''' + \frac{\alpha}{n}ff''|f''|^{1-n} = \frac{m}{n}f'^2|f''|^{1-n}, \quad \forall t > t_0.$$

By differentiation we have

$$f^{(iv)} + f'''\left[\frac{\alpha(2-n)}{n}|f''|^{1-n}f - \frac{m(1-n)}{n}|f''|^{-n-1}f''f'^2\right] = -\frac{\alpha-2m}{n}f'f''|f''|^{1-n}. \tag{19}$$

Then the function $f'''e^G$ is monotonic increasing on (t_0,∞), where

$$G' = \frac{\alpha(2-n)}{n}|f''|^{1-n}f - \frac{m(1-n)}{n}|f''|^{-n-1}f''f'^2.$$

This indicates that the function f''' has at most one zero. Because f'' is negative and goes to 0 at infinity, we deduce that $f'''(t) > 0$ on (t_1,∞), for t_1 large. On the other hand, from $(11)_1$ we deduce

$$(|f''|^{n-1}f'')'' + (\alpha - 2m)f'f'' = -\alpha ff'''.$$

Therefore the function $t \longmapsto (|f''|^{n-1}f'')' + \frac{\alpha-2m}{2}f'^2$ is positive and monotonic decreasing on $(\inf\{t_0,t_1\},\infty)$. Together with the fact that f' tends to 0 as $t \to \infty$ we deduce that

$$\lim_{t\to+\infty}\left(|f''|^{n-1}f''\right)'(t) = 0$$

and then we conclude that $ff''(t) \to 0$ as $t \to \infty$, thanks to $(11)_1$. ■

Proposition 13 *Let f be a solution to (11) where $n > 1$, $m \in (-2\alpha, 0)$. Then*

$$\lim_{t\to+\infty} ff''(t) = \begin{cases} \infty, & \text{if } m+\alpha > 0, \\ L \in (0,\infty), & \text{if } m+\alpha = 0, \\ 0, & \text{if } m+\alpha < 0. \end{cases}$$

Proof. Let f be a global solution to (11). First we claim that there exists $t_0 \geq 0$ such that

$$|f''|^{n-1}f''(t_0) + \alpha ff'(t_0) > 0.$$

Suppose not; that is

$$|f''|^{n-1}f''(t) + \alpha ff'(t) \leq 0,$$

for all $t \geq 0$. Since $f''(t) \to 0$ as $t \to \infty$ the following

$$f'' + \alpha ff' \leq 0$$

holds on some $(t_1, \infty), t_1$ large. Consequently the function $f' + \frac{\alpha}{2}f^2$ is decreasing and goes to infinity with t, which is absurd. Now we use the identity

$$|f''|^{n-1}f''(t) + \alpha ff'(t) = |f''|^{n-1}f''(t_0) + \alpha ff'(t_0) + (m+\alpha)\int_{t_0}^{t} f'^2(s)ds,$$

to deduce that ff' has a limit $L \in [0,\infty]$ at infinity. This limit is finite for $\alpha + m = 0$. Assume that $\alpha + m \neq 0$. If $L \in (0,\infty)$ we get immediately that $f' \sim \sqrt{\frac{L}{\eta}}$ at infinity which implies that $ff' \to \infty$. A contradiction. Consequently $L \in \{0,\infty\}$, we use again the above identity to conclude that $L = \infty$ if $m + \alpha > 0$ and $L = 0$ if $m + \alpha < 0$. ■

Remark 14 We stress that the condition

$$|f''|^{n-1}f''(t_0) + \alpha ff'(t_0) > 0, \quad f'(t_0) \geq 0$$

is necessary and sufficient to obtain a global solution converging to plus infinity with t in the case $m \in (-\alpha, 0)$.

Now we are ready to give the result concerning the large $t-$behavior of solutions to (11).

Theorem 15 *Suppose $n > 1, -2\alpha < m < 0$. Let f be a solution to (11) such that $f \to \infty$. Then there exists a constant, $A > 0$, such that*

$$f(t) = t^{\frac{\alpha}{\alpha-m}}(A + o(1)), \tag{20}$$

as $t \to \infty$.

Proof. Let f be a global solution to (11). First we prove the result for the case $m + \alpha > 0$.
Let t_0 be a real number such that $f'' < 0$ and $f''' > 0$ on (t_0, ∞). Dividing equation $(11)_1$ by ff' gives

$$\frac{(|f''|^{n-1}f'')'}{ff'} = m\frac{f'}{f} - \alpha\frac{f''}{f'}.$$

Integrating over (t_1, t), for $t_1 > t_0$, gives

$$\int_{t_1}^{t} \frac{(|f''|^{n-1}f'')'}{ff'}ds = \log\left(f^m(t)f'^{-\alpha}(t)\right) - \log\left(f^m(t_1) - f'^{-\alpha}(t_1)\right).$$

According to Proposition 13, ff' goes to infinity with t and then the left hand side of the above is integrable, consequently $f^m f'^{-\alpha}$ has a positive finite limit at infinity. The desired asymptotic behavior (20) follows by a simple integration. Now we deal with the case $m + \alpha < 0$. For this sake we define

$$\Psi = \varphi(f)|f''|^{n-1}f'' - \frac{1}{2}\varphi'(f)(f')^2|f''|^{n-1} + \alpha\varphi(f)ff',$$

where φ is a smooth function. Then if follows from $(11)_1$

$$\Psi'(t) = f'^2\left[\alpha f\varphi'(f) + (\alpha+m)\varphi\right] - \frac{1}{2}\varphi''(f)f'^3|f''|^{n-1} - \frac{n-1}{2}\varphi'(f)f'^2|f''|^{n-3}f''f'''.$$

Let the function φ be defined by

$$\varphi(s) = s^{-\frac{m+\alpha}{\alpha}}.$$

It satisfies the following differential equation

$$\alpha s\varphi'(s) + (\alpha+m)\varphi = 0.$$

This implies that

$$\Psi' = -\frac{1}{2}\varphi''(f)f'^3|f''|^{n-1} - \frac{n-1}{2}\varphi'(f)f'^2|f''|^{n-3}f''f''' \geq 0,$$

$$\Psi = \varphi(f)\left[|f''|^{n-1}f'' - \frac{\alpha+m}{2\alpha f}(f')^2|f''|^{n-1} + \alpha ff'\right],$$

and then

$$|\Psi'(t)| \leq \varepsilon\left[f^{-\frac{3\alpha+m}{\alpha}}f' + (n-1)(-f'')^{n-2}f'''\right],$$

for all $t \geq t_0, t_0$ large. Therefore Ψ' is integrable on $[0,\infty)$ and then Ψ has a finite limit at infinity, say L.

Next we show that $L > 0$. It will be sufficient to show that $\Psi(t_1) > 0$ for some t_1 large. Suppose not; that is for any $t > t_2,\ t_2$ large we have

$$|f''|^{n-1}f'' - \frac{\alpha+m}{2\alpha f}f'^2|f''|^{n-1} + \alpha ff' \leq 0.$$

Since $\alpha + m < 0$ then

$$f'' + \alpha ff' \leq 0,$$

from which we deduce, as above, that $f' + \frac{\alpha}{2}f^2$ is a decreasing function going to infinity with t. A contradiction. We conclude that $\lim_{t\to\infty} f^{-\frac{m}{\alpha}}f' = \frac{L}{\alpha}$. Finally, a simple integration leads to estimate (20).
To finish, we pay attention to the case $m = -\alpha$. Here the identity (14) leads to

$$|f''|^{n-1}f'' + \alpha ff' = |\gamma|^{n-1}\gamma + \alpha ab.$$

According to Theorem 5, f is global and satisfies $f \sim t^{\frac{1}{2}}$ at infinity. ■

6 Conclusion

The laminar two-dimensional steady boundary layer flow, of a non-Newtonian incompressible fluid, over a stretching surface have been considered. Using the shooting method, existence of global unbounded similarity solutions have been shown, the dependency of those solutions on the power-law index have been investigated, and their asymptotic behavior was also discussed.
Coming back to the original problem (2),(3) we find that, for $-2\alpha < m < 0$, the stream function satisfies

$$\psi(x,y) \sim y^{\frac{1+m(2n-1)}{1+m(n-2)}} \quad \text{as} \quad yx^{\frac{(2-n)m-1}{n+1}} \to \infty.$$

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