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Exact Solitary Wave Solutions of Equal Width Wave and Related Equations Using a Direct Algebraic Method

K. R. Raslan *
Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr-City, Cairo, Egypt
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Abstract: New exact solutions of some important nonlinear partial differential equations in one and two dimensions are obtained by using the direct algebraic method. The applicability of the method is demonstrated by applying it for the equal width wave (EW) equation, modified equal width wave (MEW) equation, improved Korteweg de Vries (IKdV) equation, modified regularized long wave (MRLW) equation, two dimensions Korteweg de Vries and two dimensions equal width wave equation

Keywords:direct algebraic method; solitary wave equations

1 Introduction

In various fields of science and engineering, many problems can be described by nonlinear partial differential equations (PDEs). The study of numerical methods for the solution of partial differential equations has enjoyed an intense period of activity over the last 40 years from both theoretical and practical points of view. Improvements in numerical techniques, together with the rapid advance in computer technology, have meant that many of the PDEs arising from engineering and scientific applications, which were previously intractable, can now be routinely solved [1]. In finite difference methods differential operators are approximated and difference equations are solved. In the finite element method the continuous domain is represented as a collection of a finite number N of subdomains known as elements. The collection of elements is called the finite element mesh. The differential equations for time dependent problems are approximated by the finite element method to obtain a set of ordinary differential equations (ODEs) in time. These differential equations are solved approximately by finite difference methods. In all finite difference and finite elements it is necessary to have boundary and initial conditions. However, the Adomian decomposition method, which has been developed by George Adomian [2], depends only ***on*** the initial conditions and obtains a solution in series which converges to the exact solution of the problem. In recently years, other ansatz method have been developed, such as the tanh method [3,4,5], extended tanh function method [6,7], the modified extended tanh function method [8], the generalized hyperbolic function [8,9,10], the variable separation method [11,12], and the first integral method[13-17]. In recent paper [18], they used the first integral method to find the new exact solutions of the linear Klein-Gordan equation, the MKdV equation, the Burgers' equation in two and three dimensions which are useful in the numerical studies. The technique we use in this paper is due to Hereman et al. [19]. By this method, solutions are developed as series in real exponential functions which physically corresponds to mixing of elementary solutions of the linear part due to nonlinearity. The method of Hereman et al. [19] falls into the category of direct solution methods for nonlinear partial differential equations. This method is currently restricted to traveling wave solutions. In addition, depending on the number of NL terms in the PDE with arbitrary numerical coefficients, it is sometimes necessary to specialize to particular values of the velocity in order to find closed form solutions. On the other hand, the Hereman et al. series method does give a systematic means of developing recursion relations. Any finite number of the coefficients in the series can be found manually or with the aid of

* **Corresponding author**. *E-mail address*: kamal_raslan@yahoo.com

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symbolic packages such as Mathematica. Hereman et al. direct series method can be used to solve both dissipative and nondissipative equations [19]. They take solutions of the linear equation to be of the form $\exp[-k(c)\,(x-c\,t)]$ where $k(c)$a function of the velocity is c. The velocity is assumed constant but in general is related to the wave amplitude. It is from the solutions of the linear part that the solution of the full NL PDE is synthesized. With wave number$k$, the dispersion relation $\omega = k(c)$gives the angular frequency. In this paper we apply the direct algebraic method for solving the important solitary wave equations in one and two dimensions as EW, MEW, IKdV, MRLW, 2D KdV and 2D EW equations.

2 The direct algebraic method (DAM)

Consider the nonlinear PDE:

$$F(u, u_t, u_x, u_y, u_{xx}, u_{xy}, ...) = 0, \tag{1}$$

where $u(x, y, t)$is the solution of the Eq. (1). We use the transformations

$$u(x, y, t) = f(\xi),\, \xi = x + \beta\, y - c\, t, \tag{2}$$

where c and β are constants. Based on this we obtain

$$\frac{\partial}{\partial t}(.) = -c\,\frac{d}{d\xi}(.)\,,\,\frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.)\,,\frac{\partial}{\partial y}(.) = \beta\,\frac{d}{d\xi}(.)\,,\,... \tag{3}$$

We use (3) to change the PDE (1) to ODE:

$$G(f,\ f_\xi, f_{\xi\xi}, ...) = 0, \tag{4}$$

Next, we apply the approach of Hereman et al. [19]. We solve the linear terms and then suppose the solution in the form

$$f(\xi) = \sum_{n=1}^{\infty} a_n\, g^n(\xi), \tag{5}$$

where $g(\xi)$ is a solution of linear terms and the expansion coefficients $\{a_n\}_{n=1}^{\infty}$are to be determined. To deal with the nonlinear terms, we need to apply the extension of Cauchy's product rule for multiple series.

Lemma 1 *(Extension of Cauchy's product rule) If*

$$F^{(i)} = \sum_{n_1}^{\infty} a_{n_1}^{(i)},\ \ i = 1, ..., I, \tag{6}$$

represents I infinite convergent series then

$$\prod_{i=1}^{I} F^{(i)} = \sum_{n=1}^{\infty}\sum_{r=I-1}^{n-1} \,.....\, \sum_{m=2}^{k-1}\sum_{l=1}^{m-1} a_l^{(1)}\, a_{m-l}^{(2)} .... a_{n-r}^{(I)}, \tag{7}$$

Proof. See [19]. ■

Substituting (5) into (4) yields recursion relation which gives the values of the coefficients.

3 Applications

In this section, we discuss the problems which involving some solitary wave equations in one and two dimensions by using the direct algebraic method described in section 2.

3.1 One dimension solitary wave equations

We start with solitary wave equation in one dimension which are EW, MEW, IKdV and MRLW equations

3.1.1 The equal width wave (EW) equation

We first start with the equal width wave (EW) equation in the form [5, 21-27]

$$u_t + u\,u_x - u_{xxt} = 0, \tag{8}$$

Using (2) and (3) Eq. (8) becomes

$$-c\frac{df(\xi)}{d\xi} + (f(\xi))\frac{df(\xi)}{d\xi} + c\frac{d^3 f(\xi)}{d\xi^3} = 0, \tag{9}$$

Integrating (9) gives

$$-cf(\xi) + \frac{(f(\xi))^2}{2} + c\frac{d^2 f(\xi)}{d\xi^2} = 0, \tag{10}$$

The constant of integration equal zero since the solitary wave solution and its derivatives equal zero as $\xi \to \pm\infty$. The linear equation from (10) has the solution in the form$f(\xi) = e^{k\,\xi}$, $k = \pm 1$ these solutions are obtained by direct integration of the linear part of (10). We define $g_1(\xi) = \frac{1}{g_2(\xi)} = g(\xi) = e^{k\,\xi}$ and let $f(\xi) = \sum\limits_{n=1}^{\infty} a_n\, g^n(\xi)$. From (5) and (10), we deduce the recursion relation (RR) for $n \geq 2$:

$$c(k^2\,n^2 - 1)\,a_n + \frac{1}{2}\sum_{m=1}^{n-1} a_{n-m}\,a_m = 0,\ n \geq 2, \tag{11}$$

where a_1 is an arbitrary coefficient, by applying the RR, we find the first few coefficients to be given by

$$\begin{aligned} a_2 &= -\tfrac{a_1^2}{6\,c}, \\ a_3 &= \tfrac{a_1^3}{48\,c^2}, \\ a_4 &= -\tfrac{a_1^4}{432\,c^3}, \\ a_5 &= \tfrac{5\,a_1^5}{20736\,c^4}, \\ &\ldots \end{aligned} \tag{12}$$

In general,

$$a_n = \frac{(-1)^{n-1}\,n\,a_1^n}{2^{2n-2}\,3^{n-1}\,c^{n-1}}, \tag{13}$$

Substituting (13) into (5) gives

$$f(\xi) = \sum_{n=1}^{\infty} a_n\, g_1^n(\xi) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,n\,a_1^n}{2^{2n-2}\,3^{n-1}\,c^{n-1}}\, g_1^n(\xi), \tag{14}$$

This series converges to the value

$$f(\xi) = \frac{144\,c^2\,e^{\xi}\,a_1}{(12\,c\,e^{\xi} + a_1)^2}, \tag{15}$$

Then the exact solution of Eq. (8) can be written in the form

$$u(x,t) = \frac{144\,c^2\,e^{x-c\,t}\,a_1}{(12\,c\,e^{x-c\,t} + a_1)^2}, \tag{16}$$

When we replace $g_1(\xi)$ by $g_2(\xi)$ in Eq. (14) we get another solution in the form

$$u(x,t) = \frac{144\,c^2\,e^{x-c\,t}\,a_1}{(12\,c + \,e^{x-c\,t} a_1)^2}, \tag{17}$$

In equations (16) and (17) if we choose $a_1 = 12\,c$ then the solitary wave solution of EW equation can be expressed as:

$$u(x,t) = 3\,c\,\sec h^2\left(\frac{1}{2}(x - c\,t)\right), \tag{18}$$

This solution is appears to be the same as obtained in other papers [5, 21-27]. Fig.1 shows the soliton solutions for the EW equation with increase time. Also, we calculate the conservation quantities as given in Table 1. It is noted that the obtained results are quite good comparing with that obtained in [5, 21-27].

3.1.2 The modified equal width wave (MEW) equation

Second, we consider the MEW equation in the form [22, 26]

$$u_t + u^2 u_x - u_{xxt} = 0, \tag{19}$$

Then $g_1(\xi) = \frac{1}{g_2(\xi)} = g(\xi) = e^{k\xi}$, $k = \pm 1$and the corresponding RR can be deduced to take the form

$$c\,(k^2 n^2 - 1)\, a_n + \frac{1}{3}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_{n-m}\, a_{m-l} a_l = 0,\ n \geq 3, \tag{20}$$

The general formula for the coefficients is given by,

$$a_n = \frac{(-1)^n\, a_1^{2n+1}}{2^{3n}\, 3^n\, c^n}, \tag{21}$$

Then, the exact solutions of Eq. (19) take the forms

$$u(x,t) = \frac{24c\, e^{x-ct}\, a_1}{24\, c + \ e^{2(x-ct)} a_1^2},\ \ u(x,t) = \frac{24c\, e^{x-ct}\, a_1}{a_1^2 + 24\, c\, e^{2(x-ct)}}, \tag{22}$$

If we choose $a_1 = \sqrt{24\, c}$, then the solitary wave solution of MEW equation can be given in the form

$$u(x,t) = \sqrt{6\, c}\, \sec h\, (x - c\, t)\,, \tag{23}$$

Fig.2 shows the soliton solution for the MEWE equation with increase time. Also, we can calculate the conservation quantities as given in Table 2. It is to be noted that the obtained results are quite good with that obtained [22,26].

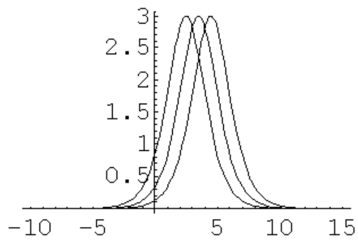

Figure 1: Soliton Solutions for EW Equation

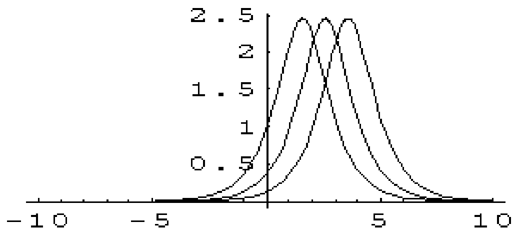

Figure 2: Soliton Solution for MEWE Equation

Table 1: The conservative quantities for the EW equation with increasing the time

T	I_1	I_2	I_3
0	12	28.8	57.6
1	12	28.8	57.6

Table 2: The conservative quantities the MEWE equation with increase time

T	I_1	I_2	I_3
0	7.6953	48	16
1	7.6953	48	16
2	7.6953	48	16

3.1.3 The improved Korteweg de Vries (IKdV) equation

We consider the IKdV equation in the form [31]

$$u_t + u\, u_x - u_{xxt} + u_{xxx} = 0, \tag{24}$$

Then $g_1(\xi) = \frac{1}{g_2(\xi)} = g(\xi) = e^{k\xi}$, $k = \sqrt{\frac{c}{c+1}}$and the corresponding RR takes the form

$$(\,(c+1)k^2 n^2 - c)\, a_n + \frac{1}{2}\sum_{m=1}^{n-1} a_{n-m}\, a_m = 0,\ n \geq 2, \tag{25}$$

Then, the general formula for the coefficients is deduced by

$$a_n = \frac{(-1)^{n-1} n\, a_1^n}{2^{2n-2}\, 3^{n-1}\, c^{n-1}}, \tag{26}$$

Then, the exact solutions of Eq. (24) take the forms

$$u(x,t) = \frac{144\, c^2\, e^{\sqrt{\frac{c}{c+1}}(x-c\,t)}\, a_1}{(12\, c + e^{\sqrt{\frac{c}{c+1}}(x-c\,t)} a_1)^2},\ u(x,t) = \frac{144\, c^2\, e^{\sqrt{\frac{c}{c+1}}(x-c\,t)}\, a_1}{(12\, c\, e^{\sqrt{\frac{c}{c+1}}(x-c\,t)} + a_1)^2}, \tag{27}$$

In Eq. (27), if we choose $a_1 = 12\, c$, then the solitary wave solution of IKdV equation can be expressed as:

$$u(x,t) = 3\, c\, \sec h^2 \left(\frac{1}{2} \sqrt{\frac{c}{c+1}} (x - c\, t) \right), \tag{28}$$

3.1.4 The modified regularized long wave (MRLW) equation

Consider the modified regularized long wave equation in the form [28-30, 32, 33]

$$u_t + u_x + u^2\, u_x - u_{xxt} = 0, \tag{29}$$

Then $g_1(\xi) = \frac{1}{g_2(\xi)} = g(\xi) = e^{k\,\xi},\ k = \sqrt{\frac{c-1}{c}}$and the corresponding RR can be expressed as:

$$(\, ck^2\, n^2 - c + 1)\, a_n + \frac{1}{3} \sum_{m=2}^{n-1} \sum_{l=1}^{m-1} a_{n-m}\, a_{m-l} a_l = 0,\ n \geq 3, \tag{30}$$

The general formula for the coefficients is given by,

$$\begin{aligned} a_{2\,n} &= 0, && n \geq 1, \\ a_{2n+1} &= \frac{a_1^{2\,n+1}}{2^{3n}\, 3^n\, (1-c)^n}, && n \geq 0, \end{aligned} \tag{31}$$

Then, the exact solutions of Eq. (29) can be deduced to take the forms

$$u(x,t) = \frac{(24 - 24\, c)\, e^{\sqrt{\frac{c-1}{c}}(x-c\,t)}\, a_1}{24 - 24\, c + e^{2\sqrt{\frac{c-1}{c}}(x-c\,t)} a_1^2},\ u(x,t) = \frac{(24 - 24\, c)\, e^{\sqrt{\frac{c-1}{c}}(x-c\,t)}\, a_1}{(24\, c - 24)\, e^{2\sqrt{\frac{c-1}{c}}(x-c\,t)} - a_1^2}, \tag{32}$$

If we choose $a_1 = \sqrt{24\, c - 24}$, then the solitary wave solution of MRLW equation is given by the form

$$u(x,t) = \sqrt{6\, c - 6}\, \sec h \left(\sqrt{\frac{c-1}{c}}\, (x - c\, t) \right), \tag{33}$$

3.2 Two dimensions solitary wave equations

In this section, we consider the two dimensions solitary wave equations.

3.2.1 The two dimension Korteweg de Vries (2D KdV) equation [20]

Consider the two dimensions Korteweg de Vries in the form

$$(u_t - 6\, u\, u_x + u_{xxx})_x + 3\, u_{yy} = 0, \tag{34}$$

We use the transformations

$$u(x,y,t) = f(\xi),\ \xi = x + \beta\, y - c\, t, \tag{35}$$

where c and β are constants, Eq. (34) reduces to the ODE:

$$(3\,\beta^2 - c)f(\xi) - 3\,f^2(\xi) + \frac{d^2 f(\xi)}{d\xi^2} = 0, \tag{36}$$

Then $g_1(\xi) = \frac{1}{g_2(\xi)} = g(\xi) = e^{k\,\xi},\ k = \sqrt{c - 3\,\beta^2}$and the corresponding RR takes the form

$$((3\,\beta^2 - c) + k^2\,n^2)\,a_n - 3\sum_{m=1}^{n-1} a_{n-m}\,a_m = 0,\ n \geq 2, \tag{37}$$

The general formula for the coefficients can be written as:

$$a_n = \frac{n\,a_1^n}{2^{n-1}\,(c - 3\beta^2)^{n-1}}, \tag{38}$$

Then, the exact solutions of Eq. (34) can be expressed in the forms

$$\begin{aligned} u(x,y,t) &= \frac{4\,e^{\sqrt{c-3\beta^2}(x+\beta\,y-c\,t)}(c-3\beta^2)\,a_1}{(-2\,c+e^{\sqrt{c-3\beta^2}(x+\beta\,y-c\,t)}a_1+6\,\beta^2)^2},\\ u(x,y,t) &= \frac{4\,e^{\sqrt{c-3\beta^2}(x+\beta\,y-c\,t)}(c-3\beta^2)\,a_1}{(a_1-2\,e^{\sqrt{c-3\beta^2}(x+\beta\,y-c\,t)}(c-3\,\beta^2))^2}, \end{aligned} \tag{39}$$

In Eq. (39), if we choose $a_1 = -2\,(c - 3\,\beta^2)$, then the solitary wave solution of 2D KdV equation takes the form

$$u(x,y,t) = \frac{(3\beta^2 - c)}{2}\ \sec h^2\left(\frac{\sqrt{c - 3\,\beta^2}}{2}\,(x + \beta\,y - c\,t)\right), \tag{40}$$

Also, if we consider the two dimensions Korteweg de Vries in the form

$$(u_t - 6\,u\,u_x + u_{xxx})_x - 3\,u_{yy} = 0, \tag{41}$$

Then, the exact solutions of Eq. (41) can be deduced in the forms

$$\begin{aligned} u(x,y,t) &= \frac{4\,e^{\sqrt{c+3\beta^2}(x+\beta\,y-c\,t)}(c+3\beta^2)\,a_1}{(2\,c-\,e^{\sqrt{c+3\beta^2}(x+\beta\,y-c\,t)}a_1+6\,\beta^2)^2},\\ u(x,y,t) &= \frac{4\,e^{\sqrt{c+3\beta^2}(x+\beta\,y-c\,t)}(c+3\beta^2)\,a_1}{(a_1-2\,e^{\sqrt{c+3\beta^2}(x+\beta\,y-c\,t)}(c+3\,\beta^2))^2}, \end{aligned} \tag{42}$$

In Eq. (42), if we choose $a_1 = -2\,(c + 3\,\beta^2)$, then the solitary wave solution of 2D KdV equation becomes in the form

$$u(x,y,t) = -\frac{(3\beta^2 + c)}{2}\ \sec h^2\left(\frac{\sqrt{c + 3\,\beta^2}}{2}\,(x + \beta\,y - c\,t)\right), \tag{43}$$

3.2.2 The two dimensions equal width wave (2D EW) equation

We consider the two dimensions equal width wave equation in the form

$$(u_t +\, u\,u_x - u_{xxt})_x +\, u_{yy} = 0, \tag{44}$$

We use the transformations

$$u(x,y,t) = f(\xi),\ \xi =\, x + \beta\,y - c\,t, \tag{45}$$

where c and β are constants, Eq. (44) changes to ODE:

$$(\,\beta^2 - c)f(\xi) + \frac{1}{2}\,f^2(\xi) + \frac{d^2 f(\xi)}{d\xi^2} = 0, \tag{46}$$

Then $g_1(\xi) = \frac{1}{g_2(\xi)} = g(\xi) = e^{k\,\xi},\; k = \sqrt{\frac{c-\beta^2}{c}}$ and the corresponding RR takes the form

$$((\beta^2 - c) + c\,k^2\,n^2)\,a_n + \frac{1}{2}\sum_{m=1}^{n-1} a_{n-m}\,a_m = 0,\; n \geq 2, \tag{47}$$

The general formula for the coefficients is given by,

$$a_n = \frac{(-1)^n n\,a_1^n}{2^{2(n-1)}\,(3)^{n-1}\,(c-\beta^2)^{n-1}}, \tag{48}$$

Then, the exact solutions of Eq. (44) take the forms

$$\begin{aligned} u(x,y,t) &= \frac{144\,e^{\sqrt{\frac{c-\beta^2}{c}}(x+\beta\,y-c\,t)}(c-\beta^2)^2\,a_1}{(12\,c+e^{\sqrt{\frac{c-\beta^2}{c}}(x+\beta\,y-c\,t)}a_1-12\,\beta^2)^2}, \\ u(x,y,t) &= \frac{144\,e^{\sqrt{\frac{c-\beta^2}{c}}(x+\beta\,y-c\,t)}(c-\beta^2)^2\,a_1}{(a_1+12\;e^{\sqrt{\frac{c-\beta^2}{c}}(x+\beta\,y-c\,t)}(c-\,\beta^2)^2}, \end{aligned} \tag{49}$$

In Eq. (49), if we choose $a_1 = 12\,(c - \;\beta^2)$, then the solitary wave solution of 2D EW equation is given by

$$u(x,y,t) = 3\,(c-\beta^2)\;\sec h^2\left(\frac{\sqrt{c-\beta^2}}{2\,\sqrt{c}}\,(x+\beta\,y-c\,t)\right), \tag{50}$$

Also, if we consider the two dimensions equal width wave equation in the form

$$(u_t +\,u\,u_x - u_{xxt})_x -\,u_{yy} = 0, \tag{51}$$

Then, the exact solution of Eq. (51) can be obtained to be

$$\begin{aligned} u(x,y,t) &= \frac{144\,e^{\sqrt{\frac{c+\beta^2}{c}}(x+\beta\,y-c\,t)}(c+\beta^2)^2\,a_1}{(12\,c+e^{\sqrt{\frac{c+\beta^2}{c}}(x+\beta\,y-c\,t)}a_1+12\,\beta^2)^2}, \\ u(x,y,t) &= \frac{144\,e^{\sqrt{\frac{c+\beta^2}{c}}(x+\beta\,y-c\,t)}(c+\beta^2)^2\,a_1}{(a_1+12\;e^{\sqrt{\frac{c-\beta^2}{c}}(x+\beta\,y-c\,t)}(c+\,\beta^2)^2}, \end{aligned} \tag{52}$$

In Eq. (52), if we choose $a_1 = 12\,(c + \;\beta^2)$, then the solitary wave solution of 2D EW equation takes the form

$$u(x,y,t) = 3\,(c+\beta^2)\;\sec h^2\left(\frac{\sqrt{c+\beta^2}}{2\,\sqrt{c}}\,(x+\beta\,y-c\,t)\right), \tag{53}$$

4 Conclusion

In this work the direct algebraic method was applied successfully for solving some solitary wave equations in one and two dimensions. Six Solitary waves' equations which are the EW, MEW, IKdV, MRLW, 2D KdV and 2D EW equations have been solved exactly. The direct algebraic method described herein is not efficient but also has the merit of being widely applicable. Thus, we deduced that the proposed method can be extended to solve many nonlinear partial differential equations problems which are arising in the theory of solitons and other areas.

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