

A Second-order Finite Difference Scheme for a Type of Black-Scholes Equation

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Abstract: In this paper we consider a backward parabolic partial differential equation, called the Black-Scholes equation, governing American and European option pricing. We present a numerical method combining the Crank-Nicolson method in the time discretization with a hybrid finite difference scheme on a piecewise uniform mesh in the spatial discretization. The difference scheme is stable for the arbitrary volatility and arbitrary asset price. It is shown that the scheme is second-order convergent with respect to both time and spatial variables. Numerical results support the theoretical results.

Key words: Black-Scholes equation; option valuation; Crank-Nicolson method; central difference scheme; piecewise uniform mesh

1 Introduction

An option is a financial contract that gives its owner the right to buy or sell a specified amount of a particular asset at a fixed price, called the exercise price, on or before a specified date, called the maturity date. Options that can be exercised at any time up to the maturity are called American, while options that can only be exercised on the maturity date are European. Options which provide the right to buy the underlying asset are known as calls, whereas options conferring the right to sell the underlying asset are referred to as puts. It was shown by Black and Scholes [1,7] that these option prices satisfy a second-order partial differential equation with respect to the time horizon t and the underlying asset price x . This equation is now known as the Black-Scholes equation, and can be solved exactly when the coefficients are constant or space-independent. However, in many practical situations, numerical solutions are normally sought. Therefore, efficient and accurate numerical algorithms are essential for solving this problem accurately. The lattice technique was proposed in Cox et al. [4] to solve the Black-Scholes equations and was improved in Hull and White [5]. That approach is equivalent to an explicit time-stepping scheme. Other numerical schemes based on classical finite difference methods applied to constant-coefficient heat equations have also been developed (cf. Rogers and Talay [9]; Schwartz [10]; Ali and Raslan [19]; Courtadon [3]; Cai and Huang [20]; Wilmott et al. [16]). The reason for this is that when the coefficients of the Black-Scholes equation are constant or space-independent, the equation can be transformed into a diffusion equation. However, when a problem is space-dependent, this transformation is impossible, and thus the Black-Scholes equation in the original form need to be solved.

The standard finite difference method is widely applied to valuating the option pricing problems, see Seydel [11], Tavella and Randall [14], Li and Xi [18], Cen et al. [17] and Wilmott et al. [16]. It is well known that when using the standard finite difference method to solve those problems involving the convection-diffusion operator, such as the Black-Scholes partial differential operator, numerical difficulty can be caused. The main reason is that when the volatility or the asset price is small, the Black-Scholes

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partial differential operator becomes a convection-dominated operator. Hence, the implicit Euler scheme with central spatial difference method will lead to nonphysical oscillations in the computed solution. This is due to a loss in stability. The implicit Euler scheme with upwind spatial difference method do not have this disadvantage, but this difference scheme is only first-order convergent. Wang [15] applied a fitted finite volume scheme to solve the Black-Scholes equation, and showed that the fitted volume scheme is also first-order convergent.

In this paper we present a numerical method combining the Crank-Nicolson method in the time discretization with a hybrid finite difference scheme on a piecewise uniform mesh in the spatial discretization. In the spatial discretization, our hybrid finite difference scheme uses central difference whenever the local mesh size is small enough to ensure the stability of the scheme; otherwise we use a simple upwind difference scheme. Without loss of generality, we shall discuss the method using the model for European options in our paper. Naturally, the method is applicable to American option if it is used together with a technique for free boundary problems. Our scheme is stable for the arbitrary volatility and the arbitrary asset price, and is second-order convergent with respect to both time and spatial variables. Another distinct feature of our method is that it can handle the degeneracy of the Black-Scholes differential operator at $x = 0$ without truncating the domain. Our hybrid difference scheme for the Black-Scholes equation is a modification of the difference scheme used in [12] and [13].

The rest of the paper is organized as follows. In the next section we discuss the continuous model of the Black-Scholes equations. In section 3 we prove the convergence of the Crank-Nicolson discretization. In section 4 we consider the spatial discretization based on a hybrid finite difference scheme on a piecewise uniform mesh. It is shown that the difference scheme is stable for the arbitrary volatility and the arbitrary asset price, and is second-order convergent with respect to both time and spatial variables. Numerical examples are presented in section 5. Finally discussion on results is indicated in section 6.

Notation 1 Throughout the paper, C will denote a generic positive constant (possibly subscripted) that is independent of the mesh. Note that C is not necessarily the same at each occurrence.

2 The continuous problem

Let V denote the value of a European call or put option and let x denote the price of the underlying asset. It is well known that V satisfies the following Black-Scholes equation (see, for example, Wilmott et al. [16]):

$$-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2(t)x^2\frac{\partial^2 V}{\partial x^2} - (r(t)x - D(x,t))\frac{\partial V}{\partial x} + rV = 0 \quad \text{for } (x,t) \in Q \quad (1)$$

with compatibility boundary and final (or payoff) conditions

$$V(0,t) = g_1(t), \quad V(X,t) = g_2(t), \quad t \in [0,T], \quad (2)$$

$$V(x,T) = g_3(x), \quad x \in \bar{\Omega}, \quad (3)$$

where $Q = \Omega \times (0,T)$, $\Omega = (0,X) \subset R$, $\sigma > 0$ denotes the volatility of the asset, $T > 0$ the expiry date, $r \geq 0$ the interest rate and D the dividend.

Remark 1 We remark that the option pricing problem (2.1)-(2.2) is usually posed in a semi-infinite domain (ie., $\Omega = (0,\infty)$). In this article, we truncate this infinite interval into $\Omega = (0,X)$ for applying the numerical method.

Wang[15] transform (2.1) with the non-homogeneous Dirichlet boundary conditions in (2.2) and (2.3) into one with the homogeneous boundary condition:

$$Lu \equiv -\frac{\partial u}{\partial t} + L_{x,t}u = f(x,t) \quad \text{for } (x,t) \in Q, \quad (4)$$

$$u(x,T) = g(x,T) \quad \text{for } x \in \bar{\Omega}, \quad (5)$$

$$u(0,t) = u(X,t) = 0 \quad \text{for } t \in [0,T], \quad (6)$$

where

$$L_{x,t}v = -a(t)x^2 \frac{\partial^2 v}{\partial x^2} - b(x,t)x \frac{\partial v}{\partial x} + c(t)v,$$

$a(t) = \frac{1}{2}\sigma^2(t)$, $b(x,t) = r(t) - \frac{D(x,t)}{x}$, $c(t) = r(t)$ and $f(x,t)$ are sufficiently smooth functions. We assume that $a(t) \geq \alpha > 0$, $\beta^* \geq b(x,t) \geq \beta > 0$. We also assume that the problem satisfies sufficient regularity and compatibility conditions which guarantee the problem has a unique solution $u(x,t) \in C^4(Q)$ satisfying u_{ttt} , xu_{xxx} and x^2u_{xxxx} are bounded on Ω . Our interest lies in constructing higher order numerical method for the Black-Scholes equation.

3 The time semidiscretization: the Crank-Nicolson method

To approximate the solution of (2.4), first we will use the Crank-Nicolson method to discretize in time. This scheme, on a uniform mesh $\bar{\omega}^K = \{j\Delta t, 0 \leq j \leq K, \Delta t = T/K\}$, reads

$$\begin{cases} u^K = u(x,T) = g(x,T), \\ \begin{cases} (I + \frac{\Delta t}{2}\tilde{L}_x^n)u^n = \Delta t f^{n+1/2} + (I - \frac{\Delta t}{2}\tilde{L}_x^n)u^{n+1}, \\ u^n(0) = u^n(X) = 0, \quad n = K-1, \dots, 1, 0, \end{cases} \end{cases} \quad (1)$$

where

$$\begin{aligned} \tilde{L}_x^n v^n &= -a(t_n + \Delta t/2)x^2 \frac{d^2 v^n}{dx^2} - b(x, t_n + \Delta t/2)x \frac{dv^n}{dx} + c(t_n + \Delta t/2)v^n, \\ f^{n+1/2} &= f(x, t_n + \Delta t/2). \end{aligned}$$

To obtain the convergence of the solution u^n to $u(x, t_n)$, we begin by studying the stability of (3.1) in the maximum norm. We remember that a semidiscretization scheme to approach (2.4)-(2.6) is stable, if a perturbation of the data $\tilde{g} = g + \tilde{\delta}^K(x)$ and $\tilde{f}^{n+1/2} = f^{n+1/2} + \delta^n(x)$ ($n = 0, 1, \dots, K-1$) provides a perturbed sequence \tilde{u}^n , whose distance from the original sequence is bounded by the maximum size of the perturbations, i.e.,

$$\|u^n - \tilde{u}^n\| \leq C(\tilde{\delta}^K + \max_{0 \leq j \leq n} \|\delta^j\|_\infty). \quad (2)$$

To study the stability of the Crank-Nicolson scheme, we introduce the notation $d^n \equiv \tilde{u}^n - u^n$. From (3.1) we deduce

$$\begin{cases} d^K = \tilde{\delta}^K, \\ \begin{cases} (I + \frac{\Delta t}{2}\tilde{L}_x^n)d^n = \Delta t \tilde{\delta}^n + (I - \frac{\Delta t}{2}\tilde{L}_x^n)d^{n+1}, \\ u^n(0) = u^n(X) = 0, \quad n = K-1, \dots, 1, 0, \end{cases} \end{cases}$$

This recurrence leads us to

$$d^n = (I + \frac{\Delta t}{2}\tilde{L}_x^n)^{-1} (\Delta t \sum_{i=n}^{K-1} R^{K-1-i} \tilde{\delta}^i) + R^{K-n} \tilde{\delta}^0, \quad (3)$$

where the operator R is defined in such way that $v \equiv Rw$ is the solution of

$$\begin{cases} (I + \frac{\Delta t}{2}\tilde{L}_x^n)v = (I - \frac{\Delta t}{2}\tilde{L}_x^n)w, \\ v(0) = v(X) = 0. \end{cases}$$

From (3.3), if

$$\|R^i\|_\infty \leq C, \quad \forall i = 0, 1, \dots, K-n, \quad (4)$$

then (3.2) holds. This result was proved by Palencia in [8] for any operator R of the form $R(L)$, where $R(z)$ is a rational A-acceptable function and L is any operator that generates an analytic semigroup e^{-tL} . In

our case, the operator \tilde{L}_x^n generate an analytic semigroup in $Y = (C^0(\bar{\Omega}), \|\cdot\|_{inf ty})$. This fact makes that condition (3.4) hold.

After the stability, we study the consistency of (3.1). We remember that the local error is defined by $e^n = u(x, t_n) - \hat{u}^n(x)$, where $\hat{u}^n(x)$ is the solution of

$$\begin{cases} \hat{u}^{n+1} = u(x, t_{n+1}), \\ \begin{cases} (I + \frac{\Delta t}{2} \tilde{L}_x^n) \hat{u}^n = \Delta t f^{n+1/2} + (I - \frac{\Delta t}{2} \tilde{L}_x^n) u(x, t_{n+1}), \\ \hat{u}^n(0) = \hat{u}^n(X) = 0, \quad n = K - 1, \dots, 1, 0. \end{cases} \end{cases} \tag{5}$$

Similarly to [6] we can prove that the differential operator $(I + (\Delta t/2)\tilde{L}_x^n)$ satisfies a maximum principle, which we will use in the analysis of the convergence of the Crank-Nicolson method.

Lemma 1 *The local error associated to the method (3.1) satisfies*

$$\|e^n\|_{\bar{\Omega}} = O((\Delta t)^3), \tag{6}$$

where $\|e^n\|_{\bar{\Omega}}$ is the maximum norm of e^n on the closed set $\bar{\Omega}$.

Proof. Following [2] we have

$$\begin{aligned} -\frac{u(x, t_{n+1}) - u(x, t_n)}{\Delta t} &= -u_t(x, t_n + \Delta t/2) + O((\Delta t)^2) \\ &= -\tilde{L}_x^n u(x, t_n + \Delta t/2) + f(x, t_n + \Delta t/2) + O((\Delta t)^2). \end{aligned} \tag{7}$$

Differentiating (2.4) twice w.r.t. t , we can obtain $|L_{x,t}u_{tt}| \leq C$ by the assumption $u(x, t) \in C^2(\bar{Q}) \cup C^4(Q)$ and therefore

$$\tilde{L}_x^n u(x, t_n + \Delta t/2) = \tilde{L}_x^n \frac{u(x, t_n) + u(x, t_{n+1})}{2} + O((\Delta t)^2). \tag{8}$$

From (3.3) and (3.4), it is straightforward to show that the local error is the solution of the problem

$$\begin{cases} (I + \frac{\Delta t}{2} \tilde{L}_x^n) e^n = O((\Delta t)^3), \\ e^n(0) = e^n(X) = 0, \end{cases}$$

and therefore the result follows from the maximum principle for the operator $(I + (\Delta t/2)\tilde{L}_x^n)$. ■

Combining the consistency result (3.6) with the stability result (3.2), in the classical way we deduce the following result.

Lemma 2 *The global error associated to the Crank-Nicolson method, given by $E^n = u(x, t_n) - u^n(x)$, satisfies*

$$\|E^n\|_{\bar{\Omega}} = O((\Delta t)^2).$$

4 The spatial semidiscretization: a hybrid difference scheme

To approximate the solution of problem (3.1), we use a hybrid finite difference scheme on a piecewise-uniform mesh Ω^N .

The use of central difference scheme on a uniform mesh may produce nonphysical oscillations in the computed solution. To overcome this oscillation we use a piecewise uniform mesh Ω^N on the spatial interval $[0, X]$:

$$x_i = \begin{cases} h, & i = 1, \\ h[1 + \frac{\alpha}{\beta^*}(i - 1)], & i = 2, \dots, N, \end{cases} \tag{1}$$

where

$$h = \frac{X}{1 + \frac{\alpha}{\beta^*}(N - 1)}.$$

It is easy to see that the mesh sizes $h_i = x_i - x_{i-1}$ satisfy

$$h_i = \begin{cases} h, & i = 1, \\ \frac{\alpha}{\beta^*} h, & i = 2, \dots, N. \end{cases}$$

We discretize (3.1) using a hybrid finite difference scheme on the above piecewise uniform mesh. Our discretization is similar to that of [12] and [13] in that it uses the central difference approximation

$$(\tilde{L}_{x,c}^N v^n)_i = -\frac{2a^{n+1/2}x_i^2}{h_i + h_{i+1}} \left(\frac{v_{i+1}^n - v_i^n}{h_{i+1}} - \frac{v_i^n - v_{i-1}^n}{h_i} \right) - b_i^{n+1/2} x_i \frac{v_{i+1}^n - v_i^n}{h_i + h_{i+1}} + c^{n+1/2} v_i^n$$

whenever the local mesh size allows us to do this without losing stability, but employs a simple upwind difference scheme otherwise:

$$(\tilde{L}_{x,u}^N v^n)_i = -\frac{2a^{n+1/2}x_i^2}{h_i + h_{i+1}} \left(\frac{v_{i+1}^n - v_i^n}{h_{i+1}} - \frac{v_i^n - v_{i-1}^n}{h_i} \right) - b_i^{n+1/2} x_i \frac{v_{i+1}^n - v_i^n}{h_{i+1}} + c^{n+1/2} v_i^n,$$

where $a^{n+1/2} = a(t_n + \Delta t/2)$, $b_i^{n+1/2} = b(x_i, t_n + \Delta t/2)$, $c^{n+1/2} = c(t_n + \Delta t/2)$. We set

$$(\tilde{L}_{x,h}^N v^n)_i = \begin{cases} (\tilde{L}_{x,u}^N v^n)_i, & i = 1, \\ (\tilde{L}_{x,c}^N v^n)_i, & i = 2, \dots, N. \end{cases}$$

Then our scheme reads:

$$\begin{cases} U_i^K = g(x_i, T), & 0 \leq i \leq N, \\ \text{For } n = K-1, \dots, 1, 0, \\ \tilde{L}^N U_i^n \equiv (I + \frac{\Delta t}{2} \tilde{L}_{x,h}^N) U_i^n = (I - \frac{\Delta t}{2} \tilde{L}_{x,h}^N) U_i^{n+1} + \Delta t f_i^{n+1/2}, \\ U_0^n = U_N^n = 0. \end{cases} \quad (2)$$

Lemma 3 The discrete operator \tilde{L}^N defined by (4.2) is uniformly stable and it satisfies a discrete maximum principle.

Proof. It is easy to verify that the matrix associated with \tilde{L}^N is an M-matrix, as in the proof of [6, Lemma 3.1].

To prove the convergence of the hybrid finite difference scheme we discretize the auxiliary problem (3.2), obtaining

$$\begin{cases} \tilde{L}^N \hat{U}_i^n \equiv (I + \frac{\Delta t}{2} \tilde{L}_{x,h}^N) \hat{U}_i^n = (I - \frac{\Delta t}{2} \tilde{L}_{x,h}^N) \hat{u}^{n+1}(x_i) + \Delta t f_i^{n+1/2}, & 0 < i < N, \\ \hat{U}_0^n = \hat{U}_N^n = 0. \end{cases} \quad (3)$$

■

Lemma 4 Let $\hat{u}^n(x)$ be the solution of (3.2) and $\{\hat{U}_i^n\}$ be the solution of (4.3). Then, the error satisfies

$$|\hat{u}^n(x_i) - \hat{U}_i^n| \leq C \Delta t N^{-2}, \quad 0 \leq i \leq N.$$

Proof. Applying the maximum principle for the differential operator $(I + (\Delta t/2) \tilde{L}_x^n)$ we can obtain

$$\left| \frac{d^k \hat{u}^n}{dx^k} \right| \leq C(1 + x^{2-k}), \quad k = 0, 1, \dots, 4.$$

by the assumption $u(x, t) \in C^2(\bar{Q}) \cup C^4(Q)$.

We remember that the local error associated to the hybrid finite difference scheme is given by

$$(I + \frac{\Delta t}{2} \tilde{L}_{x,h}^N)(\hat{u}^n(x_i) - \hat{U}_i^n) = \frac{\Delta t}{2} (\tilde{L}_{x,h}^N - \tilde{L}_x^n) \hat{u}^n(x_i).$$

Applying Taylor's formula with the integral form of the remainder, we have

$$\begin{aligned} |(\tilde{L}_{x,h}^N - \tilde{L}_x^n)\hat{u}^n(x_i)| &\leq Ch \int_{x_{i-1}}^{x_{i+1}} (x_i^2 \left| \frac{d^4 \hat{u}^n}{dx^4} \right| + x_i \left| \frac{d^3 \hat{u}^n}{dx^3} \right|) dx \\ &\leq CN^{-2}, \quad 2 \leq i < N, \\ |(\tilde{L}_{x,h}^N - \tilde{L}_x^n)\hat{u}^n(x_i)| &\leq C \int_{x_{i-1}}^{x_{i+1}} x_i^2 \left| \frac{d^3 \hat{u}^n}{dx^3} \right| dx + C \int_{x_{i-1}}^{x_i} x_i \left| \frac{d^2 \hat{u}^n}{dx^2} \right| dx \\ &\leq CN^{-2}, \quad i = 1. \end{aligned}$$

Hence, using the discrete maximum principle we have

$$|\hat{u}^n(x_i) - \hat{U}_i^n| \leq C\Delta t N^{-2}, \quad 0 \leq i \leq N,$$

which completes the proof. ■

Now we can get the main result for our difference scheme.

Theorem 1 Let $u(x, t)$ be the solution of (2.4)-(2.6) and U_i^n be the solution of (4.2). Then

$$|u(x_i, t_n) - U_i^n| \leq C(N^{-2} + (\Delta t)^2), \quad 0 \leq i \leq N, \quad 0 \leq n \leq K.$$

Proof. First, we split the global error at the time t_n in the form

$$|u(x_i, t_n) - U_i^n| \leq |u(x_i, t_n) - \hat{u}^n(x_i)| + |\hat{u}^n(x_i) - \hat{U}_i^n| + |\hat{U}_i^n - U_i^n|.$$

Using Lemmas 1 and 4, it follows

$$|u(x_i, t_n) - U_i^n| \leq C\Delta t((\Delta t)^2 + N^{-2}) + |\hat{U}_i^n - U_i^n|. \quad (4)$$

To bound the last addend, we take into account that $\hat{U}^n - U^n$ can be written as the solution of one step of (4.2), taking as source term $f = 0$ together with zero boundary conditions and $u(x_i, t_{n+1}) - U_i^{n+1}$ as final value. Then, from the uniform stability of the totally discrete scheme, it follows

$$|\hat{U}_i^n - U_i^n| \leq C|u(x_i, t_{n+1}) - U_i^{n+1}|. \quad (5)$$

The required result immediately holds from (4.4), (4.5) and a recurrence relation for the global error. ■

5 Numerical experiments

In this section we verify experimentally the theoretical results obtained in the preceding section. Errors, convergence rates and error constants for the hybrid difference scheme are presented for two test problems.

Example 1 Consider the problem

$$\begin{aligned} -\frac{\partial u}{\partial t} - 2x^2 \frac{\partial^2 u}{\partial x^2} - x \frac{\partial u}{\partial x} + u &= f(x, t) \quad \text{for } (x, t) \in (0, 1) \times (0, 1), \\ u(0, t) = u(1, t) &= 0 \quad \text{for } t \in [0, 1], \\ u(x, 1) &= (1 - x^3)(e^x - 1) + ex(1 - x) \quad \text{for } x \in [0, 1], \end{aligned}$$

where $f(x, t)$ is chosen such that $u(x, t) = (1 - x^3)(e^x - 1) + e^t x(1 - x)$.

Example 2 Consider the problem

$$\begin{aligned} -\frac{\partial u}{\partial t} - \left(\frac{1}{2} + t\right)x^2 \frac{\partial^2 u}{\partial x^2} - (1 + xt)x \frac{\partial u}{\partial x} + e^t u &= f(x, t) \quad \text{for } (x, t) \in (0, 1) \times (0, 1), \\ u(0, t) = u(1, t) &= 0 \quad \text{for } t \in [0, 1], \\ u(x, 1) &= (1 - x^3)(e^x - 1) + x(1 - x) \quad \text{for } x \in [0, 1], \end{aligned}$$

where $f(x, t)$ is chosen such that $u(x, t) = t(1 - x^3)(e^x - 1) + t^3 x(1 - x)$.

In the numerical experiment we have taken $N = K$. We measure the accuracy in the discrete maximum norm $\|u - U\|_\infty$, the convergence rate

$$r^N = \log_2 \left(\frac{\|u - U^N\|_\infty}{\|u - U^{2N}\|_\infty} \right)$$

and the constant in the error estimate

$$C^N = \frac{\|u - U^N\|_\infty}{N^{-2}}.$$

The Table 1 and 2 correspond to the above problems respectively. The numerical results are clear illustrations of the convergence estimate of Theorem 1. They indicate that the theoretical results are fairly sharp.

Table 1: Numerical results for example 1

$N=K$	error	rate	constant
32	1.1204e-3	2.029	1.1472
64	2.7454e-4	2.014	1.1245
128	6.7952e-5	2.007	1.1133
256	1.6902e-5	2.004	1.1077
512	4.2150e-6	-	1.1049

Table 2: Numerical results for example 2

$N=K$	error	rate	constant
32	2.2737e-3	1.646	2.3282
64	7.2634e-4	1.808	2.9751
128	2.0736e-4	1.893	3.3975
256	5.5827e-5	1.938	3.6587
512	1.4566e-5	-	3.8184

6 Conclusion

In this paper a second-order difference scheme for the Black-Scholes equation governing option pricing has been proposed. The key to the success of difference scheme is discretized the Black-Scholes equation by a hybrid difference scheme. It combines the Crank-Nicolson method in the time discretization with a hybrid finite difference scheme on a piecewise uniform mesh in the spatial discretization. The difference scheme is stable for the arbitrary volatility and arbitrary asset price. And it can handle the degeneracy of the Black-Scholes differential operator at $x = 0$ without truncating the domain. Our hybrid difference scheme for the Black-Scholes equation is a modification of the difference scheme used in [12] and [13]. Numerical experiments, performed to demonstrate the effectiveness of the method, showed that the approach is stable and accurate.

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