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On the Stability of Solutions of Certain Non-Linear Third-Order Delay Differential Equations

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Abstract: In this paper, we study equation (1.6) and (1.7) and establish some sufficient conditions for the asymptotic stability of the zero solution of them.

Key words: asymptotic stability; Lyapunov functional; delay differential equations; third-order delay differential equations

1 Introduction

The Lyapunov function or functional approach has been a powerful tool to ascertain the stability of solutions of certain differential equations. Up to today, perhaps, the most effective method to determine the stability of solutions of non-linear differential equations is still the Lyapunov's direct (or second) method. The major advantage of this method is that the stability of solutions can be obtained without any prior knowledge of solutions.

For over four decades many authors made use of the Lyapunov's direct method by considering Lyapunov functionals and obtained the conditions which ensure the stability of the problem, for example, [1-5]. But, how do we construct those appropriate Lyapunov functionals? No author has discussed them thus far. It is in general a difficult task. Similar problem is shared with ordinary differential equations of high orders, [6]. Obviously, it is even more difficult to construct Lyapunov functionals for differential equations of higher orders with delay. In particular, Zhu[7], obtained sufficient conditions which ensure the stability of systems

$$\dddot{x} + a\ddot{x} + b\dot{x} + f(x(t-r)) = 0, \tag{1.1}$$

and

$$\dddot{x} + a\ddot{x} + \phi(\dot{x}(t-r)) + f(x) = 0, \tag{1.2}$$

where: r, a and b are positive constants.

In 2003, Sadek [8] investigated the asymptotic stability of the trivial solution of the third order non-linear differential with delay of the form

$$\dddot{x} + a\ddot{x} + g(\dot{x}(t-r(t))) + f(x(t-r(t))) = 0. \tag{1.3}$$

In 2006, Cemil Tunc [9] obtained the asymptotic stability of the solution of the delay differential equation of the form

$$\dddot{x} + \varphi(x,\dot{x})\ddot{x} + g(\dot{x}(t-r(t))) + f(x(t-r(t))) = 0. \tag{1.4}$$

More recently, Afuwape and Omeike [10] obtained sufficient conditions for the stability of the delay differential equation of the form

$$\dddot{x} + h(\dot{x})\ddot{x} + g(x(t-r(t)), \dot{x}(t-r(t))) + f(x(t-r(t))) = 0. \tag{1.5}$$

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In the current paper, we consider the third-order nonlinear differential equations with delay

$$\dddot{x} + \varphi(\ddot{x}) + g(x(t-r(t)), \dot{x}(t-r(t))) + f(x(t-r(t))) = 0 \tag{1.6}$$

and

$$\dddot{x} + \varphi(x,\dot{x})\ddot{x} + g(x(t-r(t)), \dot{x}(t-r(t))) + f(x(t-r(t))) = 0, \tag{1.7}$$

$0 \le r(t) \le \gamma$, γ is a positive constant which will be determined later.

Essentially, our subjects are to establish some sufficient conditions for the asymptotic stability of the zero solution of (1.6) and (1.7).

2 Preliminaries

In order to reach our main results, we will give the preliminary definitions and results for the general autonomous delay differential system. It should be noted that the symbol$\|\cdot\|$ denotes the Euclidean norm throughout this paper. First we consider the general autonomous delay differential system

$$\dot{x} = f(x_t), x_t = x(t+\theta), -r \le \theta \le 0, t \ge 0, \tag{2.1}$$

where $f : C_H \to R^n$ is a continuous mapping, $f(0) = 0$, $C_H = \{\phi \in (C[-r,0], R^n) : \|\phi\| \le H\}$ and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(\phi)| \le L(H_1)$ when $\|\phi\| \le H_1$.

Definition 2.1 *An element $\psi \in C$ is in the ω-limit set of ϕ, say, $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,\infty)$ and there is a sequence $\{t_n\}$, $t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n+\theta, 0, \phi)$ for $-r \le \theta \le 0$.*

Definition 2.2 $(See[7])$ *A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (2.1), $x(t,0,\phi)$, is defined on $[0,\infty)$, and $x_t(\phi) \in Q$ for $t \in [0,\infty)$.*

Lemma 2.1 $(See[2,7])$ *If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (2.1) with $x_0(\phi) = \phi$ is defined on $[0,\infty)$ and $\|x_t(\phi)\| \le H_1 < H$ for $t \in [0,\infty)$, then $\Omega(\phi)$ is a nonempty, compact, invariant set and $dist(x_t(\phi), \Omega(\phi)) \to 0$, as $t \to \infty$.*

Lemma 2.2 $(See[7])$*Let $V(\phi) : C_H \to R$ be a continuous functional satisfying a local Lipschitz condition, $V(0) = 0$ and such that:*

(1) $W_1(|\phi(0)|) \le V(\phi) \le W_2(||\phi||)$, where $W_1(r), W_2(r)$ are wedges.

(2) $\dot{V}_{(2.1)}(\phi) \le 0$ for $\phi \in C_H$.

Then the zero solution of (2.1) is uniformly stable. If we define $Z = \left\{\phi \in C_H : \dot{V}_{(2.1)}(\phi) = 0\right\}$, then the zero solution of (2.1) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3 Main results

First, we write equation (1.6) as the following equivalent system:

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = z(t) \\ \dot{z}(t) = -\varphi(z) - g(x,y) - f(x) + \int_{t-r(t)}^{t} f'(x(s))y(s)ds \\ \qquad + \int_{t-r(t)}^{t} g'_x(x(s),y(s))y(s)ds + \int_{t-r(t)}^{t} g'_y(x(s),y(s))z(s)ds \end{cases} \tag{3.1}$$

The following will be our main stability result for (3.1).

Theorem 3.1 *Consider system (3.1) with $\varphi(0) = g(0,0) = f(0) = 0, \varphi(z), g(x,y), f'(x)$ continuous in their respective arguments, and $g'_x(x,y), g'_y(x,y)$ continuous for all x, y.*

Suppose further that

(1) There are constants $a > 0, b > \frac{1}{2}, c > 0, L > 0, M > 0$ and $N > 0$, such that $ab > a + c$;

(2) $f(x)\operatorname{sgn} x > 0$ *for all* $x \neq 0$, $\sup\{f'(x)\} = c$, *and* $|f'(x)| \leqslant L$ *for all* x*;*
(3) $\frac{g(x,y)}{y} \geqslant b$ *for all* $x, y \neq 0$, $\left|g'_y(x,y)\right| \leqslant M$, $|g'_x(x,y)| \leqslant N$ *and* $g'_x(x,y) \leqslant 0$ *for all* x, y*;*
(4) $0 \leqslant r(t) \leqslant \gamma$, $r'(t) \leqslant \beta$, $0 < \beta < 1$*;*
and
(5) $\frac{(\mu^2+1)-\sqrt{1-\mu(2\delta+L+M+N)\gamma}}{\mu} < \frac{\varphi(z)}{z} < \frac{(\mu^2+1)+\sqrt{1-\mu(2\delta+L+M+N)\gamma}}{\mu}$ *for all* $z \neq 0$.
Then the zero solution of (3.1) is asymptotically stable, provided that

$$\gamma < \min\left\{\frac{1-\beta}{\mu(1-\beta)(L+N)+\mu(\mu+2-\beta)M}, \frac{(2\mu b-2c-\mu)(1-\beta)}{(2\mu+1-\mu\beta)(L+N)+\mu(1-\beta)M}\right\}$$

with $\delta = \frac{(\mu+1)M}{2(1-\beta)} > 0$, $\mu = \frac{ab+c}{2b-1}$.

Proof. Our main tool in the proof of the theorem just stated above is a Lyapunov functional $V_1(x_t, y_t, z_t)$ defined by

$$\begin{aligned} V_1(x_t, y_t, z_t) &= \mu\int_0^x f(\xi)d\xi + f(x)y + \int_0^y g(x,\xi)d\xi + \frac{1}{2}(\mu y + z)^2 \\ &\quad + \lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds + \delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds, \end{aligned} \tag{3.2}$$

where λ, μ and δ are positive constants which will be determined later in the proof. Since $\varphi(0) = g(0,0) = f(0)$, it is immediate that $V(0,0,0) = 0$.

Let $\frac{d}{dt}V_1(x_t, y_t, z_t) = \frac{d}{dt}V_1$ denote the derivative of $V_1 = V_1(x_t, y_t, z_t)$ along the solution of (3.1). Then, by a straightforward calculation from (3.2) and (3.1), we observe that

$$\begin{aligned} \frac{d}{dt}V_1(x_t, y_t, z_t) &= f'(x)y^2 + y\int_0^y g'_x(x,\xi)d\xi + \mu^2 yz + \mu z^2 + \lambda y^2 r(t) \\ &\quad - \lambda(1-r'(t))\int_{t-r(t)}^t y^2(\theta)d\theta - \mu y\varphi(z) - \mu y g(x,y) - z\varphi(z) + \delta z^2 r(t) \\ &\quad - \delta(1-r'(t))\int_{t-r(t)}^t z^2(\theta)d\theta + (\mu y + z)\int_{t-r(t)}^t f'(x(s))y(s)ds \\ &\quad + \int_{t-r(t)}^t g'_x(x(s),y(s))y(s)ds + \int_{t-r(t)}^t g'_y(x(s),y(s))z(s)ds. \end{aligned} \tag{3.3}$$

Suppose $|f'(x)| \leqslant L$, $\left|g'_y(x,y)\right| \leqslant M$, $|g'_x(x,y)| \leqslant N$ and using $2uv \leqslant u^2 + v^2$ we obtain

$$\begin{aligned} &(\mu y + z)\int_{t-r(t)}^t f'(x(s))y(s)ds \\ &\le \mu L\int_{t-r(t)}^t |y||y(s)|ds + L\int_{t-r(t)}^t |z||y(s)|ds \\ &\le \frac{\mu L}{2}y^2 r(t) + \frac{\mu L}{2}\int_{t-r(t)}^t y^2(s)ds + \frac{L}{2}z^2 r(t) + \frac{L}{2}\int_{t-r(t)}^t y^2(s)ds. \end{aligned} \tag{3.4}$$

Similarly, we obtain

$$\begin{aligned} &(\mu y + z)\int_{t-r(t)}^t g'_y(x(s),y(s))z(s)ds \\ &\le \frac{\mu M}{2}y^2 r(t) + \frac{\mu M}{2}\int_{t-r(t)}^t z^2(s)ds + \frac{M}{2}z^2 r(t) + \frac{M}{2}\int_{t-r(t)}^t z^2(s)ds, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &(\mu y + z)\int_{t-r(t)}^t g'_x(x(s),y(s))y(s)ds \\ &\le \frac{\mu N}{2}y^2 r(t) + \frac{\mu N}{2}\int_{t-r(t)}^t y^2(s)ds + \frac{N}{2}z^2 r(t) + \frac{N}{2}\int_{t-r(t)}^t y^2(s)ds \end{aligned} \tag{3.6}$$

(3.7)

and

$$\begin{aligned}\mu^2 yz - \mu y\varphi(z) &= \mu y(\mu z - \varphi(z)) \\ &\leq \frac{1}{2}\mu y^2 + \mu^2 z^2 - 2\mu\varphi(z)z + \varphi^2(z) \\ &= \frac{\mu}{2}y^2 + \frac{\mu^3}{2}z^2 - \mu^2\varphi(z)z + \frac{\mu}{2}\varphi^2(z). \end{aligned} \tag{3.8}$$

Thus, from (3.3), (3.4), (3.5), (3.6) and (3.7), we have

$$\begin{aligned}\frac{d}{dt}V_1(x_t, y_t, z_t) \leq & -\mu y g(x,y) + f'(x) + \frac{\mu}{2} + \lambda r(t) + \frac{\mu L}{2}r(t) + \frac{\mu M}{2}r(t) + \frac{\mu N}{2}r(t)y^2 \\ & + \frac{\mu}{2}\varphi^2(z) - (\mu^2+1)z\varphi(z) + \frac{\mu^3}{2} + \mu + \delta r(t) + \frac{L}{2}r(t) + \frac{M}{2}r(t) + \frac{N}{2}r(t)z^2 \\ & + \frac{\mu L}{2} + \frac{L}{2} + \frac{\mu N}{2} + \frac{N}{2} - \lambda(1-r'(t))\int_{t-r(t)}^{t} y^2(s)ds \\ & + \frac{\mu M}{2} + \frac{M}{2} - \delta(1-r'(t))\int_{t-r(t)}^{t} z^2(s)ds + y\int_0^y g'_x(x,\xi)d\xi. \end{aligned} \tag{3.9}$$

Assume that $\sup\{f'(x)\} = c > 0$, $r'(t) \leqslant \beta$, $0 < \beta < 1$, $\frac{g(x,y)}{y} \geqslant b > 0$ ($y \neq 0$), $ab > a + c$, $0 \leqslant r(t) \leqslant \gamma$. Also, suppose that $g'_x(x,y) \leqslant 0$ for all x, y , we get $y\int_0^y g'_x(x,\xi)d\xi \leqslant 0$. Hence, it follows from (3.8) that

$$\begin{aligned}\frac{d}{dt}V_1(x_t, y_t, z_t) \leq & -\left(\mu b - c - \frac{\mu}{2} - \lambda\gamma - \frac{\mu L}{2}\gamma - \frac{\mu M}{2}\gamma - \frac{\mu N}{2}\gamma\right)y^2 \\ & + \frac{\mu}{2}\varphi^2(z) - (\mu^2+1)z\varphi(z) + (\frac{\mu^3}{2} + \mu + \delta\gamma + \frac{L}{2}\gamma + \frac{M}{2}\gamma + \frac{N}{2}\gamma)z^2 \\ & + \left[\frac{1}{2}(\mu+1)(L+N) - \lambda(1-\beta)\right]\int_{t-r(t)}^{t} y^2(s)ds \\ & + \left[\frac{1}{2}(\mu+1)M - \delta(1-\beta)\right]\int_{t-r(t)}^{t} z^2(s)ds. \end{aligned} \tag{3.10}$$

If we take $\mu = \frac{ab+c}{2b-1} > 0$, $\lambda = \frac{(\mu+1)(L+N)}{2(1-\beta)} > 0$ and $\delta = \frac{(\mu+1)M}{2(1-\beta)} > 0$, we obtain

$$\begin{aligned}\frac{d}{dt}V_1(x_t, y_t, z_t) \leq & -\left(\mu b - c - \frac{\mu}{2} - \lambda\gamma - \frac{\mu L}{2}\gamma - \frac{\mu M}{2}\gamma - \frac{\mu N}{2}\gamma\right)y^2 \\ & + \frac{\mu}{2}\varphi^2(z) - (\mu^2+1)z\varphi(z) + (\frac{\mu^3}{2} + \mu + \delta\gamma + \frac{L}{2}\gamma + \frac{M}{2}\gamma + \frac{N}{2}\gamma)z^2. \end{aligned}$$

Let

$$H(z) = \frac{\mu}{2}\varphi^2(z) - (\mu^2+1)z\varphi(z) + (\frac{\mu^3}{2} + \mu + \delta\gamma + \frac{L}{2}\gamma + \frac{M}{2}\gamma + \frac{N}{2}\gamma)z^2,$$

then $H(z)$ can be considered as a quadratic function about $\varphi(z)$ for all $z \neq 0$. We obtain

$$\begin{aligned}\Delta &= \left[-(\mu^2+1)z\right]^2 - 4\cdot\frac{\mu}{2}\cdot(\frac{\mu^3}{2} + \mu + \delta\gamma + \frac{L}{2}\gamma + \frac{M}{2}\gamma + \frac{N}{2}\gamma)z^2 \\ &= \left[1 - \mu(2\delta + L + M + N)\gamma\right]z^2. \end{aligned}$$

Therefore, if

$$\gamma < \frac{1}{\mu(2\delta + L + M + N)} = \frac{1-\beta}{\mu(1-\beta)(L+N) + \mu(\mu+2-\beta)M},$$

then

$$\Delta = \left\{1 - \left[\frac{\mu(1-\beta)(L+N) + \mu(\mu+2-\beta)M}{1-\beta}\right]\gamma\right\}z^2 > 0.$$

Hence, the solutions of the quadratic equation $H(z) := H(\varphi(z)) = 0$ are

$$\varphi(z)=\frac{(\mu^2+1)z+\sqrt{1-\mu(2\delta+L+M+N)\gamma}|z|}{\mu}=\begin{cases}\frac{(\mu^2+1)z+z\sqrt{1-\mu(2\delta+L+M+N)\gamma}}{\mu}(z>0),\\ \frac{(\mu^2+1)z-z\sqrt{1-\mu(2\delta+L+M+N)\gamma}}{\mu}(z<0),\end{cases}$$

or

$$\varphi(z)=\frac{(\mu^2+1)z-\sqrt{1-\mu(2\delta+L+M+N)\gamma}|z|}{\mu}=\begin{cases}\frac{(\mu^2+1)z-z\sqrt{1-\mu(2\delta+L+M+N)\gamma}}{\mu}(z>0),\\ \frac{(\mu^2+1)z+z\sqrt{1-\mu(2\delta+L+M+N)\gamma}}{\mu}(z<0).\end{cases}$$

Therefore, if

$$\frac{(\mu^2+1)-\sqrt{1-\mu(2\delta+L+M+N)\gamma}}{\mu}<\frac{\varphi(z)}{z}<\frac{(\mu^2+1)+\sqrt{1-\mu(2\delta+L+M+N)\gamma}}{\mu},$$

we have $H(z)<0$. We can easily obtain $H(0)=0$. Therefore, $H(z)\leqslant 0$ and $H(z)=0$ if and only if $z=0$.

If

$$\gamma<\frac{\mu b-c-\frac{\mu}{2}}{\lambda+\frac{\mu L}{2}+\frac{\mu M}{2}+\frac{\mu N}{2}}=\frac{(2\mu b-2c-\mu)(1-\beta)}{(2\mu+1-\mu\beta)(L+N)+\mu(1-\beta)M},$$

then

$$\begin{aligned}&-\left(\mu b-c-\tfrac{\mu}{2}-\lambda\gamma-\tfrac{\mu L}{2}\gamma-\tfrac{\mu M}{2}\gamma-\tfrac{\mu N}{2}\gamma\right)y^2\\ &=-\left[\mu b-c-\tfrac{\mu}{2}-\left(\lambda+\tfrac{\mu L}{2}+\tfrac{\mu M}{2}+\tfrac{\mu N}{2}\right)\gamma\right]y^2\\ &=-\left[\mu b-c-\tfrac{\mu}{2}-\left(\tfrac{(2\mu+1-\mu\beta)(L+N)+\mu(1-\beta)M}{2(1-\beta)}\right)\gamma\right]y^2\\ &\leq 0\end{aligned}$$

and $-\left(\mu b-c-\frac{\mu}{2}-\lambda\gamma-\frac{\mu L}{2}\gamma-\frac{\mu M}{2}\gamma-\frac{\mu N}{2}\gamma\right)y^2=0$ if and only if $y=0$.

Therefore, if

$$\gamma<\min\left\{\frac{1-\beta}{\mu(1-\beta)(L+N)+\mu(\mu+2-\beta)M},\frac{(2\mu b-2c-\mu)(1-\beta)}{(2\mu+1-\mu\beta)(L+N)+\mu(1-\beta)M}\right\},$$

we have $\frac{d}{dt}V_1(x_t,y_t,z_t)\leqslant-\alpha(y^2+z^2)$ for some $\alpha>0$.

By $\frac{d}{dt}V_1(x_t,y_t,z_t)=0$, $f(x)\operatorname{sgn}x>0$ ($x\neq 0$) and system (3.1), we can easily obtain $x=y=z=0$. Thus, the condition (2) of Lemma 2.2 is satisfied.

Next, we have to show that the condition (1) of Lemma 2.2 is satisfied.

In view of $f(x)\operatorname{sgn}x>0$ ($x\neq 0$) and $\frac{g(x,y)}{y}\geqslant b$ ($y\neq 0$), we see that

$$\begin{aligned}V_1(x_t,y_t,z_t)&=\mu\int_0^x f(\xi)d\xi+f(x)y+\int_0^y\frac{g(x,\eta)}{\eta}\eta d\eta+\frac{1}{2}(\mu y+z)^2\\ &+\lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds+\delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds\\ &\geq\mu\int_0^x f(\xi)d\xi+f(x)y+\frac{1}{2}by^2+\frac{1}{2}(\mu y+z)^2\\ &+\lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds+\delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds\\ &=\frac{1}{2by^2}\left[4\int_0^x f(\xi)\left\{\int_0^y(\mu b-f'(\xi))\eta d\eta\right\}d\xi\right]+\frac{1}{2b}(by+f(x))^2\\ &+\frac{1}{2}(\mu y+z)^2+\lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds+\delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds, for\ \ all\ \ y\neq 0.\end{aligned}$$

Now, recall that $\mu b-f'(x)\geqslant\frac{b(ab-c)+c}{2b-1}>0$. Hence, it can be easily seen that the functional $V_1(x_t,y_t,z_t)$ satisfies the condition (1) of Lemma 2.2. The proof is complete. ■

Remark 3.1 *If $g(x(t-r(t)), \dot{x}(t-r(t))) = g(\dot{x}(t-r(t)))$ in (1.6), then we can obtain a corollary from Theorem 3.1 about the system $\dddot{x} + \varphi(\ddot{x}) + g(\dot{x}(t-r(t))) + f(x(t-r(t))) = 0$. Because it is similar to Theorem 3.1, we don't write it out.*

Now, we will consider the stability of the zero solution of equation (1.7). First, we write equation (1.7) as the following equivalent system:

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = z(t) \\ \dot{z}(t) = -\varphi(x,y)z - g(x,y) - f(x) + \int_{t-r(t)}^{t} f'(x(s))y(s)ds \\ \qquad + \int_{t-r(t)}^{t} g'_x(x(s),y(s))y(s)ds + \int_{t-r(t)}^{t} g'_y(x(s),y(s))z(s)ds \end{cases} \tag{3.11}$$

Next, we shall state and prove our another main result.

Theorem 3.2 *Consider system (3.10) with $\varphi(0,0) = g(0,0) = f(0) = 0$, $\varphi(x,y), g(x,y), f'(x)$ continuous in their respective arguments, and $\varphi'_x(x,y), g'_x(x,y), g'_y(x,y)$ continuous for all x, y.*

Suppose further that

(1) There are positive constants a, b, c, L, M and N, such that $ab - c > 0$;

(2) $\varphi(x,y) \geqslant a$, $y\varphi'_x(x,y) \leqslant 0$ for all x, y;

(3) $\frac{g(x,y)}{y} \geqslant b$ for all $x, y \neq 0$, $|g'_y(x,y)| \leqslant M$, $|g'_x(x,y)| \leqslant N$ and $g'_x(x,y) \leqslant 0$ for all x, y ;

(4) $f(x)\operatorname{sgn} x > 0$ for all $x \neq 0$, $\sup\{f'(x)\} = c$, and $|f'(x)| \leqslant L$ for all x ;

and

(5) $0 \leqslant r(t) \leqslant \gamma, r'(t) \leqslant \beta, 0 < \beta < 1$.

Then the zero solution of (3.10) is asymptotically stable, provided that

$$\gamma < \min\left\{ \frac{(ab-c)(1-\beta)}{b(1-\beta)(L+N) + bM(2+\mu-\beta)}, \frac{(ab-c)(1-\beta)}{\mu(L+M+N)(1-\beta) + L(1+\mu)} \right\}$$

with $\mu = \frac{ab+c}{2b}$.

Proof. Define the Lyapunov functional $V_2(x_t, y_t, z_t)$ as

$$\begin{aligned} V_2(x_t,y_t,z_t) = {}& \mu\int_0^x f(\xi)d\xi + f(x)y + \mu\int_0^y \varphi(x,\eta)\eta d\eta + \int_0^y g(x,\xi)d\xi \\ & + \mu yz + \frac{1}{2}z^2 + \lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \delta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds, \end{aligned} \tag{3.12}$$

where λ, μ and δ are positive constants which will be determined later in the proof. Since $\varphi(0,0) = g(0,0) = f(0)$, it is immediate that $V(0,0,0) = 0$.

Let $\frac{d}{dt}V_2(x_t,y_t,z_t) = \frac{d}{dt}V_2$ denote the derivative of $V_2 = V_2(x_t,y_t,z_t)$ along the solution of (3.10). Then, by a straightforward calculation from (3.11) and (3.10), we observe that

$$\begin{aligned} &\frac{d}{dt}V_2(x_t,y_t,z_t) = f'(x)y^2 + \mu z^2 - \mu y g(x,y) - \varphi(x,y)z^2 \\ &+ \mu y\int_0^y \varphi'_x(x,\eta)\eta d\eta + y\int_0^y g'_x(x,\xi)d\xi + \lambda y^2 r(t) + \delta z^2 r(t) \\ &- \lambda(1-r'(t))\int_{t-r(t)}^{t} y^2(\theta)d\theta - \delta(1-r'(t))\int_{t-r(t)}^{t} z^2(\theta)d\theta \\ &+ (\mu y + z)\left[\int_{t-r(t)}^{t} f'(x(s))y(s)ds + \int_{t-r(t)}^{t} g'_x(x(s),y(s))y(s)ds + \int_{t-r(t)}^{t} g'_y(x(s),y(s))z(s)ds\right]. \end{aligned} \tag{3.13}$$

Thus, from (3.4),(3.5),(3.6) and (3.12), we obtain

$$\begin{aligned}\frac{d}{dt}V_2(x_t,y_t,z_t) \le{}& f'(x)y^2+\mu z^2-\mu y g(x,y)-\varphi(x,y)z^2+\frac{\mu}{2}(L+M+N+\frac{2\lambda}{\mu})y^2r(t)\\ &+\frac{1}{2}(L+M+N+2\delta)z^2r(t)+\frac{(L+N)(\mu+1)}{2}\int_{t-r(t)}^{t}y^2(s)ds\\ &+\frac{M(\mu+1)}{2}\int_{t-r(t)}^{t}z^2(s)ds-\lambda(1-r'(t))\int_{t-r(t)}^{t}y^2(\theta)d\theta\\ &-\delta(1-r'(t))\int_{t-r(t)}^{t}z^2(\theta)d\theta+\mu y\int_0^y\varphi'_x(x,\eta)\eta d\eta+y\int_0^y g'_x(x,\xi)d\xi. \end{aligned}\tag{3.14}$$

Also, from the hypotheses (2) and (3), we get

$$\mu y\int_0^y\varphi'_x(x,\eta)\eta d\eta\leqslant 0,\quad y\int_0^y g'_x(x,\xi)d\xi\leqslant 0,$$

and with the hypothesis (5), we have

$$\begin{aligned}\frac{d}{dt}V_2(x_t,y_t,z_t)\le{}&-\left[(\mu b-c)-\frac{\gamma\mu}{2}(L+M+N+\frac{2\lambda}{\mu})\right]y^2\\ &-\left[(a-\mu)-\frac{\gamma}{2}(L+M+N+2\delta)\right]z^2\\ &+\frac{1}{2}\{(\mu+1)(L+N)-2\lambda(1-\beta)\}\int_{t-r(t)}^{t}y^2(s)ds\\ &+\frac{1}{2}\{M(\mu+1)-2\delta(1-\beta)\}\int_{t-r(t)}^{t}z^2(s)ds.\end{aligned}\tag{3.15}$$

Following a same way arising in the course of [10, Theorem 2.1] it can be easily concluded from (3.14) that

$$\begin{aligned}\frac{d}{dt}V_2(x_t,y_t,z_t)\le{}&-\left[\frac{ab-c}{2}-\frac{\mu(L+M+N)(1-\beta)+L(1+\mu)}{2(1-\beta)}\gamma\right]y^2\\ &-\left[\frac{ab-c}{2b}-\frac{(1-\beta)(L+N)+M(2+\mu-\beta)}{2(1-\beta)}\gamma\right]z^2.\end{aligned}\tag{3.16}$$

Hence, there is a positive constant α such that

$$\frac{d}{dt}V_2(x_t,y_t,z_t)\leqslant-\alpha(y^2+z^2)$$

provided that

$$\gamma<\min\left\{\frac{(ab-c)(1-\beta)}{b(1-\beta)(L+N)+bM(2+\mu-\beta)},\frac{(ab-c)(1-\beta)}{\mu(L+M+N)(1-\beta)+L(1+\mu)}\right\}.$$

By $\frac{d}{dt}V_2(x_t,y_t,z_t)=0$, $f(x)\operatorname{sgn}x>0$ $(x\neq 0)$ and system (3.10), we can easily obtain $x=y=z=0$. Thus, the condition (2) of Lemma 2.2 is satisfied.

In view of $f(x)\operatorname{sgn}x>0$ $(x\neq 0)$, $\frac{g(x,y)}{y}\geqslant b$ $(y\neq 0)$ and $\varphi(x,y)\geqslant a$, we see that

$$\begin{aligned}V_2(x_t,y_t,z_t)\ge{}&\mu\int_0^x f(\xi)d\xi+f(x)y+\frac{1}{2}\mu a y^2+\frac{1}{2}by^2+\mu yz+\frac{1}{2}z^2\\ &+\lambda\int_{-r(t)}^{0}\int_{t+s}^{t}y^2(\theta)d\theta ds+\delta\int_{-r(t)}^{0}\int_{t+s}^{t}z^2(\theta)d\theta ds\\ ={}&\frac{1}{2b}\left[by+f(x)\right]^2+\frac{1}{2by^2}\left[4\int_0^x f(\xi)\left\{\int_0^y(\mu b-f'(\xi))\eta d\eta\right\}d\xi\right]+\frac{1}{2}(\mu y+z)^2\\ &+\frac{1}{2}\mu(a-\mu)y^2+\lambda\int_{-r(t)}^{0}\int_{t+s}^{t}y^2(\theta)d\theta ds+\delta\int_{-r(t)}^{0}\int_{t+s}^{t}z^2(\theta)d\theta ds,\ for\ \ all\ \ y\neq 0.\end{aligned}\tag{3.17}$$

Now, recall that $a-\mu=\frac{ab-c}{2b}>0$, and $\mu b-f'(x)\geqslant\frac{ab-c}{2}>0$. Hence, it can be easily seen that the functional $V_2(x_t,y_t,z_t)$ satisfies the condition (1) of Lemma 2.2. The proof is complete. ■

Remark 3.2 *If $\varphi(x,\dot{x})=\varphi(\dot{x})$ in (1.7), then Theorem 3.2 reduces to Theorem 2.1 of [10]. But it should be pointed out that the condition (5) of Theorem 2.1 [10] should be changed to $g'_x(x,y)\leq 0$ for all x,y. Otherwise, we cannot obtain $y\int_0^y g'_x(x,\xi)d\xi\leqslant 0$ from it.*

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