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A Boundedness Criterion for Fourth Order Nonlinear Ordinary Differential Equations with Delay

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Abstract: In this paper, by using Lyapunov's second method, we study a certain fourth order nonlinear ordinary delay differential equation and obtain some sufficient conditions for the boundedness of solutions of this equation.

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1 Introduction

Especially, since 1960s many good books, most of them are in Russian literature, dealt with delay differential equations. For a comprehensive treatment of subject we refer the reader to the book by Burton [1], Èl'sgol'ts [2], Èl'sgol'ts and Norkin [3], Hale [4], Hale and Verduyn Lunel [5], Kolmanovskii and Myshkis [6], Kolmanovskii and Nosov [7], Krasovskii [8], Yoshizawa [13] and the references citied in these books. With respect to our observations from the literature, it is only found two works achieved on the boundedness of solutions of fourth order nonlinear delay differential equations. These works can be summarized as follows: First, in 1989, Okoronkwo [10] considered the fourth-order nonlinear delay differential equation of the form

$$x^{(4)}(t) + f(x''(t))x'''(t) + \alpha_2 x''(t) + \beta_2 x''(t-h) + g(x'(t-h)) + \alpha_4 x(t) + \beta_4 x(t-h) = p(t).$$

Subject specified conditions imposed on the functions f, g, p and the constants α_2, α_4, β_2, β_4 appeared in this equation, he established some sufficient conditions that guarantee the boundedness of the solutions of the equation. Later, in 2000, Tejumola&Tchegnani [11] took into consideration the fourth-order nonlinear delay differential equation

$$x^{(4)}(t) + \varphi(t, x(t), x'(t), x''(t), x'''(t))x'''(t) + \psi(t, x'(t-\tau), x''(t-\tau)) + \chi(t, x(t-\tau), x'(t-\tau))$$

$$+h(x(t-\tau)) = p_2(t, x(t), x'(t), x''(t), x'''(t), x(t-\tau), x'(t-\tau), x''(t-\tau)).$$

They proved a result [11, Theorem 2.4] on the uniformly bounded and uniformly ultimately bounded of solutions of this equation.

In this paper we are concerned with the fourth order nonlinear delay differential equations of the type

$$\begin{aligned} &x^{(4)}(t) + \varphi(x''(t))x'''(t) + h(x''(t-r)) + \phi(x'(t-r)) + f(x(t-r)) \\ &= p(t, x(t), x'(t), x''(t), x'''(t), x(t-r), x'(t-r), x''(t-r)) \end{aligned} \tag{1}$$

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in which φ, h, ϕ, f and p depend only on the variables displayed explicitly and r is a positive constant, fixed delay; the primes in equation (1) denote differentiation with respect to t, $t \in [0, \infty)$. It is assumed as basic that the functions φ , h , ϕ , f and p are continuous in their respective arguments and satisfy a Lipschitz condition in $x(t)$, $x'(t)$, $x''(t)$, $x'''(t)$, $x(t-r)$, $x'(t-r)$ and $x''(t-r)$; $h(0) = \phi(0) = f(0) = 0$ and the derivatives $\frac{d\phi}{dx'} \equiv \phi'(x')$ and $\frac{df}{dx} \equiv f'(x)$ exist and are also continuous. Equation (1) can be transformed into an equivalent system of the form

$$x' = y, y' = z, z' = u,$$

$$\begin{aligned} u' = &- \varphi(z)u - h(z) - \phi(y) - f(x) + \int_{t-r}^{t} h'(z(s))u(s)ds + \int_{t-r}^{t} \phi'(y(s))z(s)ds \\ &+ \int_{t-r}^{t} f'(x(s))y(s)ds + p(t, x(t), y(t), z(t), u(t), x(t-r), y(t-r), z(t-r)) \end{aligned} \quad (2)$$

where $x(t)$, $y(t)$, $z(t)$ and $u(t)$ are respectively abbreviated as x, y, z and u throughout the paper. All solutions considered are also assumed to be real valued.

It should be noted in proving the main result of this paper we make use of Lyapunov's second method [9] as in Okoronkwo [10] and Tejumola and Tchegnani [11]. Our assumptions and the Lyapunov functional used here will be completely different than that in Okoronkwo [10] and Tejumola &Tchegnani [11].

2 Preliminaries

In order to reach our main result, we give some important basic information for the general non-autonomous delay differential system (see also Èl'sgol'ts [2], Èl'sgol'ts and Norkin [3], Hale [4], Hale and Verduyn Lunel [5], Kolmanovskii and Myshkis [6], Kolmanovskii and Nosov [7], Krasovskii [8]).

Now, we consider the general non-autonomous delay differential system

$$\dot{x} = f(t, x_t), x_t(\theta) = x(t + \theta), -r \le \theta \le 0, t \ge 0, \quad (3)$$

where $f : [0, \infty) \times C_H \to \Re^n$ is a continuous mapping, $f(t, 0) = 0$, and we suppose that f takes closed bounded sets into bounded sets of $\Re^n$. Here $(C, \|.\|)$ is the Banach space of continuous function $\phi : [-r, 0] \to \Re^n$ with supremum norm, $r > 0$,C_H is the open H -ball in C ;
$C_H := \{\phi \in (C[-r, 0], \Re^n) : \|\phi\| < H\}$. Standard existence theory, see Burton [1, pp.312], shows that if $\phi \in C_H$ and $t \ge 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ such that on $[t_0, t_0 + \alpha)$ satisfying equation (3) for $t > t_0$, $x_{t_0}(s, t, \phi) = \phi_{t_0}(s)$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B , then $\alpha = \infty$. Further, the symbol $|.|$ will denote the norm in $\Re^n$ with $|x| = \max_{1 \le i \le n} |x_i|$.

Definition 1 *(See [1, pp.223].) A continuous function $W : [0, \infty) \to [0, \infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i an integer.)*

Definition 2 *(See [1, pp. 260].) Let $V(t, \phi)$ be a continuous functional defined for $t \ge 0$, $\phi \in C_H$. The derivative of V along solutions of (3) will be denoted by $\dot{V}_{(3)}$ and is defined by the following relation*

$$\dot{V}_{(3)}(t, \phi) = \limsup_{h \to 0} \frac{V(t + h, x_{t+h}(t_0, \phi)) - V(t, x_t(t_0, \phi))}{h},$$

where $x(t_0, \phi)$ is the solution of (3) with $x_{t_0}(t_0, \phi) = \phi$.

Definition 3 *(See [13, pp.184].) A function $x(t_0, \phi)$ is said to be a solution of (3) with the initial condition $\phi \in C_H$ at $t = t_0$, $t_0 \ge 0$, if there is a constant $A > 0$ such that $x(t_0, \phi)$ is a function from $[t_0 - h, t_0 + A]$ into $\Re^n$ with the properties:*

(i) *$x_t(t_0, \phi) \in C_H$ for $t_0 \le t < t_0 + A$,*
(ii) *$x_{t_0}(t_0, \phi) = \phi$,*
(iii) *$x(t_0, \phi)$ satisfies (3) for $t_0 \le t < t_0 + A$.*

Theorem 1 *(See [13, pp.184].) If* $f(t,\phi)$ *in (3) is continuous in* t , ϕ *, for every* $\phi \in C_{H_1}$, $H_1 < H$, *and* t_0, $0 \leq t_0 < c$, *where c is a positive constant, then there exist a solution of (3) with initial value* ϕ *at* $t = t_0$, *and this solution has a continuous derivative for* $t > t_0$.

3 Main result

First, we introduce the following notations:

$$\varphi_1(z) = \begin{cases} \frac{1}{z}\int\limits_0^z \varphi(\tau)d\tau, z \neq 0 \\ \varphi(0), z = 0 \end{cases}$$

and

$$\phi_1(y) = \begin{cases} \frac{\phi(y)}{y}, y \neq 0 \\ \phi'(0), y = 0. \end{cases}$$

Our main result is the following theorem.

Theorem 2 *In addition to the basic assumptions imposed on* φ, h, ϕ, f *and* p, *we assume the following conditions are satisfied:*

(i) *There are positive constants* α_1, α_2, α_3, α_4, Δ, L, d_1, d_2, d_3 *and* ε *such that*

$$\alpha_1\alpha_2\alpha_3 - \alpha_3\phi'(y) - \alpha_1\alpha_4\varphi(z) \geq \Delta > 0 \text{ for all } y \text{ and } z,$$

in which $\varepsilon \leq \frac{\Delta}{2\alpha_1\alpha_3 D}$, $D = \alpha_1\alpha_2 + \frac{\alpha_2\alpha_3}{\alpha_4}$;

(ii) $0 < \alpha_4 - \frac{\alpha_1\Delta}{4\alpha_3} < f'(x) \leq \alpha_4$ *for all x;*

(iii) $\phi'(y) \geq \alpha_3$ *and* $0 \leq \phi_1(y) - \alpha_3 < \frac{\Delta}{8\alpha_3}\sqrt{\frac{\alpha_4}{2\alpha_1\alpha_3}}$ *for all y;*

(iv) $0 \leq \frac{h(z)}{z} - \alpha_2 \leq \frac{\alpha_3}{8\alpha_4}\sqrt{\frac{\varepsilon\Delta}{\alpha_1}}$ *for all z* , $(z \neq 0)$ *and* $|h'(z)| \leq L$ *for all z;*

(v) $\varphi(z) \geq \alpha_1$, $\varphi_1(z) - \varphi(z) < \frac{\Delta}{2\alpha_1^2\alpha_3}$ *for all z;*

(vi) $|p(t, x(t), y(t), z(t), u(t), x(t-r), y(t-r), z(t-r))| \leq q(t)$, *where* $\max q(t) < \infty$ *and* $q \in L^1(0,\infty)$, $L^1(0,\infty)$ *is space of integrable Lebesgue functions.*

Then, there exists a finite positive constant K such that the solution $x(t)$ of equation (1) defined by the initial functions

$$x(t) = \phi(t), x'(t) = \phi'(t), x''(t) = \phi''(t), x'''(t) = \phi'''(t)$$

satisfies the inequalities

$$|x(t)| \leq K, |x'(t)| \leq K, |x''(t)| \leq K, |x'''(t)| \leq K$$

for all $t \geq t_0$, where $\phi \in C^3\left([t_0 - r, t_0]\,,\ \Re\right)$, provided that

$$r < 2\min\left\{\frac{\varepsilon\alpha_3}{2(d_2\alpha_4 + d_2L + d_2\alpha_1\alpha_2 + 2\lambda)}, \frac{\Delta}{8\alpha_1\alpha_3(\alpha_1\alpha_2 + L + \alpha_4 + 2\mu)}, \frac{\varepsilon\alpha_1}{(d_1\alpha_1\alpha_2 + d_1L + d_1\alpha_4 + 2\rho)}\right\}$$

with $\lambda = \frac{\alpha_4}{2}(d_1 + d_2 + 1) > 0$, $\mu = \frac{\alpha_1\alpha_2}{2}(d_1 + d_2 + 1) > 0$ and $\rho = \frac{L}{2}(d_1 + d_2 + 1) > 0$.

Remark 3 *Making use of conditions(i),(iii) and (v) of Theorem 2 we obtain that*

$$\varphi(z) < \frac{\alpha_2\alpha_3}{\alpha_4}, \phi'(y) < \alpha_1\alpha_2.$$

Proof. Now, to verify Theorem 2, we introduce the Lyapunov functional

$$\begin{aligned}2V(x_t,y_t,z_t,u_t) = \; & 2d_2\int_0^x f(\xi)d\xi+\alpha_2 d_2 y^2-d_1\alpha_4 y^2+2\int_0^y \phi(\eta)d\eta+2d_1\int_0^z h(\zeta)d\zeta \\ & +2\int_0^z \varphi(\tau)\tau d\tau-d_2 z^2+d_1u^2+2f(x)y+2d_1 f(x)z+2d_1\phi(y)z \\ & +2d_2 y\int_0^z \varphi(\tau)d\tau+2d_2yu+2zu+2\lambda\int_{-r}^0\int_{t+s}^t y^2(\theta)d\theta ds \\ & +2\mu\int_{-r}^0\int_{t+s}^t z^2(\theta)d\theta ds+2\rho\int_{-r}^0\int_{t+s}^t u^2(\theta)d\theta ds\end{aligned} \tag{4}$$

where

$$d_1=\varepsilon+\frac{1}{\alpha_1}, d_2=\varepsilon+\frac{\alpha_4}{\alpha_3},$$

and λ, μ and μ are some positive constants which will be determined later in the proof.

In view of the assumptions of Theorem 2, one can easily obtain that

$$\begin{aligned}2V \; & \geq \varepsilon\left(\alpha_4-\tfrac{\alpha_1\Delta}{4\alpha_3}\right)x^2+\left(\tfrac{\Delta\alpha_4}{4\alpha_1\alpha_3^2}\right)y^2+\left(\tfrac{\Delta}{8\alpha_1^2\alpha_3}\right)z^2+\varepsilon u^2 \\ & +2\lambda\int_{-r}^0\int_{t+s}^t y^2(\theta)d\theta ds+2\mu\int_{-r}^0\int_{t+s}^t z^2(\theta)d\theta ds+2\rho\int_{-r}^0\int_{t+s}^t u^2(\theta)d\theta ds \\ & \geq D_1x^2+D_2y^2+D_3z^2+D_4u^2+2\lambda\int_{-r}^0\int_{t+s}^t y^2(\theta)d\theta ds \\ & +2\mu\int_{-r}^0\int_{t+s}^t z^2(\theta)d\theta ds+2\rho\int_{-r}^0\int_{t+s}^t u^2(\theta)d\theta ds \\ & \geq D_1x^2+D_2y^2+D_3z^2+D_4u^2\geq 2D_5\left(x^2+y^2+z^2+u^2\right),\end{aligned} \tag{5}$$

where $D_1=\frac{1}{2}\varepsilon\left(\alpha_4-\frac{\alpha_1\Delta}{4\alpha_3}\right)$, $D_2=\frac{\Delta\alpha_4}{8\alpha_1\alpha_3^2}$, $D_3=\frac{\Delta}{16\alpha_1^2\alpha_3}$, $D_4=\varepsilon$ and $D_5=\min\{D_1,D_2,D_3,D_4\}$. (See, also, for the details of the operations to Tunç [12]).

Now, differentiating the functional $V=V(x_t,y_t,z_t,u_t)$ in (4), we have

$$\begin{aligned}\tfrac{d}{dt}V(x_t,y_t,z_t,u_t) = & -[\alpha_4-f'(x)]\,.\left[y+\tfrac{d_1z}{2}\right]^2-\left[\tfrac{h(z)}{z}-d_1\phi'(y)-d_2\varphi_1(z)\right]z^2 \\ & +\tfrac{d_1^2}{4}[\alpha_4-f'(x)]z^2-[d_1\varphi(z)-1]u^2-\left[d_2\tfrac{\phi(y)}{y}-\alpha_4\right]y^2 \\ & -d_2\left[\tfrac{h(z)}{z}-\alpha_2\right]\,yz-d_1[\alpha_4-f'(x)]\,yz \\ & +(d_1u+z+d_2y)p(t,x(t),y(t),z(t),u(t),x(t-r),y(t-r),z(t-r)) \\ & +(d_1u+z+d_2y)\int_{t-r}^t h'(z(s))u(s)ds+(d_1u+z+d_2y)\int_{t-r}^t \phi'(y(s))z(s)ds \\ & +(d_1u+z+d_2y)\int_{t-r}^t f'(x(s))y(s)ds+\lambda y^2r-\lambda\int_{t-r}^t y^2(s)ds \\ & +\mu z^2r-\mu\int_{t-r}^t z^2(s)ds+\rho u^2r-\rho\int_{t-r}^t u^2(s)ds.\end{aligned} \tag{6}$$

By following some similar lines taken place in Tunç [12], one can easily obtain that

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t,u_t) \;&\leq -\left(\tfrac{\varepsilon\alpha_3}{2}\right)y^2-\left(\tfrac{\Delta}{8\alpha_1\alpha_3}\right)z^2-(\varepsilon\alpha_1)u^2+(d_1u+z+d_2y)\int\limits_{t-r}^{t}h'(z(s))u(s)ds\\
&+(d_1u+z+d_2y)\int\limits_{t-r}^{t}\phi'(y(s))z(s)ds+(d_1u+z+d_2y)\int\limits_{t-r}^{t}f'(x(s))y(s)ds\\
&+(d_1u+z+d_2y)p(t,x(t),y(t),z(t),u(t),x(t-r),y(t-r),z(t-r))\\
&+\lambda y^2r-\lambda\int\limits_{t-r}^{t}y^2(s)ds+\mu z^2r-\mu\int\limits_{t-r}^{t}z^2(s)ds+\rho u^2r-\rho\int\limits_{t-r}^{t}u^2(s)ds.\end{aligned}$$

Now, in view of the assumptions $f'(x)\leq\alpha_4$, $|\phi'(y)|\leq\alpha_1\alpha_2$, $|h'(z)|\leq L$ and $2\,|ab|\leq a^2+b^2$, it follows the following inequalities for some terms contained in (6):

$$\begin{aligned}(d_1u+z+d_2y)\int\limits_{t-r}^{t}h'(z(s))u(s)ds\leq\; &\tfrac{d_1L}{2}ru^2(t)+\tfrac{L}{2}rz^2(t)+\tfrac{d_2L}{2}ry^2(t)\\
&+\tfrac{L}{2}(d_1+d_2+1)\int\limits_{t-r}^{t}u^2(s)ds,\end{aligned}$$

$$\begin{aligned}(d_1u+z+d_2y)\int\limits_{t-r}^{t}\phi'(y(s))z(s)ds\leq\; &\tfrac{d_1\alpha_1\alpha_2}{2}ru^2(t)+\tfrac{\alpha_1\alpha_2}{2}rz^2(t)+\tfrac{d_2\alpha_1\alpha_2}{2}ry^2(t)\\
&+\tfrac{\alpha_1\alpha_2}{2}(d_1+d_2+1)\int\limits_{t-r}^{t}z^2(s)ds\end{aligned}$$

and

$$\begin{aligned}(d_1u+z+d_2y)\int\limits_{t-r}^{t}f'(x(s))y(s)ds\leq\; &\tfrac{d_1\alpha_4}{2}ru^2(t)+\tfrac{\alpha_4}{2}rz^2(t)+\tfrac{d_2\alpha_4}{2}ry^2(t)\\
&+\tfrac{\alpha_4}{2}(d_1+d_2+1)\int\limits_{t-r}^{t}y^2(s)ds.\end{aligned}$$

Substituting these estimates into (6) we get

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t,u_t)\leq\; &-\left[\tfrac{\varepsilon\alpha_3}{2}-\tfrac{1}{2}(d_2\alpha_4+d_2L+d_2\alpha_1\alpha_2+2\lambda)r\right]y^2\\
&-\left(\tfrac{\Delta}{8\alpha_1\alpha_3}-\tfrac{1}{2}(\alpha_1\alpha_2+L+\alpha_4+2\mu)r\right)z^2\\
&+(d_1u+z+d_2y)p(t,x(t),y(t),z(t),u(t),x(t-r),y(t-r),z(t-r))\\
&-\left(\varepsilon\alpha_1-\tfrac{1}{2}(d_1\alpha_1\alpha_2+d_1L+d_1\alpha_4+2\rho)r\right)u^2\\
&+\left[\tfrac{\alpha_4}{2}(d_1+d_2+1)-\lambda\right]\int\limits_{t-r}^{t}y^2(s)ds\\
&+\left[\tfrac{\alpha_1\alpha_2}{2}(d_1+d_2+1)-\mu\right]\int\limits_{t-r}^{t}z^2(s)ds\\
&+\left[\tfrac{L}{2}(d_1+d_2+1)-\rho\right]\int\limits_{t-r}^{t}u^2(s)ds.\end{aligned}$$

Let us choose

$$\lambda = \frac{\alpha_4}{2}(d_1 + d_2 + 1) > 0,$$
$$\mu = \frac{\alpha_1\alpha_2}{2}(d_1 + d_2 + 1) > 0$$

and

$$\rho = \frac{L}{2}(d_1 + d_2 + 1) > 0.$$

Hence

$$\begin{aligned}\tfrac{d}{dt}V(x_t, y_t, z_t, u_t) \le\ & -\left[\tfrac{\varepsilon\alpha_3}{2} - \tfrac{1}{2}(d_2\alpha_4 + d_2L + d_2\alpha_1\alpha_2 + 2\lambda)r\right] y^2 \\ & -\left(\tfrac{\Delta}{8\alpha_1\alpha_3} - \tfrac{1}{2}(\alpha_1\alpha_2 + L + \alpha_4 + 2\mu)r\right) z^2 \\ & -\left(\varepsilon\alpha_1 - \tfrac{1}{2}(d_1\alpha_1\alpha_2 + d_1L + d_1\alpha_4 + 2\rho)r\right) u^2 \\ & +(d_1u + z + d_2y)p(t, x(t), y(t), z(t), u(t), x(t-r), y(t-r), z(t-r)).\end{aligned}$$

Now, in fact, we can obtain

$$\begin{aligned}\tfrac{d}{dt}V(x_t, y_t, z_t, u_t) \le\ & -\tau(y^2 + z^2 + u^2) \\ & + |d_1u + z + d_2y|\,|p(t, x(t), y(t), z(t), u(t), x(t-r), y(t-r), z(t-r))|\end{aligned}$$

for some constant $\tau > 0$ provided that

$$r < 2\min\left\{\frac{\varepsilon\alpha_3}{2(d_2\alpha_4 + d_2L + d_2\alpha_1\alpha_2 + 2\lambda)}, \frac{\Delta}{8\alpha_1\alpha_3(\alpha_1\alpha_2 + L + \alpha_4 + 2\mu)}, \frac{\varepsilon\alpha_1}{(d_1\alpha_1\alpha_2 + d_1L + d_1\alpha_4 + 2\rho)}\right\}$$

Therefore

$$\begin{aligned}\tfrac{d}{dt}V(x_t, y_t, z_t, u_t) &\le |d_1u + z + d_2y|\,|p(t, x(t), y(t), z(t), u(t), x(t-r), y(t-r), z(t-r))| \\ &\le D_6\left(|y| + |z| + |u|\right)\ q(t)\end{aligned}$$

for a constant $D_6 > 0$ by (vi), where $D_6 = \max\{d_1,\ 1,\ d_2\}$.

Now, making use of the inequalities $|y| < 1 + y^2$, $|z| < 1 + z^2$ and $|u| < 1 + u^2$, it is clear that

$$\frac{d}{dt}V(x_t, y_t, z_t, u_t) \le D_6\left(3 + y^2 + z^2 + u^2\right)\ q(t).$$

By (5), we also have

$$\left(y^2 + z^2 + u^2\right) \le \left(x^2 + y^2 + z^2 + u^2\right) \le D_5^{-1}\,V(x_t, y_t, z_t, u_t).$$

Hence

$$\begin{aligned}\tfrac{d}{dt}V(x_t, y_t, z_t, u_t) &\le D_6\left(3 + D_5^{-1}V(x_t, y_t, z_t, u_t)\right) q(t) \\ &= 3D_6q(t) + D_6D_5^{-1}V(x_t, y_t, z_t, u_t)q(t).\end{aligned}$$

Now, integrating the last inequality from 0 to t , using the assumption $q \in L^1(0, \infty)$ and Gronwall-Reid-Bellman inequality, we obtain

$$\begin{aligned}V(x_t, y_t, z_t, u_t) &\le V(x_0, y_0, z_0, u_0) + 3D_6A + D_6D_5^{-1}\int_0^t \left(V(x_s, y_s, z_s, u_s)\right) q(s)ds \\ &\le \left(V(x_0, y_0, z_0, u_0) + 3D_6A\right)\exp\left(D_6D_5^{-1}\int_0^t q(s)ds\right) \\ &\le \left(V(x_0, y_0, z_0, u_0) + 3D_6A\right)\exp\left(D_6D_5^{-1}A\right) = K_1 < \infty,\end{aligned} \tag{7}$$

where $K_1 > 0$ is a constant, $K_1 = (V(x_0, y_0, z_0, u_0) + 3D_6A) \exp\left(D_6D_5^{-1}A\right) = K_1 < \infty$ and $A = \int\limits_0^\infty q(s)ds$.

Now, the inequalities (5) and (7) together yields that

$$x^2(t) + y^2(t) + z^2(t) + u^2(t) \leq 2D_5^{-1}V(x_t, y_t, z_t, u_t) \leq K,$$

where $K = 2K_1D_5^{-1}$. Thus, we can conclude that

$$|x(t)| \leq K, |y(t)| \leq K, |z(t)| \leq K, |u(t)| \leq K$$

for all $t \geq t_0$. That is,

$$|x(t)| \leq K, \left|x'(t)\right| \leq K, \left|x''(t)\right| \leq K, \left|x'''(t)\right| \leq K$$

for all $t \geq t_0$. The proof of Theorem 2 is completed. ■

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