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Boundary Control on the Modified b-family Equation

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Abstract: This paper investigates the boundary control of the modified b-family equation on [0,1]. The existence of its solution is proved in a short time interval under the given boundary condition. Meanwhile, we prove the global exponential stability of the solutions in four different spaces, i.e. in L^2, H^1, H^2, H^3.

Key words: the modified b-family equation; boundary control;global exponential stability

1 Introduction

In [1], D.D.Holm and M.F.Struley introduced the b-family equation which describes the balance between the convection and the stretching for small viscosity in the dynamics of 1D nonlinear wave in fluids:

$$m_t + \underbrace{um_x}_{convection} + \underbrace{bu_x m}_{stretching} = \underbrace{\varepsilon m}_{vis\cos ity}, u = g * m, \tag{1.1}$$

where $u = g * m$ denotes $u(x) = \int_{-\infty}^{\infty} g(x-y)m(y)dy$. The convolution relates velocity u to momentum density m by integration against the kernel $g(x)$. Here the kernel g is chosen to be the Green's function for the Helmholtz operator on the line, that is, $g(x) = \frac{1}{2}e^{-|x|}$. This means $m = u - u_{xx}$. We have found it convenient to rewrite Eq.(1.1) as the following form

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, t > 0, x \in R. \tag{1.2}$$

Recently, Lixin Tian et al studied the attractor and optimal control and boundary control on the b-family equation [2-4].

Boundary control on different equations has been constantly investigated. Byrnes et al studied local stability of Burgers equation. Vanly et al further consummated this result, while the slution is still in the local sense. Miroslav Kristic studied global stability of Burgers equation [5]. Biler Rassel and Zhang Bingyu studied KdVB equation under periodical boundary condition[6-8]. Liu and Kristic studied the stability of KdVB equation in a limited area [9].Wei-jiu Liu and Miroslav Krstic studied global stability of K-S equation [10]. Haixia Chao also did some studies on boundary control of K-S equation [11]. In this paper, we will study the modified viscosity b-family equation with the following boundary

$$\begin{cases} u_t - u_{xxt} - \varepsilon\left(u - u_{xx}\right)_{xx} + (b+1)u^2 u_x = bu_x u_{xx} + uu_{xxx} \\ u_0 = u(x,0) \\ u(0,t) = u(1,t) = u_{xx}(0,t) = 0 \\ u_{xxx}(1,t) = u_x(1,t) = u_x(0,t) \end{cases} \tag{1.3}$$

where $t > 0$, $x \in \Omega, \Omega = [0,1]$.

We discuss the global well-posedness of (1.3). It will be shown that for given $T > 0$, $u_0 \in H^3$, (1.3) admits a unique solution $u \in C\left((0,\infty), H^3(0,1)\right) \cap C'\left((0,\infty), H^2(0,1)\right)$.

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2 Main Theorem

Denote $A = -\Delta$, where Δ is the Laplace operator. Let $v = u + Au$. Denote $B(u, v) = uv_x$. Eq. (1.5) can be denoted as

$$\begin{cases} \frac{dv}{dt} + \varepsilon Av + B(u,v) + bB(v,u) - (b+1)uu_x + (b+1)u^2u_x = 0 \\ u_0 = u(x,0), \\ u(0,t) = u(1,t) = u_{xx}(0,t) = 0 \\ u_{xxx}(1,t) = u_x(1,t) = u_x(0,t) \end{cases} \tag{2.1}$$

where A is a self-adjoint positive operator with compact inverse. Hence the space H has an orthonormal basis $\{\phi_j\}_{j=1}^{\infty}$ of eigenfunctions of A, i.e. $A\phi_j = \lambda_j\phi_j$, with $0 < \lambda_1 \le \lambda_2 \le \cdots, \lambda_j \to \infty$ when $j \to \infty$.

Theorem 1 *With $u_0 \in H^3$ Eq.(2.1) has a global solution*

$$u \in C\left((0,\infty), H^3(0,1)\right) \cap C'\left((0,\infty), H^2(0,1)\right) \quad ,$$

and satisfies the following L^2, H^1, H^2, H^3 stability:

(1)

$$\|u\|_{L^2}^2 + \|u\|_{H^1}^2 \le (\|u_0\|_{L^2}^2 + \|u_0\|_{H^1}^2)\exp(M_1(b-2) - 2\varepsilon)\lambda_3 t, \tag{2.2}$$

where $\varepsilon \in \left(\frac{M_1(b-2)}{2}, +\infty\right)$, $\|u_0\|_{L^2}^2 + \|u_0\|_{H^1}^2 \triangleq M_1^2$.

(2)

$$\|u\|_{H^1}^2 + \|u\|_{H^2}^2 \le \left(\|u_0\|_{H^1}^2 + \|u_0\|_{H^2}^2\right)\exp\left((b+1)(M_1^2 + 5M_1) - 2\varepsilon\lambda_4\right)t, \tag{2.3}$$

where $\varepsilon \in (\frac{(b+1)(M_1^2+5M_1)}{2\lambda_4}, +\infty)$, λ_4 *is the Poincare constant.*

(3)

$$\|u\|_{H^2}^2 + \|u\|_{H^3}^2 \le \left(\|u_0\|_{H^2}^2 + \|u_0\|_{H^3}^2\right)\exp\left((b+1)C_1(5M_1C_2 + 9)M_1^{\frac{1}{2}}M_7^{\frac{1}{2}} - 2\varepsilon\lambda_5\right)t, \tag{2.4}$$

where $\varepsilon \in (\frac{(b+1)C_1(5M_1C_2+9)M_1^{\frac{1}{2}}M_7^{\frac{1}{2}}}{2\lambda_5}, +\infty)$, λ_5 *is thePoincareconstant,* C_1, C_2, M_1, M_7 *are the positive constants.*

We will use the Galerkin procedure to prove its global existence.

Let $\{\phi_j\}_{j=1}^{\infty}$be an orthonormal basis of Hconsisting of eigenfunctions of the operator A. The Galerkin Procedure for Eq.(2.1) is the ordinary differential system,

$$\begin{cases} \frac{dv_m}{dt} + \varepsilon Av_m + p_mB(u_m, v_m) + p_mbB(v_m, u_m) + (b+1)u_m^2u_{m,x} - (b+1)u_mu_{m,x} = 0 \\ u_m(0) = p_mu(0) \end{cases} \tag{2.5}$$

where $v_m = u_m + Au_m$. Since the nonlinear term is quadratic in u_m, then by the classical theory of ordinary differential equations, the system (2.5) has a unique solution for a short interval of time $(0, T_m)$. Our purpose is to show the existence of the solution under given boundary control.

3 Proof of Theorem

First, we prove the global existence of the solution.

We take the inner product of (2.5) with u_m in Ω to obtain,

$$\frac{1}{2}\frac{d}{dt}\left(\|u_m\|_{L^2}^2 + \|u_m\|_{H^1}^2\right) + \varepsilon\left(\|u_m\|_{H^1}^2 + \|Au_m\|_{L^2}^2\right) + p_m(b-2)\left(B(u_m,u_m), Au_m\right) = 0. \tag{3.1}$$

By Amgon inequality ,Young inequality and Poincare inequality, we get

$$|p_m(B(u_m,u_m), Au_m)| \le \frac{1}{2}\|u_{m,x}\|_{L^\infty}\|u_{m,x}\|_{L^2}^2 \le \frac{k_1}{2}\|Au_m\|_{L^2}\|u_m\|_{H^1}^2. \tag{3.2}$$

It follows from Young inequality

$$\begin{array}{l} \frac{1}{2}\frac{d}{dt}\left(\|u_m\|_{L^2}^2+\|u_m\|_{H^1}^2\right)+\varepsilon\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right) \\ \le \frac{k_1(b-2)}{2}\|Au_m\|_{L^2}\|u_m\|_{H^1}^2 \\ \le \varepsilon\|u_m\|_{H^1}^2+\frac{k_1^2(b-2)^2}{4\varepsilon}\|u_m\|_{H^1}^4+\varepsilon\|Au_m\|_{L^2}^2 \end{array}, \tag{3.3}$$

$$\frac{d}{dt}\left(\|u_m\|_{L^2}^2+\|u_m\|_{H^1}^2\right)\le\frac{k_1^2(b-2)^2}{2\varepsilon}(\|u_m\|_{L^2}^2+\|u_m\|_{H^1}^2)^2. \tag{3.4}$$

Then $\forall t\in[0,T]$ and $T<\frac{1}{M(\|u_0\|_{L^2}^2+\|u_0\|_{H^1}^2)}$. It thus transpires that

$$\|u_m\|_{L^2}^2+\|u_m\|_{H^1}^2\le\frac{\|u_0\|_{L^2}^2+\|u_0\|_{H^1}^2}{1-M(\|u_0\|_{L^2}^2+\|u_0\|_{H^1}^2)t}\le\|u_0\|_{L^2}^2+\|u_0\|_{H^1}^2\triangleq M_1^2,$$

where $M=\frac{2\varepsilon}{k_1^2(b-2)^2}$

From the above analysis, we know that $\|u_m\|_{L^2}\le M_1$, $\|u_m\|_{H^1}\le M_1$, where M_1 is a positive constant. So

$$\begin{array}{l} |p_m(B(u_m,u_m),Au_m)|\le\frac{1}{2}\|u_{m,x}\|_{L^\infty}\|u_{m,x}\|_{L^2}^2\le\frac{k_1}{2}\|Au_m\|_{L^2}\|u_m\|_{H^1}^2 \\ \le\frac{k_1M_1}{4}\ (\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2). \end{array}$$

Integrating (3.1) over the interval $[t,t+r]$, then

$$\int_t^{t+r}\left(\|u_m(s)\|_{H^1}^2+\|Au_m(s)\|_{L^2}^2\right)ds\le\frac{M_1^2r}{\varepsilon-\frac{k_1M_1}{4}}\triangleq M_2. \tag{3.5}$$

Now, take the inner product of (2.5) with Au_m in$(0,1)$ to obtain

$$\begin{array}{l} \frac{1}{2}\frac{d}{dt}\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)+\varepsilon\left(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2\right)+p_m(B(u_m,v_m),Au_m)+ \\ bp_m(B(v_m,u_m),Au_m)+((b+1)u_m^2u_{m,x},Au_m)-((b+1)u_mu_{m,x},Au_m)=0. \end{array} \tag{3.6}$$

By computing, we have

$$|p_m(B(u_m,v_m),Au_m)+bp_m(B(v_m,u_m),Au_m)|$$

$$=|(b+1)(B(u_m,u_m),Au_m)+b(B(Au_m,u_m),Au_m)+(B(u_m,Au_m),Au_m)|$$

By Agmon inequality, we get

$$|p_m(B(u_m,u_m),Au_m)|\le\frac{1}{2}\|u_{m,x}\|_{L^\infty}\|u_{m,xx}\|_{L^2}^2\le\frac{C_1}{2}\|u_m\|_{H^1}^{\frac{5}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}},$$

$$|p_m(B(u_m,Au_m),Au_m)|\le\frac{1}{2}\|u_{m,x}\|_{L^\infty}\|Au_m\|_{L^2}^2\le\frac{C_1}{2}\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{5}{2}},$$

$$|p_m(B(Au_m,u_m),Au_m)|\le\|u_{m,x}\|_{L^\infty}\|Au_m\|_{L^2}^2\le C_1\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{5}{2}},$$

$$\left|((b+1)u_m^2u_{m,x},Au_m)\right|\le(b+1)\|u_m\|_{L^\infty}\|u_{m,x}\|_{L^\infty}\|u_m\|_{H^1}^2\le(b+1)C_1C_2M_1\|Au_m\|_{L^2}^{\frac{1}{2}}\|u_m\|_{H^1}^{\frac{5}{2}},$$

$$((b+1)u_mu_{m,x},Au_m)\le\frac{b+1}{2}\|u_{m,x}\|_{L^\infty}\|u_m\|_{H^1}^2\le\frac{C_1(b+1)}{2}\|u_m\|_{H^1}^{\frac{5}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}},$$

From above inequalities and Young inequality, we get

$$\begin{array}{l} \left|p_m(B(u_m,v_m),Au_m)+bp_m(B(v_m,u_m),Au_m)+((b+1)u_m^2u_{mx},Au_m)-((b+1)u_mu_{mx},Au_m)\right| \\ \le(b+1)C_1(1+C_1C_2M_1)\|u_m\|_{H^1}^{\frac{5}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}+\left(bC_1+\frac{C_1}{2}\right)\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{5}{2}} \\ \le C_3\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right) \\ \le\varepsilon\lambda_1\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)+C_4\|u_m\|_{H^1}\|Au_m\|_{L^2}\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right) \end{array} \tag{3.7}$$

where $C_4 = \frac{C_3^2}{4\varepsilon\lambda_1}$.

By (3.6) and (3.7), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)+\varepsilon\left(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2\right)\\&\le\varepsilon\lambda_1\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)+C_4\|u_m\|_{H^1}\|Au_m\|_{L^2}\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right).\end{aligned}\tag{3.8}$$

By Poincare inequality $\|Au_m\|_{L^2}^2 > \lambda_1\|u_m\|_{H^1}^2$, $\|Au_m\|_{H^1}^2 > \lambda_1\|Au_m\|_{L^2}^2$and (3.8) and Young inequality, we get

$$\frac{d}{dt}\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)\le C_4\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)^2.\tag{3.9}$$

Denote $y=\|u_m(s)\|_{H^1}^2+\|Au_m(s)\|_{L^2}^2$, $g=C_4\left(\|u_m(s)\|_{H^1}^2+\|Au_m(s)\|_{L^2}^2\right)$. By inequality (3.5), we have $\int_t^{t+r}y(s)ds\le M_2$, $\int_t^{t+r}g(s)ds\le C_4M_2$.

By the uniform Gronwall lemma, we have

$$\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)\le\frac{M_2}{r}\exp\left(C_4M_2\right)\underline{\underline{\Delta}}M_3, t>t_0.\tag{3.10}$$

Integrating (3.8) over the interval $[t,t+r]$, we have

$$\begin{aligned}&\varepsilon\int_t^{t+r}\left(\|Au_m(s)\|_{L^2}^2+\|Au_m(s)\|_{H^1}^2\right)ds\\&\le\int_t^{t+r}\left(\varepsilon\lambda_1\left(\|u_m(s)\|_{H^1}^2+\|Au_m(s)\|_{L^2}^2\right)+\frac{C_4}{2}\left(\|u_m(s)\|_{H^1}^2+\|Au_m(s)\|_{L^2}^2\right)^2\right)ds+M_3\\&\le\left(\varepsilon\lambda_1M_3+\frac{C_4M_3^2}{2}\right)r+M_3\underline{\underline{\Delta}}M_4.\end{aligned}\tag{3.11}$$

Take the inner product of (2.5) with A^2u_m in Ω to obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2\right)+\varepsilon\left(\|Au_m\|_{H^1}^2+\left\|A^2u_m\right\|_{L^2}^2\right)+p_m\left(B\left(u_m,v_m\right),A^2u_m\right)+\\&bp_m\left(B\left(v_m,u_m\right),A^2u_m\right)+((b+1)u_m^2u_{m,x},A^2u_m)-((b+1)u_mu_{m,x},A^2u_m)=0\end{aligned}\tag{3.12}$$

According to Agmon inequality when $n=1$, then we have

$$\left|P_m\left(B\left(u_m,u_m\right),A^2u_m\right)\right|\le\|u_{m,x}\|_{L^\infty}\|Au_m\|_{L^2}^2\le C_1\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2),$$

$$\left|P_m\left(B\left(Au_m,u_m\right),A^2u_m\right)\right|\le\|u_{m,x}\|_{L^\infty}\|Au_m\|_{H^1}^2\le C_1\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2),$$

$$\left|P_m\left(B\left(u_m,Au_m\right),A^2u_m\right)\right|\le\|u_{m,x}\|_{L^\infty}\|Au_m\|_{H^1}^2\le C_1\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2),$$

$$\begin{aligned}&((b+1)u_m^2u_{m,x},A^2u_m)\le5\left|\int_0^1u_mu_{m,x}u_{m,xx}^2dx\right|\le5\|u_m\|_{L^\infty}\|u_{m,x}\|_{L^\infty}\|Au_m\|_{L^2}^2\\&\le5(C_2\|u_m\|_{L^2}^{\frac{1}{2}}\|u_m\|_{H^1}^{\frac{1}{2}})(C_1\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}})(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2)\\&\le5C_1C_2M_1\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2),\end{aligned}$$

$$((b+1)u_mu_{m,x},A^2u_m)\le\frac{5}{2}\|u_{m,x}\|_{L^\infty}\|Au_m\|_{L^2}^2\le\frac{5}{2}C_1\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2).$$

From above inequalities, we can obtain

$$\begin{aligned}&\left|P_m\left(B\left(u_m,v_m\right),A^2u_m\right)+bP_m\left(B\left(v_m,u_m\right),A^2u_m\right)+((b+1)u_m^2u_{mx},A^2u_m)-((b+1)u_mu_{mx},A^2u_m)\right|\\&=\big|(b+1)P_m(B(u_m,u_m),A^2u_m)+(B(u_m,Au_m),A^2u_m)+b(B(Au_m,u_m),A^2u_m)\\&+((b+1)u_m^2u_{m,x},A^2u_m)-((b+1)u_mu_{m,x},A^2u_m)\big|\\&\le(b+1)C_1(\frac{9}{2}+5C_2M_1)\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2)\\&=C_5\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2),\end{aligned}\tag{3.13}$$

By (3.12) and above inequality and Young inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2\right)+\varepsilon\left(\|Au_m\|_{H^1}^2+\left\|A^2u_m\right\|_{L^2}^2\right)\\
&\le C_5\,\|u_m\|_{H^1}^{\frac{1}{2}}\,\|Au_m\|_{L^2}^{\frac{1}{2}}\,(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2)\\
&\le \varepsilon\lambda_2\left(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2\right)+C_6\,\|u_m\|_{H^1}\,\|Au_m\|_{L^2}\,(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2)
\end{aligned} \tag{3.14}$$

where $C_6=\frac{C_5^2}{4\varepsilon\lambda_2}$.

By Poincare inequality and Young inequality, we have

$$\begin{aligned}
&\tfrac{d}{dt}\left(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2\right)\le 2C_6\,\|u_m\|_{H^1}\,\|Au_m\|_{L^2}\,(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2)\\
&\le C_6\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)\left(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2\right)
\end{aligned}$$

From (3.5) and (3.11) and uniform Gronwall lemma, we get

$$\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2\le\frac{M_4}{r\varepsilon}\exp C_6M_2\underline{\underline{\Delta}}M_5 \tag{3.15}$$

Integrating (3.14) over the interval $[t,t+r]$, we get

$$\varepsilon\int_t^{t+r}\left(\|Au_m\|_{H^1}^2+\left\|A^2u_m\right\|_{L^2}^2\right)ds\le\left(\varepsilon\lambda_2M_5+\frac{c_6M_3M_5}{2}\right)r+M_5 \tag{3.16}$$

Take the inner product of (3.5) with A^3u_m in Ω to obtain,

$$\|Au_m\|_{H^1}^2+\left\|A^2u_m\right\|_{L^2}^2\le M_6, \tag{3.17}$$

Similarly (3.17) stands by using the above method and the uniform Gronwall lemma.

Now we get that $\|u_m\|_{L^2}$, $\|u_m\|_{H^1}$, $\|Au_m\|_{L^2}$, $\|Au_m\|_{H^1}$and $\left\|A^2u_m\right\|_{L^2}$are bounded. Then $\|v_m\|_{L^2}$, $\|v_m\|_{H^1}$ and $\|Av_m\|_{L^2}$ are bounded. Hence we get $\left\|\frac{du_m}{dt}\right\|_{L^2}$ and $\left\|\frac{dv_m}{dt}\right\|_{L^2}$ are also bounded. By $Aubin's$ Compactness Theorem, we conclude that there is a subsequence u'_m, such that $u'_m\to u$, or equivalently $v'_m\to v$. Let us replace u'_m and v'_m by u_m and v_m. Now we prove that u and v satisfy equation (2.1).

Let $w\in D(A)$. We know $\|w\|_{L^2}$ is bounded from the above discussion. From ordinary differential equation (2.5), we get

$$\begin{aligned}
&(v_m(t),w)+\varepsilon\textstyle\int_{t_0}^t(v_m(s),p_mAw)ds+b\int_{t_0}^t(B(v_m(s),u_m(s)),p_mw)\,ds+\\
&\textstyle\int_{t_0}^t(B(u_m(s),v_m(s)),p_mw)\,ds+(b+1)\int_{t_0}^t u_m^2u_{m,x}ds-(b+1)\int_{t_0}^t u_mu_{m,x}ds=(v_m(t_0),w)\,.
\end{aligned}$$

Now, it is clear that $\lim\limits_{m\to\infty}\int_{t_0}^t(v_m(s),Aw)ds=\int_{t_0}^t(v(s),Aw)\,ds$. And so that

$$\lim_{m\to\infty}|p_mw-w|=0,\ \lim_{m\to\infty}|p_mAw-Aw|=0.$$

Thus, $(v_m(t),w)\to(v(t),w)\,,\int_{t_0}^t(v_m(s),p_mAw)ds\to\int_{t_0}^t(v(s),Aw)ds$, as $m\to\infty$.

$$\left|\int_{t_0}^t(B(v_m(s),u_m(s)),p_mw)\,ds-\int_{t_0}^t(B(v(s),u(s)),w)\,ds\right|\le I_m^1+I_m^2+I_m^3,$$

where

$$I_m^1=\left|\int_{t_0}^t(B(v_m(s),u_m(s)),p_mw-w)\,ds\right|\le\int_{t_0}^t\|v_m(s)\|_{L^2}\,\|u_m(s)\|_{H^1}\,\|p_mw-w\|_{L^2}\,ds\to0,$$

$$I_m^2=\left|\int_{t_0}^t(B(v_m(s)-v(s),u_m(s)),w)\,ds\right|\le\int_{t_0}^t\|v_m(s)-v(s)\|_{L^2}\,\|u_m(s)\|_{H^1}\,\|w\|_{L^2}\,ds\to0,$$

$$I_m^3 = \left| \int_{t_0}^{t} \left(B\left(v\left(s \right), u_m\left(s \right) - u\left(s \right) \right), w \right) ds \right| \le \int_{t_0}^{t} \left\| v\left(s \right) \right\|_{L^2} \left\| u_m\left(s \right) - u\left(s \right) \right\|_{H^1} \left\| w \right\|_{L^2} ds \to 0.$$

From the above discussion and by Lebesgue Theorem we get

$$\lim_{m \to \infty} \int_{t_0}^{t} \left(B\left(v_m\left(s \right), u_m\left(s \right) \right), p_m w \right) ds = \int_{t_0}^{t} \left(B\left(v\left(s \right), u\left(s \right) \right), w \right) ds.$$

Similarity, we have

$$\lim_{m \to \infty} \int_{t_0}^{t} \left(B\left(u_m\left(s \right), v_m\left(s \right) \right), p_m w \right) ds = \int_{t_0}^{t} \left(B\left(u\left(s \right), v\left(s \right) \right), w \right) ds.$$

We can also get

$$\begin{aligned} &\left| \int_{t_0}^{t} (u_m^2 u_{m,x}, P_m \omega) ds - \int_{t_0}^{t} (u^2 u_x, \omega) ds \right| \le \tfrac{1}{3} \int_{t_0}^{t} \left\| u_m^3 \right\|_{H^1} \left\| P_m \omega - \omega \right\|_{L^2} ds \\ &+ \tfrac{1}{3} \int_{t_0}^{t} \left\| (u_m - u)(u_m^2 + u_m u + u^2) \right\|_{H^1} \left\| \omega \right\|_{L^2} ds, \end{aligned}$$

$$\begin{aligned} &\left| \int_{t_0}^{t} u_m u_{m,x}, P_m \omega) ds - \int_{t_0}^{t} u u_x, \omega) ds \right| \\ &\le \int_{t_0}^{t} \left\| \tfrac{1}{2} u_m^2 \right\|_{H^1} \left\| P_m \omega - \omega \right\|_{L^2} ds + \int_{t_0}^{t} \left\| \tfrac{1}{2} (u_m - u)(u_m + u) \right\|_{H^1} \left\| \omega \right\|_{L^2} ds \to 0 \end{aligned}$$

Above all, for $w \in D\left(A \right)$, we get

$$\begin{aligned} &\left(v\left(t \right), w \right) + \varepsilon \int_{t_0}^{t} (v\left(s \right), Aw) ds + b \int_{t_0}^{t} \left(B\left(v\left(s \right), u\left(s \right) \right), w \right) ds + \\ &\int_{t_0}^{t} \left(B\left(u\left(s \right), v\left(s \right) \right), w \right) ds + (b+1) \int_{t_0}^{t} u^2 u_x ds - (b+1) \int_{t_0}^{t} u u_x ds = \left(v\left(t_0 \right), w \right). \end{aligned}$$

Summing up, it is reasonable to construct a Galerkin sequence $u_m\left(x, t \right)$, which converges to the weak solution to (2.5).

We take the inner product of (2.1) with u in $(0, 1)$ to obtain,

$$\frac{1}{2} \frac{d}{dt} \left(\left\| u \right\|_{L^2}^2 + \left\| u \right\|_{H^1}^2 \right) + \varepsilon \left(\left\| u \right\|_{H^1}^2 + \left\| Au \right\|_{L^2}^2 \right) + (b-2) \left(B\left(u, u \right), Au \right) = 0. \tag{3.18}$$

By Amgon inequality and young inequality and Poincare inequality, we get

$$\left| \left(B\left(u, u \right), Au \right) \right| \le \frac{1}{2} \left\| u_x \right\|_{L^\infty} \left\| u_x \right\|_{L^2}^2 \le \frac{k_2}{2} \left\| u \right\|_{H^2} \left\| u \right\|_{H^1}^2.$$

It follows from Young's inequality

$$\begin{aligned} &\tfrac{1}{2} \tfrac{d}{dt} \left(\left\| u \right\|_{L^2}^2 + \left\| u \right\|_{H^1}^2 \right) + \varepsilon \left(\left\| u \right\|_{H^1}^2 + \left\| Au \right\|_{L^2}^2 \right) \\ &\le \tfrac{k_2(b-2)}{2} \left\| Au \right\|_{L^2} \left\| u \right\|_{H^1}^2 \le \varepsilon \left\| u \right\|_{H^1}^2 + \tfrac{k_2^2(b-2)^2}{4\varepsilon} \left\| u \right\|_{H^1}^4 + \varepsilon \left\| Au \right\|_{L^2}^2 \end{aligned}$$

$$\tfrac{d}{dt} \left(\left\| u \right\|_{L^2}^2 + \left\| u \right\|_{H^1}^2 \right) \le \tfrac{k_2^2(b-2)^2}{2\varepsilon} (\left\| u \right\|_{L^2}^2 + \left\| u \right\|_{H^1}^2)^2.$$

Then $\forall t \in [0, T]$ and $T < \frac{1}{M(\|u_0\|_{L^2}^2 + \|u_0\|_{H^1}^2)}$. It thus transpires that

$$\left\| u \right\|_{L^2}^2 + \left\| u \right\|_{H^1}^2 \le \frac{\left\| u_0 \right\|_{L^2}^2 + \left\| u_0 \right\|_{H^1}^2}{1 - M(\left\| u_0 \right\|_{L^2}^2 + \left\| u_0 \right\|_{H^1}^2) t} \le \left\| u_0 \right\|_{L^2}^2 + \left\| u_0 \right\|_{H^1}^2 \triangleq M_1^2,$$

where $M = \frac{2\varepsilon}{k_2^2(b-2)^2}$.

From the above analysis, we know that $\left\| u \right\|_{L^2} \le M_1, \left\| u \right\|_{H^1} \le M_1,$ where M_1 is a positive constant.

$$\left| \left(B\left(u, u \right), Au \right) \right| \le \left\| u \right\|_{L^2} \left\| u \right\|_{H^1} \left\| Au \right\|_{L^2} \le M_1 \left\| u \right\|_{H^1} \left\| Au \right\|_{L^2} \le \frac{M_1}{2} (\left\| u \right\|_{H^1}^2 + \left\| Au \right\|_{L^2}^2).$$

By Poincare inequality, we have $\|u\|_{H^1}^2 \geq \lambda_3 \|u\|_{L^2}^2$, $\|Au\|_{L^2}^2 \geq \lambda_3 \|u\|_{H^1}^2$

$$\frac{d}{dt}\left(\|u\|_{L^2}^2+\|u\|_{H^1}^2\right) \leq (M_1(b-2)-2\varepsilon)\lambda_3\left(\|u\|_{L^2}^2+\|u\|_{H^1}^2\right).$$

Thus we get (2.2).

Now, take the inner product of (2.2) with Au in $(0,1)$ to obtain,

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right)+\varepsilon\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right)+\left(B\left(u,v\right),Au\right)+\\&b\left(B\left(v,u\right),Au\right)+((b+1)u^2u_x,Au)-((b+1)uu_x,Au)=0.\end{aligned} \tag{3.19}$$

By computing, we have

$$\left|((b+1)u^2u_x,Au)\right| \leq (b+1)\|u\|_{L^2}^2\|u\|_{H^1}\|Au\|_{L^2} \leq \frac{(b+1)M_1^2}{2}\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right),$$

$$((b+1)uu_x,Au) \leq (b+1)\|u\|_{L^2}\|u\|_{H^1}\|Au\|_{L^2} \leq \frac{(b+1)M_1}{2}\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right),$$

$$|(B(u,v),Au)+b(B(v,u),Au)| = (b+1)(B(u,u),Au)+b(B(Au,u),Au)+(B(u,Au),Au),$$

$$|(B(u,u),Au)| \leq \|u\|_{L^2}\|u\|_{H^1}\|Au\|_{L^2} \leq M_1\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right),$$

$$|(B(Au,u),Au)| \leq \|u\|_{H^1}\|Au\|_{L^2}^2 \leq M_1\|Au\|_{L^2}^2 \leq M_1(\|u\|_{H^1}^2+\|Au\|_{L^2}^2),$$

$$|(B(u,Au),Au)| \leq \|u\|_{L^2}\|Au\|_{L^2}^2 \leq M_1\|Au\|_{L^2}^2 \leq M_1\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right).$$

From (3.19) and above inequalities and Poincare inequality , we get

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right) \leq \left(\frac{(b+1)(M_1^2+5M_1)}{2}-\varepsilon\lambda_4\right)\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right),$$

$$\|u\|_{H^1}^2+\|Au\|_{L^2}^2 \leq \left(\|u_0\|_{H^1}^2+\|Au_0\|_{L^2}^2\right)\exp\left((b+1)(M_1^2+5M_1)-2\varepsilon\lambda_4\right)t, \tag{3.20}$$

where $\varepsilon \in (\frac{(b+1)(M_1^2+5M_1)}{2\lambda_4}, +\infty)$, λ_4 is the Poincare constant. Then we get (2.3).

Take the inner product of (2.1) with A^2u in $(0,1)$ to obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right)+\varepsilon\left(\|Au\|_{L^2}^2+\left\|A^2u\right\|_{L^2}^2\right)+\left(B\left(u,v\right),A^2u\right)+\\&b\left(B\left(v,u\right),A^2u\right)+((b+1)u^2u_x,A^2u)-((b+1)uu_x,A^2u)=0.\end{aligned} \tag{3.21}$$

By computing, we have

$$\begin{aligned}&\left|((b+1)u^2u_x,A^2u)\right| = \left|5(b+1)\int_0^1 uu_xu_{xx}^2dx\right| \leq 5(b+1)\|u\|_{L^\infty}\|u_x\|_{L^\infty}\|Au\|_{L^2}^2\\&\leq 5(b+1)(C_2\|u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}})(C_1\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}})(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2)\\&\leq 5(b+1)M_1C_1C_2\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}}(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2),\end{aligned}$$

$$\begin{aligned}&\left|((b+1)uu_x,A^2u)\right| \leq \tfrac{5(b+1)}{2}\|u_x\|_{L^\infty}\|Au\|_{L^2}^2\\&\leq \tfrac{5(b+1)}{2}C_1\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}}(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2)\left|\left(B\left(u,v\right),A^2u\right)+b\left(B\left(v,u\right),A^2u\right)\right|\\&=\left|(b+1)(B(u,u),A^2u)+(B(u,Au),A^2u)+b(B(Au,u),A^2u)\right|,\end{aligned}$$

$$\left|\left(B\left(u,u\right),A^2u\right)\right| \leq \|u_x\|_{L^\infty}\|Au\|_{L^2}^2 \leq C_1\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}}\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right),$$

$$\left|\left(B\left(Au,u\right),A^2u\right)\right| \leq \|u_x\|_{L^\infty}\|Au\|_{H^1}^2 \leq C_1\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}}\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right),$$

$$\left|\left(B\left(u,Au\right),A^2u\right)\right| \leq \|u_x\|_{L^\infty}\|Au\|_{L^2}^2 \leq C_1\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}}\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right),$$

By (3.20) , we get $\|u\|_{H^1} \leq M_1, \|Au\|_{L^2} \leq M_7$.

From above inequalities, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|Au\|_{L^2}^2 + \|Au\|_{H^1}^2\right) + \varepsilon\left(\|Au\|_{H^1}^2 + \left\|A^2u\right\|_{L^2}^2\right) \\
&\leq \frac{(b+1)C_1(5M_1C_2+9)}{2}\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}}\left(\|Au\|_{L^2}^2 + \|Au\|_{H^1}^2\right) \\
&\leq \frac{(b+1)C_1(5M_1C_2+9)}{2}M_1^{\frac{1}{2}}M_7^{\frac{1}{2}}\left(\|Au\|_{L^2}^2 + \|Au\|_{H^1}^2\right)
\end{aligned}$$

By Poincare inequality, we have

$$\|Au\|_{L^2}^2 + \|Au\|_{H^1}^2 \leq \left(\|Au_0\|_{L^2}^2 + \|Au_0\|_{H^1}^2\right)\exp\left((b+1)C_1(5M_1C_2+9)M_1^{\frac{1}{2}}M_7^{\frac{1}{2}} - 2\varepsilon\lambda_5\right)t,$$

where $\varepsilon \in (\frac{(b+1)C_1(5M_1C_2+9)M_1^{\frac{1}{2}}M_7^{\frac{1}{2}}}{2\lambda_5}, +\infty)$, λ_5 is the $Poincare$constant, C_1, C_2, M_1, M_7 are the positive constants.

Then we get (2.4).The proof of theorem is completed.

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