

Optimal Control of the Viscous Fifth Order Shallow Water Equation

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Abstract: This paper studies the problem of optimal control of the viscous fifth order shallow water equation. According to the optimal control theories and distributed parameter system control theories, it is proved that in the special Banach space, the norm of weak solution is related to the control item and initial value. The optimal control of the viscous fifth order shallow water equation under boundary condition is given in L^2 space and the existence of optimal solution is proved.

Keywords: the viscous fifth order shallow water equation; optimal control; optimal solution; distributed optimal control

1 Introduction

A number of basic equations in the study of nonlinear waves take the form

$$u_t + (f(u))_x + Lu = 0 \quad (1)$$

where $f(u)$ is a function of u and L is a linear operator with constant coefficient. When $f(u) = \frac{3}{2}u^2, L = \frac{\partial^3}{\partial x^3}$, (1) becomes the Korteweg-de Vries equation. In [1], Bona, J.L. and Smith, R.S. considered the following fifth order shallow water equation

$$u_t + \alpha u_{xxxxx} + \beta u_{xxx} + \gamma u_x + \mu(u^2)_x = 0 \quad (2)$$

which is given as $f(u) = \mu u^2, L = \alpha \frac{\partial^5}{\partial x^5} + \beta \frac{\partial^3}{\partial x^3} + \mu \frac{\partial}{\partial x}$ in (1), where $\alpha \neq 0, \beta$ and γ are real constants and μ is complex constant. The model described in (2) arises from the study of water waves with surface tension in which the Bond number takes on the critical value, where the Bond number represents a dimensionless magnitude of surface tension in the shallow water regime. In [2], by using the Fourier restriction norm method, the local well-posedness of the Cauchy problem for the above equation is established for low regularity data in Sobolev spaces $H^s (s \geq -\frac{3}{8})$.

In this paper, we study the existence of the optimal control of the viscous fifth order shallow water equation under the periodic boundary condition. The equation is as follows:

$$\begin{cases} u_t - \varepsilon u_{xx} + u_{xxxxx} + u_{xxx} + u_x + (u^2)_x = 0 \\ u(x, 0) = u_0(x) \end{cases} \quad (3)$$

where $t > 0, x \in \Omega, \Omega = [0, 1], u_0(x) \in H = L^2(\Omega)$.

On the other hand, the optimal control which is an important component of modern control theories has a wider application in modern engineering. The optimal control theories with PDE are much more difficult to deal with. Especially, there are no united theories and methods for nonlinear control theories with PDE. Two methods are introduced to study the control theories with PDE: one is using low model method, and then changing into ODE model ([3]); the other is using quasi-optimal control method ([4]). No matter

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which one to choose, it's necessary to prove the existence of optimal solution according to the basic theories ([5-7]). The optimal control of distributed parameter system now has become much more active in academic field. Especially, the optimal control of nonlinear solitary wave equation which occupies the front of the intersection of math, engineering and computer science. Zhifeng Zhao studied the optimal control of Kuramoto-sivashing equation ([8]). Lixin Tian and Chunyu Shen studied the optimal control of the b-family equation([9]).

In this research we are concerned with distributed control applied to the viscous fifth order shallow water equation. We take the distributed optimal control problem as a model

$$(P) \begin{cases} \min J(u, \varpi) = \frac{1}{2} \|Cu - z\|_S^2 + \frac{\delta}{2} \|\varpi\|_{L^2(Q_o)}^2 \\ s.t. \quad u_t - \varepsilon u_{xx} + u_{xxxxx} + u_{xxx} + u_x + (u^2)_x = f + B^* \varpi \quad \text{in } (0, T) \times (0, 1) \\ D^n u_m(0, t) = D^n u_m(1, t) = 0, D^n = \frac{\partial^n}{\partial x^n}, n = 0, 1, 2, 3, 4 \\ u(0) = \phi(x) \end{cases},$$

Here, the control target is to match the given desired state z in L^2 -sense by adjusting the body force, ϖ in a control volume $Q_o \subseteq Q = (0, T) \times \Omega$ in the L^2 -sense, i.e. with minimal energy and work, the first term in the cost functional measures the physical objective, the second one is the size of the control, where the parameter $\delta > 0$ plays the role of weight.

2 Notations

For fixed $T > 0$, we set $\Omega = (0, 1)$ and $Q = (0, T) \times \Omega$. Let $Q_o \subseteq Q$ be an open set with positive measure. Let $V = H_0^1(0, 1)$, $H = L^2(0, 1)$, $f \in L^2(H)$ and $\phi(x) \in H$. We supply V with the inner product $\langle \varphi, \psi \rangle_V = \langle \varphi_x, \psi_x \rangle_H$. $\forall \varphi, \psi \in V$.

Further, the extension operator $B^* \in L(L^2(Q_o), L^2(H))$ is given by $B^* q = \begin{cases} q, & \text{in } Q_o \\ 0, & \text{in } Q \setminus Q_o \end{cases}$.

We denote $u(t)$ and $f(t)$ the functions $u(t, \cdot)$ and $f(t, \cdot)$ respectively, considered as functions of x only when t is fixed.

Define $\|u\|_{H^m(\Omega)} = \|D^m u\|_H$, where $D^m = \frac{\partial^m}{\partial x^m}$, $m = 0, 1, \dots$, and $\|u\|_V = \|u\|_{H^1(\Omega)}$.

For $T > 0$ the space $L^2(0, T; V)$ and $C(0, T; H)$ denote the space of square integrable and continuous functions, respectively, in the sense of Bochner from $[0, T]$ to V and $[0, T]$ to H . The space $W(0, T; V)$ is defined by $W(0, T; V) = \{\varphi; \varphi \in L^2(V), \varphi_t \in L^2(V^*)\}$, which is a Hilbert space endowed with common inner product. For brevity we write $L^2(V)$, $C(H)$ and $W(V)$ in place of $L^2(0, T; V)$, $C(0, T; H)$ and $W(0, T; V)$ respectively.

3 The existence of weak solution to the viscous fifth order shallow water equation

For a control $\varpi \in L^2(Q_o)$, the state $u \in W(V)$ is given by the weak solution of the viscous fifth order shallow water equation

$$u_t - \varepsilon u_{xx} + u_{xxxxx} + u_{xxx} + u_x + (u^2)_x = f + B^* \varpi, f + B^* \varpi \in L^2(H) \quad (4)$$

$$u(x, 0) = \phi(x) \quad , \quad (5)$$

$$D^n u(0, t) = D^n u(1, t) = 0, D^n = \frac{\partial^n}{\partial x^n}, n = 0, 1, 2, 3, 4 \quad (6)$$

Definition 1 A function $u \in W(V)$ is called a weak solution to (4), if

$$\begin{aligned} \frac{d}{dt} (u, \varphi)_H - \varepsilon (u_{xx}, \varphi)_H + (u_{xxxxx}, \varphi)_H + (u_{xxx}, \varphi)_H + (u_x, \varphi)_H + (2uu_x, \varphi)_H \\ = (f + B^* \varpi, \varphi)_H, \forall \varphi \in V, t \in [0, T]. \end{aligned}$$

Theorem 2 With $\phi \in H$, $f + B^* \varpi \in L^2(H)$ holding, Eq.(4) admits a weak solution $u(x, t) \in W(0, T; V)$ in the interval $[0, T]$.

Proof. Performing the Galerkin procedure for equation (4), (5) and (6), we obtain

$$u_{m,t} - \varepsilon u_{m,xx} + u_{m,xxxxx} + u_{m,xxx} + u_{m,x} + (u_m^2)_x = f + B^* \varpi, \tag{7}$$

$$u_m(x, 0) = \phi_m(x), \phi_m(x) \in H, \tag{8}$$

$$D^n u_m(0, t) = D^n u_m(1, t) = 0, D^n = \frac{\partial^n}{\partial x^n}, n = 0, 1, 2, 3, 4 \tag{9}$$

P_m is projective from V to $V_m = span\{\omega_1, \omega_2, \dots, \omega_m\}$.

Equation (7) is an ordinary differential equation, and according to ODE theory, there is a unique solution to (7) in the interval $(0, t_m)$. We should show that the solution is uniformly bounded and $t_m \rightarrow T$ using inner product. ■

In the proof of the Theorem 2, we define $\|\cdot\|_{L^2}$ as $|\cdot|$ and $\|A^{1/2}(\cdot)\|_{L^2}$ as $\| \cdot \|$. Taking inner product in (7) to be u_m on Ω , we have

$$\frac{1}{2} \frac{d}{dt} |u_m|^2 + \|u_m\|^2 = (f + B^* \varpi, u_m). \tag{10}$$

We know that V is compactly embedded into H . By identifying H and H^* , V and V^* , we can derive that $H = H^*$ is compactly embedded into V^* .

Since $f + B^* \varpi \in L^2(H)$ is a control item, we can assume $\|f + B^* \varpi\|_{L^2(0,1)} = |f + B^* \varpi| \leq M$, where M is a positive constant.

By Poincare inequality and Young's inequality, we can construct

$$\frac{d}{dt} |u_m|^2 + 2\varepsilon \lambda_1 |u_m|^2 \leq 2|f + B^* \varpi| |u_m| \leq 2\varepsilon \lambda_1 |u_m|^2 + \frac{|f|^2}{2\varepsilon \lambda_1},$$

$$\frac{d}{dt} |u_m|^2 + 2\varepsilon \lambda_1 |u_m|^2 \leq 2\varepsilon \lambda_1 |u_m|^2 + \frac{M^2}{2\varepsilon \lambda_1},$$

where λ_1 is Poincare coefficient. Then, we can get

$$|u_m|^2 \leq \frac{M^2 t}{2\varepsilon \lambda_1} + (|u_{m,0}|^2) \triangleq r_1 \quad \forall t \in [0, T], \tag{11}$$

where r_1 is a positive constant. Let r be a positive constant.

Integrating (10) in the interval $[t, t+r] \subset [0, T]$, we have

$$\varepsilon \int_t^{t+r} \|u_m\|^2 ds \leq \frac{r_1}{2} + \int_t^{t+r} \frac{M^2 + r_1}{2} ds \leq \frac{r_1}{2} + \frac{(M^2 + r_1)r}{2} \triangleq r_2. \tag{12}$$

Taking inner product in (7) to be Au_m on Ω , we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_m\|^2 + \varepsilon |Au_m|^2 - \int_0^1 u_{m,x}^3 dx = (f + B^* \varpi, Au_m) \tag{13}$$

According to Agmon inequality when $n = 1$, then we have

$$\left| \int_0^1 (u_{m,x})^3 dx \right| \leq \|\nabla u_m\|_{L^\infty(\Omega)} \|u_m\|^2 \leq C_1 \|u_m\|^{\frac{5}{2}} |Au_m|^{\frac{1}{2}}$$

Applying Young's inequality and Poincare inequality, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_m\|^2 + \varepsilon |Au_m|^2 &\leq C_1 \|u_m\|^{\frac{5}{2}} |Au_m|^{\frac{1}{2}} + \frac{M^2}{2} + \frac{1}{2} |Au_m|^2 \\ &\leq C_1 \lambda_2 \|u_m\|^2 \|Au_m\| + \frac{M^2}{2} + \frac{1}{2} |Au_m|^2 \\ &\leq \frac{1}{2} C_1 \lambda_2 \left(\frac{C_1 \lambda_2 \|u_m\|^4}{2(\varepsilon - \frac{1}{2})} + \frac{2(\varepsilon - \frac{1}{2}) |Au_m|^2}{C_1 \lambda_2} \right) + \frac{1}{2} |Au_m|^2 + \frac{M^2}{2} \end{aligned}$$

Then we have $\frac{1}{2} \frac{d}{dt} \|u_m\|^2 \leq C_2 \|u_m\|^4 + \frac{M^2}{2}$, where $C_2 = \frac{C_1^2 \lambda_2^2}{4(\varepsilon - \frac{1}{2})}$ is constant, λ_2 is Poincare coefficient and $\varepsilon > \frac{1}{2}$.

Let $y = \|u_m\|^2$, $g = C_2 \|u_m\|^2$. Then, we can derive $\int_t^{t+r} y(s) ds \leq \frac{r_2}{\varepsilon}$, $\int_t^{t+r} g(s) ds \leq C_2 \frac{r_2}{\varepsilon}$ for $r > 0$.

According to the uniform Gronwall inequality, we have $\|u_m\|^2 \leq (\frac{r_2}{\varepsilon} + M^2 r) \exp(C_2 \frac{r_2}{\varepsilon}) \triangleq r_3$. Taking inner product in (7) to be $A^2 u_m$ on Ω , we can have $|Au_m|^2 \leq r_4$ from the method discussed above and uniform Gronwall lemma similarly. Now, we have obtained that $|u_m|, \|u_m\|, |Au_m|$ are bounded. From this, we can conclude that $|u_{m,t}|$ is bounded.

From those mentioned above, it is easy to prove $u_m \in L^2[(0, T), V]$ and $u_{m,t} \in L^2[(0, T), V^*]$, therefore u_m is bounded in $W(V)$.

Due to Aubin theory (see [7] page2), we know that there exists subsequence u_{m_k} which converges to u . For convenience, we define the subsequence with u_m as well.

Then it is easy to construct a Galerkin sequence $u_m(x, t)$, which converges to the weak solution to (4).

Lemma 3 With $f \in L^2(H)$ and $\phi \in H$ holding, there exist two constants $C_1 > 0$ and $C_2 > 0$ such that $\|u\|_{W(V)}^2 \leq 2C_1 \left[(\|\phi\|_H + \|f\|_{L^2(H)})^2 + \|\varpi\|_{L^2(Q_o)}^2 \right] + C_2$.

Proof. Multiplying u on both sides of equation (4). We have

$$uu_t - \varepsilon uu_{xx} + uu_{xxxx} + uu_{xxx} + uu_x + u(u^2)_x = (f + B^* \varpi)u.$$

Integrating the resulting equation with respect to x over the interval $(0, 1)$. Then we get

$$\frac{1}{2} \frac{d}{dt} \|u\|_H^2 + \varepsilon \int_0^1 u_x^2 dx = (f + B^* \varpi, u)_H \quad (14)$$

From the above, we have

$$\frac{1}{2} \frac{d}{dt} \|u\|_H^2 + \varepsilon \int_0^1 u_x^2 dx \leq (f + B^* \varpi, u)_H. \quad (15)$$

Integrating (15) with respect to t over the interval $(0, T)$, we thus obtain

$$\frac{1}{2} \|u(T)\|_H^2 - \frac{1}{2} \|\phi\|_H^2 + \varepsilon \|u\|_{L^2(V)}^2 \leq \int_0^T (f + B^* \varpi, v)_H dt. \quad (16)$$

It is supposed that V is dense in H so that, we have V is embedded into H .

From Holder inequality, here gives

$$\begin{aligned} \int_0^T (f + B^* \varpi, u)_H dt &\leq \int_0^T \|f + B^* \varpi\|_H \|u\|_H dt \\ &\leq \int_0^T \|f + B^* \varpi\|_H \|u\|_V dt \leq \|f + B^* \varpi\|_{L^2(H)} \|u\|_{L^2(V)} \end{aligned} \quad (17)$$

Substituting (17) into (16), we can get

$$\|u(T)\|_H^2 - \|\phi\|_H^2 + 2\varepsilon \|u\|_{L^2(V)}^2 \leq 2 \|f + B^* \varpi\|_{L^2(H)} \|u\|_{L^2(V)}. \quad (18)$$

Young's inequality gives

$$\|f + B^* \varpi\|_{L^2(H)} \|u\|_{L^2(V)} \leq \frac{1}{2\varepsilon} \|f + B^* \varpi\|_{L^2(H)}^2 + \frac{\varepsilon}{2} \|u\|_{L^2(V)}^2. \quad (19)$$

From (18) and (19) we have

$$\|u(T)\|_H^2 - \|\phi\|_H^2 + 2\varepsilon \|u\|_{L^2(V)}^2 \leq \frac{1}{\varepsilon} \|f + B^* \varpi\|_{L^2(H)}^2 + \varepsilon \|u\|_{L^2(V)}^2,$$

$$\begin{aligned} \varepsilon \|u\|_{L^2(V)}^2 &\leq \|\phi\|_H^2 + \frac{1}{\varepsilon} \|f + B^* \varpi\|_{L^2(H)}^2, \\ \|u\|_{L^2(V)}^2 &\leq \frac{1}{\varepsilon} \|\phi\|_H^2 + \frac{1}{\varepsilon^2} \|f + B^* \varpi\|_{L^2(H)}^2 \leq \max \left\{ \frac{1}{\varepsilon}, \frac{1}{\varepsilon^2} \right\} \left(\|\phi\|_H + \|f + B^* \varpi\|_{L^2(H)} \right)^2 \\ &\leq C_0 \left(\|\phi\|_H + \|f + B^* \varpi\|_{L^2(H)} \right)^2, \end{aligned} \tag{20}$$

where $C_0 = \max \left\{ \frac{1}{\varepsilon}, \frac{1}{\varepsilon^2} \right\}$.

Since H is embedded into V^* , from (7), we have $\|u_t\|_{V^*} + \|u_{xxxx}\|_{V^*} \leq \|f + B^* \varpi\|_{V^*} + \varepsilon \|u\|_V + \|u_{xxx}\|_{V^*} + \|(u^2)_x\|_{V^*} + \|u\|_H$,

$$\begin{aligned} \|u_t\|_{V^*} &\leq \|f + B^* \varpi\|_H + \varepsilon \|u\|_V + \|u_{xx}\|_H + \|u^2\|_H + \|u\|_H, \\ \|u^2\|_H &\leq \|u\|_{L^\infty(\Omega)} \|u\|_H \leq K_1 \|u_0\|_V \|u\|_H, \end{aligned} \tag{21}$$

where K_1 is nonnegatively embedded constant.

Because $u_0 \in G$, we can assume $\|u_0\|_V \leq M_1$, where M_1 is positive constant. Then we can get

$$\|u_t\|_{V^*} \leq \|f + B^* \varpi\|_H + \varepsilon \|u\|_V + M_2 \tag{22}$$

where $M_2 = r_4^{\frac{1}{2}} + K_1 M_1 r_1 + r_1$.

$$\|u_t\|_{V^*}^2 \leq 3 \|f + B^* \varpi\|_H^2 + 3\varepsilon^2 \|u\|_V^2 + 3M_2^2 \tag{23}$$

Integrating (23) over the interval $(0, T)$, we have

$$\begin{aligned} \|u_t\|_{L^2(V^*)}^2 &\leq 3 \|f + B^* \varpi\|_{L^2(H)}^2 + 3\varepsilon^2 \|u\|_{L^2(V)}^2 + 3M_2^2 T \\ &\leq 3 \|f + B^* \varpi\|_{L^2(H)}^2 + 3\varepsilon^2 \left[C_0 \left(\|\phi\|_H + \|f + B^* \varpi\|_{L^2(H)} \right)^2 \right] + 3M_2^2 T \\ &\leq (3 + 3\varepsilon^2 C_0) \left(\|\phi\|_H + \|f + B^* \varpi\|_{L^2(H)} \right)^2 + C_2, \end{aligned} \tag{24}$$

where $C_2 = 3M_2^2 T$. From (20) and (24), we can

$$\begin{aligned} \|u\|_{W(V)}^2 &= \|u\|_{L^2(V)}^2 + \|u_t\|_{L^2(V^*)}^2 \leq (C_0 + 3 + 3\varepsilon^2 C_0) \left(\|\phi\|_H + \|f + B^* \varpi\|_{L^2(H)} \right)^2 + C_2 \\ &\leq 2C_1 \left[\left(\|\phi\|_H + \|f\|_{L^2(H)} \right)^2 + \|B^* \varpi\|_{L^2(H)}^2 \right] + C_2 \\ &\leq 2C_1 \left[\left(\|\phi\|_H + \|f\|_{L^2(H)} \right)^2 + \|\varpi\|_{L^2(Q_o)}^2 \right] + C_2, \end{aligned} \tag{25}$$

where $C_1 = (C_0 + 3 + 3\varepsilon^2 C_0)$. ■

4 The distributed optimal control of the viscous fifth order shallow water equation

Let a control $\varpi \in L^2(Q_o)$, $v \in W(V)$ be a weak solution to the following equation

$$u_t - \varepsilon u_{xx} + u_{xxxx} + u_{xxx} + u_x + (u^2)_x = f + B^* \varpi, f + B^* \varpi \in L^2(H) \tag{26}$$

$$u(x, 0) = \phi(x) \tag{27}$$

$$D^n u(0, t) = D^n u(1, t) = 0, D^n = \frac{\partial^n}{\partial x^n}, n = 0, 1, 2, 3, 4 \tag{28}$$

We know that there exists weak solution u to (26)-(28) from Theorem 2.

Given an observation operator $C \in L(W(V), S)$, in which S is a real Hilbert space and C is continuous.

We choose performance index to these control equations $J(u, \varpi) = \frac{1}{2} \|Cu - z\|_S^2 + \frac{\delta}{2} \|\varpi\|_{L^2(Q_o)}^2$, where z is a desired state and $\delta > 0$ is fixed. Control problem about (26)-(28) is $\min J(u, \varpi)$, where (u, ϖ) satisfies (26)-(28).

We set $X = W(V) \times L^2(Q_o)$ and $Y = L^2(V) \times H$, we define an operator

$$e = e(e_1, e_2) : X \rightarrow Y. \quad e(u, m) = \begin{bmatrix} G \\ u(x, 0) - \phi(x) \end{bmatrix},$$

where

$$G = (-\Delta)^{-1} (u_t - \varepsilon u_{xx} + u_{xxxxx} + u_{xxx} + u_x + (u^2)_x - f - B^* \varpi)$$

and Δ is an operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. Then we write (28) in following form

$$\min J(u, \varpi) \text{ subject to } e(u, \varpi) = 0. \tag{29}$$

Theorem 4 *There exists an optimal control solution of problem (P).*

Proof. Let $(u, \varpi) \in X$ satisfy the equation $e(u, \varpi) = 0$. We have $J(u, \varpi) \geq \frac{\delta}{2} \|\varpi\|_{L^2(Q_o)}^2$. From Lemma 3, we can conclude that $\|u\|_{W(V)} \rightarrow \infty$ yields $\|\varpi\|_{L^2(Q_o)} \rightarrow \infty$. Hence, we can get

$$J(u, \varpi) \rightarrow +\infty, \text{ when } \|(u, \varpi)\|_X \rightarrow \infty. \tag{30}$$

Since C is continuous, the norm is weakly semi-continuous and inner product $\langle \cdot, z \rangle_S$ is weakly continuous. Let $\{(u^n, \varpi^n)\}_{n \in N}$ be a sequence in X converging weakly to (u^*, ϖ^*) .

It follows as: $\lim_{n \rightarrow \infty} J(u^n, \varpi^n) \geq \frac{1}{2} \|Cu^*\|^2 - \langle Cu^*, z \rangle_S + \frac{1}{2} \|z\|^2 + \frac{\delta}{2} \|\varpi^*\|_{L^2}^2 = J(u^*, \varpi^*)$. Then J is weak lowered semi-continuous in X . Since $J(u, \varpi) \geq 0$, for all $(u, \varpi) \in X$ holds, there exists $\zeta \geq 0$ with: $\zeta = \inf \{J(u, \varpi) \mid (u, \varpi) \in X \text{ with } e(u, \varpi) = 0\}$. This implies the existence of a minimizing sequence $\{(u^n, \varpi^n)\}_{n \in N}$ in X such that $\zeta = \lim_{n \rightarrow \infty} J(u^n, \varpi^n)$ and $e(u^n, \varpi^n) = 0$ for all $n \in N$.

Due to (29), we infer that there exists an element $(u^*, \varpi^*) \in X$ with

$$u^n \xrightarrow{weak} u^*, n \rightarrow \infty, u \in W(V) \quad , \tag{31}$$

$$\varpi^n \xrightarrow{weak} \varpi^*, n \rightarrow \infty, \varpi \in L^2(Q_o) \quad . \tag{32}$$

We can infer from (31) that: $\lim_{n \rightarrow \infty} \int_0^T (u_t^n(t) - u_t^*, \varphi(t))_{V^*, V} dt = 0, \quad \forall \varphi \in L^2(V)$.

Since $W(V)$ is compactly embedded into $L^2(L^\infty)$ we have $u^n \rightarrow u^*$ strongly in $L^2(L^\infty)$, as $n \rightarrow \infty$. Since $W(V)$ is continuously embedded into $C(H)$, we can derive that $u^n \rightarrow u^*$ strongly into $C(H)$, as $n \rightarrow \infty$.

As the sequence $\{u^n\}_{n \in N}$ converges weakly, $\|u^n\|_{W(V)}$ is bounded. $\|u^n\|_{L^2(L^\infty)}$ is also bounded. Since $u^n \rightarrow u^*$ strongly in $L^2(L^\infty)$, we can infer that $\|u^*\|_{L^2(L^\infty)}$ is bounded.

Thus, it follows by Holder inequality that

$$\begin{aligned} \left| \int_0^T \int_0^1 ((u^2)_x^n - (u^2)_x^*) \varphi dx dt \right| &\leq \int_0^T \|(u^2)^n - (u^2)^*\|_H \|\varphi\|_V dt \\ &\leq \|u^n - u^*\|_{C(H)} \|u^n + u^*\|_{C(H)} \|\varphi\|_{L^2(V)} \rightarrow 0, \text{ as } n \rightarrow \infty, \text{ for } \forall \varphi \in L^2(V). \end{aligned} \tag{33}$$

From (32), we can infer $\left| \int_0^T \int_0^1 (B^* \varpi^n - B^* \varpi^*) \varphi dx dt \right| \xrightarrow{n \rightarrow \infty} 0$ for all $\varphi \in L^2(V)$.

Based on the above, we can conclude that $e_1(u^*, \varpi^*) = 0, \forall n \in N$. From $u^* \in W(V)$ we can derive that $u^*(0) \in H$. From $u^n \xrightarrow{weak} u^*$, we can infer $u^n(0) \xrightarrow{weak} u^*(0)$, when $n \rightarrow \infty$. Thus we have

$$\langle u^n(0) - u^*(0), \varphi \rangle_H \rightarrow 0, (n \rightarrow \infty), \forall \varphi \in H.$$

From the above, we can get $e(u^*(0), \varpi^*(0)) = 0$. There exists an optimal solution (u^*, ϖ^*) to problem (28). ■

5 Conclusion

This paper proves the existence of weak solution in the interval to the viscous fifth order shallow water equation. According to the variational inequality, optimal control theories and distributed parameter system control theories and choosing suitable performance index, it is proved that in the special Banach space, the norm of solution is related to the control item and initial value. At last, we prove the existence of optimal solution of the viscous fifth order shallow water equation.

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