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Precise Asymptotics in the Law of the Iterated Logarithm under Dependence

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Abstract: Let $\{X_n; n \geq 1\}$ be a strictly stationary negatively associated sequence which satisfies $EX_1 = 0, Var(X_1) < \infty$. Set $S_n = \sum_{k=1}^{n} X_k, n \geq 1$, $\sigma^2 = \mathsf{E}X_1^2 + 2\sum_{k=2}^{\infty} \mathsf{E}X_1X_k$. In this paper, we prove that, for $b > -1$,

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\{|S_n| \geq \varepsilon\sigma\sqrt{n \log\log n}\} = \frac{2^{b+1}}{(b+1)\sqrt{\pi}}\Gamma(b+\frac{3}{2})$$

holds if

$$EX_1^2(1+\log\log|X_1|)^{b-1} < \infty.$$

The result of Gut and Spătaru [2] is a special case of ours.

Key words:precise asymptotics; law of the iterated logarithm; negatively associated random variables
MSC:60F15; 60G50

1 Introduction

Let $\{X_n;\ n \geq 1\}$ be a sequence of random variables with common distribution function F, $\mathsf{E}X_1 = 0$ and $0 < \mathsf{E}X_1^2 < \infty$, and set $S_n = \sum_{k=1}^{n} X_k,\ n \geq 1$. Also denote $\log n = \ln(n \vee e)$, $\log\log n = \log\log(n \vee e^e)$. When $\{X_n;\ n \geq 1\}$ is a sequence of i.i.d. random variables, by the well-known law of the iterated logarithm (LIL) we have

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = \sigma, \quad a.s.$$

Gut and Spătaru [2] proved its precise asymptotics and we cite their result as follows.

Theorem 1 *([2]) Suppose that* $\mathsf{E}X_1 = 0$ *and* $\mathsf{E}X_1^2 = \sigma^2 < \infty$*. Then,*

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n\log n} P(|S_n| \geq \varepsilon\sqrt{n\log\log n}) = \sigma^2.$$

In this note, we shall extend Theorem 1 to the case of dependence. Importantly, we prove it by a different method.

A finite sequence of random variables $\{X_k; 1 \leq k \leq n\}$ is said to be negatively associated (NA), if for every disjoint subsets A and B of $\{1, 2, \cdots, n\}$, we have

$$\mathsf{Cov}\{f(X_i; i \in A), g(X_j; j \in B)\} \leq 0,$$

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whenever f and g are coordinatewise non-decreasing and the covariance exists. An infinite family is NA if every finite subfamily is NA. The notation of NA was first introduced by Alam and Saxena [1]. Joag-Dev and Proschan [3] showed that many well-known multivariate distributions possess the NA property.

Throughout this paper, let $\{X_n;\ n \geq 1\}$ be a sequence of strictly stationary NA random variables, $\mathsf{E}X_1 = 0$, $0 < \mathsf{E}X_1^2 < \infty$, and set $0 < \sigma^2 = \mathsf{E}X_1^2 + 2\sum_{k=2}^{\infty} \mathsf{E}X_1X_k < \infty$ unless it is specially mentioned. Now, we state our result as follows.

Theorem 2 *Suppose that $EX_1 = 0$ and $EX_1^2 < \infty$. For $b > -1$, if $EX_1^2(1+\log\log|X_1|)^{b-1} < \infty$, then*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\{|S_n| \geq \varepsilon\sigma\sqrt{n\log\log n}\} = \frac{2^{b+1}}{(b+1)\sqrt{\pi}}\Gamma(b+\frac{3}{2}). \tag{1}$$

Here $\Gamma(\cdot)$ denotes a gamma function.

2 Some lemmas

First, we give some lemmas which will be used in the following proofs. And the following lemma comes from Chun Su, Lincheng Zhao and Yuebao Wang [6].

Lemma 3 *([6]) Under the condition above Theorem 2, we have*

$$\frac{S_n}{\sigma\sqrt{n}} \to N \text{ in distribution.}$$

The following lemma will be useful. A proof appears in Shao [4].

Lemma 4 *([4]) Let $\{Y_i; 1 \leq i \leq n\}$ be a sequence of NA r.v.s with mean zero and finite variance. Denote $S_k = \sum_{i=1}^{k} Y_i, 1 \leq k \leq n, B_n = \sum_{i=1}^{n} \mathsf{E}Y_i^2$, then for any $x > 0, y > 0$,*

$$\mathsf{P}\Big(\max_{k\leq n}|S_k| \geq x\Big) \leq 2\mathsf{P}\Big(\max_{k\leq n}|Y_k| \geq y\Big) + 4\exp\Big\{-\frac{x^2}{8B_n}\Big\} + 4\Big(\frac{B_n}{4(xy+B_n)}\Big)^{x/(12y)}.$$

3 Proof of Theorem 2

In this section, let $C,\ M$ denote positive constants, whose values can differ at different places. Also set $B(\varepsilon) = \exp\{\exp(M/\varepsilon^2)\},\ M > 4$ and $0 < \varepsilon < 1/4$. Without loss of generality, we assume that $\sigma = 1$. And also let N be a standard normal variable. Theorem 2 will be proved via the following propositions:

Proposition 5 *For any $b > -1$, we have*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\{|N| \geq \varepsilon\sqrt{\log\log n}\} = \frac{2^{b+1}}{(b+1)\sqrt{\pi}}\Gamma(b+\frac{3}{2}).$$

Proof. Observe that , $P(|N| \geq x) = 2P(N \geq x)$. Then

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\{|N| \geq \varepsilon\sqrt{\log\log n}\} \\
&= 2\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\{N \geq \varepsilon\sqrt{\log\log n}\} \\
&= 2\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \int_{e^e}^{\infty} \frac{(\log\log x)^b}{x \log x} \mathsf{P}\{N \geq \varepsilon\sqrt{\log\log x}\} dx \\
&= 2\lim_{\varepsilon \searrow 0} \int_{\varepsilon^2}^{\infty} y^b \mathsf{P}\{N \geq \sqrt{y}\} dy \\
&= \frac{2}{b+1} \int_0^{\infty} \mathsf{P}\{N \geq \sqrt{y}\} dy^{b+1} \\
&= \frac{2}{b+1} \mathsf{P}\{N \geq \sqrt{y}\} y^{b+1}\Big|_0^{\infty} + \frac{2^{b+1}}{(b+1)\sqrt{\pi}} \int_0^{\infty} y^{b+\frac{1}{2}} e^{-y} dy \\
&= \frac{2^{b+1}}{(b+1)\sqrt{\pi}} \Gamma(b + \frac{3}{2}).
\end{aligned}$$

Thus the Proposition is now proved. ■

Proposition 6 *Uniformly for* $0 < \varepsilon < 1/4$*, we have*

$$\lim_{M \to \infty} \varepsilon^{2(b+1)} \sum_{n > B(\varepsilon)} \frac{(\log\log n)^b}{n \log n} P\{|N| \geq \varepsilon\sqrt{\log\log n}\} = 0.$$

Proof. Note that if $M > 4$ and $0 < \varepsilon < 1/4$, then $B(\varepsilon) - 1 \geq \sqrt{B(\varepsilon)}$. Thus

$$\begin{aligned}
&\varepsilon^{2(b+1)} \sum_{n > B(\varepsilon)} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\{|N| \geq \varepsilon\sqrt{\log\log n}\} \\
=& 2\varepsilon^{2(b+1)} \sum_{n > B(\varepsilon)} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\{N \geq \varepsilon\sqrt{\log\log n}\} \\
\leq& C\varepsilon^{2(b+1)} \int_{B(\varepsilon)-1}^{\infty} \frac{(\log\log x)^b}{x \log x} \mathsf{P}\{N \geq \varepsilon\sqrt{\log\log x}\} dx \\
\leq& C\varepsilon^{2(b+1)} \int_{\sqrt{B(\varepsilon)}}^{\infty} \frac{(\log\log x)^b}{x \log x} \mathsf{P}\{N \geq \varepsilon\sqrt{\log\log x}\} dx,
\end{aligned}$$

hence, by letting $y = \varepsilon^2 \log\log x$, we have

$$\begin{aligned}
&\varepsilon^{2(b+1)} \sum_{n > B(\varepsilon)} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\{|N| \geq \varepsilon\sqrt{\log\log n}\} \\
\leq& \frac{2C}{b+1} \mathsf{P}\{N \geq \sqrt{y}\} y^{b+1}\Big|_{\sqrt{M}}^{\infty} + \frac{C2^{b+1}}{(b+1)\sqrt{\pi}} \int_{\sqrt{M}}^{\infty} y^{b+\frac{1}{2}} e^{-y} dy \to \quad 0,
\end{aligned}$$

when $M \to \infty$, uniformly for $0 < \varepsilon < 1/4$. ■

Proposition 7 *If* $\mathsf{E}X_1^2 < \infty$ *and* $\mathsf{E}X_1^2(1 + \log\log|X_1|)^{b-1} < \infty$*, then for any* $\delta > 0, M > 4$ *and* $b > -1$*,*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n > B(\varepsilon)} \frac{(\log\log n)^b}{\log n} P\{|X_1| \geq \delta\varepsilon\sqrt{n \log\log n}\} = 0.$$

Proof. When $-1 < b \le 0$, for $\delta > 0$ and $M > 4$, we can get that

$$\begin{aligned}
&\sum_{n>B(\varepsilon)} \frac{(\log\log n)^b}{\log n} \mathsf{P}\{|X_1| \ge \delta\varepsilon\sqrt{n\log\log n}\} \\
\le& C \sum_{n>B(\varepsilon)} (\log\log n)^b \mathsf{P}\{|X_1| \ge \delta\varepsilon\sqrt{n\log\log n}\} \\
\le& C \sum_{k>B(\varepsilon)} \mathsf{P}\Big(\delta\varepsilon\sqrt{k\log\log k} \le |X_1| < \delta\varepsilon\sqrt{(k+1)\log\log(k+1)}\Big) \sum_{B(\varepsilon)<n\le k} (\log\log n)^b \\
\le& C \sum_{k>B(\varepsilon)} (\log\log k)^{-1} k\log\log k \mathsf{P}\Big(\delta\varepsilon\sqrt{k\log\log k} \le |X_1| < \delta\varepsilon\sqrt{(k+1)\log\log(k+1)}\Big) \\
\le& C(\frac{M}{\varepsilon^2})^{-1} \sum_{k>B(\varepsilon)} k\log\log k \mathsf{P}\Big(\delta\varepsilon\sqrt{k\log\log k} \le |X_1| < \delta\varepsilon\sqrt{(k+1)\log\log(k+1)}\Big) \\
\le& CM^{-1}\delta^{-2}\mathsf{E}X_1^2.
\end{aligned}$$

When $b > 0$, for $\delta > 0$ and $M > 4$ we have

$$\begin{aligned}
&\sum_{n>B(\varepsilon)} \frac{(\log\log n)^b}{\log n} \mathsf{P}\{|X_1| \ge \delta\varepsilon\sqrt{n\log\log n}\} \\
\le& C \sum_{n>B(\varepsilon)} (\log\log n)^b \mathsf{P}\{|X_1| \ge \delta\varepsilon\sqrt{n\log\log n}\} \\
\le& C \sum_{n>B(\varepsilon)} (\log\log n)^b \sum_{k\ge n} \mathsf{P}\Big(\delta\varepsilon\sqrt{k\log\log k} \le |X_1| < \delta\varepsilon\sqrt{(k+1)\log\log(k+1)}\Big) \\
\le& C \sum_{k>B(\varepsilon)} \mathsf{P}\Big(\delta\varepsilon\sqrt{k\log\log k} \le |X_1| < \delta\varepsilon\sqrt{(k+1)\log\log(k+1)}\Big) \sum_{B(\varepsilon)<n\le k} (\log\log n)^b \\
\le& C \sum_{k>B(\varepsilon)} k(\log\log k)^b \mathsf{P}\Big(\sqrt{k\log\log k} \le \frac{|X_1|}{\delta\varepsilon} < \sqrt{(k+1)\log\log(k+1)}\Big) \\
\le& C\mathsf{E}\left[\frac{|X_1|^2}{(\delta\varepsilon)^2}\left(\log\log\frac{|X_1|^2}{(\delta\varepsilon)^2}\right)^{b-1}\right] \\
\le& C(\delta\varepsilon)^{-2}\left[\left(\log\log(\delta\varepsilon)^{-1}\right)^{b-1} \vee 1\right] \mathsf{E}X_1^2(1+\log\log|X_1|)^{b-1}.
\end{aligned}$$

The result follows. ■

Proposition 8 *If* $\mathsf{E}X_1^2 < \infty$ *and* $\mathsf{E}X_1^2(1+\log\log|X_1|)^{b-1} < \infty$*, then for* $b > -1$*, we have*

$$\lim_{M\to\infty}\lim_{\varepsilon\searrow 0} \varepsilon^{2(b+1)} \sum_{n>B(\varepsilon)} \frac{(\log\log n)^b}{n\log n} P\{|S_n| \ge \varepsilon\sqrt{n\log\log n}\} = 0.$$

Proof. Denote $\theta = \sqrt{1/\mathsf{E}X_1^2}$. Note that if $M > 4$, then $B(\varepsilon) - 1 > \sqrt{B(\varepsilon)}$. By using Lemma 4 (where $x = \varepsilon\sqrt{n\log\log n}$ and $y = \frac{x}{12(b+2)}$) and Proposition 7, we have

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n>B(\varepsilon)}\frac{(\log\log n)^b}{n\log n}\mathsf{P}\{|S_n|\geq\varepsilon\sqrt{n\log\log n}\}\\
\leq&\lim_{\varepsilon\searrow 0}2\varepsilon^{2(b+1)}\sum_{n>B(\varepsilon)}\frac{(\log\log n)^b}{\log n}\mathsf{P}\{|X_1|\geq\frac{\varepsilon}{12(b+2)}\sqrt{n\log\log n}\}\\
&+\lim_{\varepsilon\searrow 0}4\varepsilon^{2(b+1)}\sum_{n>B(\varepsilon)}\frac{(\log\log n)^b}{n\log n}\exp\left\{-\frac{\theta^2\varepsilon^2\log\log n}{8}\right\}\\
&+\lim_{\varepsilon\searrow 0}4\varepsilon^{2(b+1)}\sum_{n>B(\varepsilon)}\frac{(\log\log n)^b}{n\log n}\left(\frac{n/\theta^2}{4\times\varepsilon^2 n\log\log n\frac{1}{12(b+2)}}\right)^{b+2}\\
\leq&\lim_{\varepsilon\searrow 0}4\varepsilon^{2(b+1)}\int_{B(\varepsilon)-1}^{\infty}\frac{(\log\log x)^b}{x\log x}\exp\left\{-\frac{\theta^2\varepsilon^2\log\log x}{8}\right\}dx+\lim_{\varepsilon\searrow 0}C\varepsilon^{-2}\sum_{n>B(\varepsilon)}\frac{1}{n\log n(\log\log n)^2}\\
\leq&\lim_{\varepsilon\searrow 0}4\varepsilon^{2(b+1)}\int_{\sqrt{B(\varepsilon)}}^{\infty}\frac{(\log\log x)^b}{x\log x}\exp\left\{-\frac{\theta^2\varepsilon^2\log\log x}{8}\right\}dx+\lim_{\varepsilon\searrow 0}C\varepsilon^{-2}\frac{1}{\log\log B(\varepsilon)}\\
\leq&4\int_{M/2}^{\infty}y^b\exp\left\{-\frac{\theta^2 y}{8}\right\}dy+CM^{-1}.
\end{aligned}$$

Thus we have the desired result by letting $M\to\infty$ ■

Proposition 9 *For any $M > 4$, we have*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n\leq B(\varepsilon)}\frac{(\log\log n)^b}{n\log n}\left|P(|S_n|\geq\varepsilon\sqrt{n\log\log n})-P(|N|\geq\varepsilon\sqrt{\log\log n})\right|=0.$$

Proof. Denote

$$\Delta_n=\sup_x|\mathsf{P}(|S_n|\geq x\sqrt{n})-\mathsf{P}(|N|\geq x)|.$$

We can easily get that $\Delta_n\to 0$ as $n\to\infty$ by Lemma 3. So, via Toeplitz Lemma

$$\frac{1}{(\log\log m)^{b+1}}\sum_{n=1}^{m}\frac{\Delta_n(\log\log n)^b}{n\log n}\to 0,\ \ as\ \ m\to\infty.$$

Hence,

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n\leq B(\varepsilon)}\frac{(\log\log n)^b}{n\log n}\left|\mathsf{P}(|S_n|\geq\varepsilon\sqrt{n\log\log n})-\mathsf{P}(|N|\geq\varepsilon\sqrt{\log\log n})\right|\\
\leq&\varepsilon^{2(b+1)}\sum_{n\leq B(\varepsilon)}\frac{\Delta_n}{n\log n}(\log\log n)^b\\
=&\varepsilon^{2(b+1)}(\log\log[B(\varepsilon)])^{b+1}\frac{1}{(\log\log[B(\varepsilon)])^{b+1}}\sum_{n\leq B(\varepsilon)}\frac{\Delta_n}{n\log n}(\log\log n)^b\\
\leq&M^{b+1}\frac{1}{(\log[B(\varepsilon)])^{b+1}}\sum_{n\leq B(\varepsilon)}\frac{\Delta_n}{n\log n}(\log\log n)^b\to 0,\ \ as\ \ \varepsilon\searrow 0.
\end{aligned}$$

■

Proposition 10 *For any $b > -1$, we have*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n\log n}\left|P\{|S_n|\geq\varepsilon\sqrt{n\log\log n}\}-P\{|N|\geq\varepsilon\sqrt{\log\log n}\}\right|=0.$$

Proof. This is trivial via Proposition 6, 8 and 9 ■

By using the Propositions 5 and 10, we can easily obtain Theorem 2.

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