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Global Well-posedness of the Viscous b-Family of Equations

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Abstract: In this paper,the global well-posedness of the viscous b-family equations are studied. Get the global existence and uniqueness of solution to the viscous b-family equation for $u_0 \in L^2(\mathrm{R})$. According to the energy estimate, we result the local well-posedness for $u_0 \in L^2(\mathrm{R})$.By using this theorem and the usual extension theory the global well-posedness is proved.

Key words: viscosity; b-family equation; well-posedness

1 Introduction

In this paper, we are interested in the global well-posedness of the initial value problem (IVP) associated to the viscous version of the one-dimensional shallow water equation as follows

$$\partial_t u + \frac{1}{2}\partial_x(u^2) + \frac{b}{2}\partial_x(1-\partial_x^2)^{-1}(u^2) - \frac{b-3}{2}\partial_x(1-\partial_x^2)^{-1}(\partial_x u)^2 = 0 \qquad (1.1)$$

where u=(x, t), (x, t)$\in$ $\mathrm{R} \times \mathrm{R}$.

For $b = 2$, (1.1) becomes the Camassa-Holm equation, it has a bi-Hamiltionian structure and is completely integrable (see [1]). In [2] Dangping Ding and Lixin Tian researched solution of dissipative Camassa-Holm equation on total space. Tian, Song,Yin [3,4] considered the generalized Camassa-Holm equation and derived some new exact peakons and compactons.The Cauchy problem of nonlinear equation was proved.[17-19].

With $b = 3$ in (1.1), we find the Degasperis-Procesi equation

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0\,, \quad x \in \mathrm{R} \qquad (1.2)$$

Degasperis, Holm and Hone[6] proved the integrability of (1.2) by constructing a Lax pair. They also showed that (1.2) has bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admit exact peakon solutions which are analogous to the Camassa-Holm equation. After the Degasperis-Procesi equation (1.2) was derived, many papers were devoted to its study. For example, Yin proved local well-posedness to Eq (1.2) with initial data $u_0 \in H^s(\mathrm{R})\,, \left(s > \frac{3}{2}\right)$ [9] and derived the precise blow-up scenario and a blow-up result. The global existence of strong solutions and global weak solutions to Eq(1.2) was also investigated in [10,11].Recently, Lenells [12] classified all weak traveling wave solutions. Matsuno [15] studied multi-soliton solutions and their peakon limit. Coclite and Karlsen [16] proved there exists a unique global entropy weak solution in $L^1(\mathrm{R}) \cap BV(\mathrm{R})$ and $L^2(\mathrm{R}) \cap L^4(\mathrm{R})$.

It is completely integrable(see [7-12]). Despite the similarities to the Camassa-Holm equation, we would like to point out that these two equations are truly different. One of the important features of Eq(1.2) is that it has not only peaked solitons $u(t,x) = ce^{-|x-ct|}, c > 0$ but also shock peakons (see [13,14]) of the form

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$u\left(t,x\right) = \frac{1}{t+k} sgn\left(x\right) e^{-|x|}, k > 0$. On the other hand, the Lax pair and conservation laws of the two equations are also different (see [6]).

We have find it is convenient to rewriteEq(1.1)as the following form

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, t > 0, x \in \mathrm{R} \tag{1.3}$$

For a real parameter b, which includs both the Camassa-Holm equation,and the Degasperis-Procesi equation as special cases, since it arises form(1.1) when the peakon kernel $g(x) = \frac{1}{2}e^{-|x|}$ is chosen, we refer to(1.3)as the peakon b-family of equations.

It is shown in [5] that all these equations in the peakon b-family have not only the peakon solutions $u(x,t) = ce^{-|x-ct|}, c > 0$,but also multipeakon solutions $u(x,t) = \sum\limits_{k=1}^{N} p_k(t)e^{-|x-q_k|}$.For an arbitary constant b, p_k and q_k are not canonical variables but satisfy the dynamical system $p_k = -(b-1)\frac{\partial G_N}{\partial q_k}, q_k = \frac{\partial G_N}{\partial q_k}$. Where the generating function G_N is given by $G_N = \frac{1}{2}\sum\limits_{j,k=1}^{N} p_k p_j e^{-|q_j-q_k|}$.

Here, we intend to sharpen the global result to less regular initial data. We consider Eq (1.1) with an additional viscosity term

$$\begin{cases} \partial_t u = \varepsilon\partial_x^2 u + \frac{b-3}{2}\partial_x(1-\partial_x^2)^{-1}((\partial_x u)^2) - \frac{b}{2}\partial_x(1-\partial_x^2)^{-1}(u^2) - \frac{1}{2}\partial_x(u)^2 - k\partial_x u \\ u(x,0) = u_0(x) \end{cases} \tag{1.4}$$

where $\varepsilon > 0$, u=u(x, t), (x, t)$\in \mathrm{R}^2$, k is constant. We intend to show global existence and uniqueness of solution to (1.4) for $u_0 \in L^2(\mathrm{R})$. Denotes $L_x^2(\mathrm{R}), L_T^2(\mathrm{R})$ for $L_2(\mathrm{R})$ under x, T, respectively and $H_x^1(\mathrm{R})$, $L_T^\infty(\mathrm{R})$ for $H_1(\mathrm{R})$, $L_\infty(\mathrm{R})$under x, t, respectively.

Theorem 1 *For any $\varepsilon > 0$ and $u_0 \in L^2(\mathrm{R})$, the (IVP) (1.4) has solution in*

$$X = C([0,\infty); L_x^2(\mathrm{R})) \cap C((0,\infty); H_x^1(\mathrm{R})).$$

To show Theorem 1, we will first establish local well-posedness for $u_0 \in L^2(\mathrm{R})$

Theorem 2 *For any $\varepsilon > 0$ and $u_0 \in L^2(\mathrm{R})$, there exist a $T = T(u_0,\varepsilon) > 0$ and a unique solution u_ε of (1.4) such that*

$$u_\varepsilon(x,t) \in C([0,T]; L_x^2(\mathrm{R})) \cap C((0,T]; H_x^1(\mathrm{R}))$$

The rest of the paper is organized as follows. In section 2, we will set up and introduce a useful estimate concerning the operator $\partial_x(1-\partial_x^2)^{-1}$. In section 3, we will consider some estimates for the nonlinear part of the equation. Together with the crucial estimate in Lemma 4, we also obtain some estimates for $\partial_x u$. In section 4, we will prove Theorem 2 and show the local existence of (1.4) using the contraction argument. The last section is a sketch of the proof of Theorem 1.

2 Preliminaries

Consider the IVP, or, viscous b-family equation

$$\begin{cases} \partial_t u = \varepsilon\partial_x^2 u + f(u,\partial_x u), x, t \in \mathrm{R} \\ u(x,0) = u_0(x) \end{cases}$$

where $f(u,\partial_x u) = \frac{b-3}{2}\partial_x(1-\partial_x^2)^{-1}((\partial_x u)^2) - \frac{b}{2}\partial_x(1-\partial_x^2)^{-1}(u^2) - \frac{1}{2}\partial_x(u)^2 - k\partial_x u$.

The integral equivalent form of the equation is as follows:

$$u(x,t) = e^{\varepsilon t\partial_x^2}u_0 + \int_0^t e^{\varepsilon(t-t')\partial_x^2} f(u,\partial_x u)dt',$$

where $e^{\varepsilon t\partial_x^2}u_0 = (e^{-4\pi^2\varepsilon t\xi}\widehat{u_0}(\xi))^\vee$.

We will list some lemmas needed in the proof of the Theorem 2. First, we state the following lemma which consists of the crucial inequality involving the operator $\partial_x(1-\partial_x^2)^{-1}$. From [20] Lemma 1 for $g,h\in L^2(\mathrm{R})$,

$$\left\|\partial_x(1-\partial_x^2)^{-1}(gh)\right\|_{L_x^2}\le c\left\|g\right\|_{L_x^2}\left\|h\right\|_{L_x^2}$$

$$\left\|\left|\partial_x\right|^s(1-\partial_x^2)^{-1}(gh)\right\|_{L_x^2}\le c\left\|g\right\|_{L_x^2}\left\|h\right\|_{L_x^2}$$

for all $s<\frac{3}{2}$. The next two lemmas are regarding the nonlinear part of (1.4).

Lemma 1 *The next inequality is regarding the nonlinear part of (1.4).*

$$\begin{aligned}&\left\|\textstyle\int_0^t e^{\varepsilon(t-t')\partial_x^2}(-\frac{b}{2}\partial_x(1-\partial_x^2)^{-1}(u^2)-\frac{1}{2}\partial_x(u^2)-k\partial_x u)(x,t')dt'\right\|_{L_T{}^{\infty}L_x{}^{2}}\\&\le c(\left\|u\right\|_{L_T^2L_x^2}^2+\left\|\partial_x u\right\|_{L_T^2L_x^2}^2+\left\|\partial_x u\right\|_{L_T^2L_x^2}),\end{aligned}\tag{2.1}$$

$$\begin{aligned}&\left\|\textstyle\int_0^t e^{\varepsilon(t-t')\partial_x^2}(\frac{b-3}{2}\partial_x(1-\partial_x^2)^{-1}(\partial_x u)^2)(x,t')dt'\right\|_{L_T^\infty L_x^2}\\&\le c\left\|\partial_x u\right\|_{L_T^2L_x^2}^2,\end{aligned}\tag{2.2}$$

$$\left\|u(x,t)-e^{\varepsilon t\partial_x^2}u_0\right\|_{L_T^\infty L_x^2}\le c(\left\|u\right\|_{L_T^2L_x^2}^2+\left\|\partial_x u\right\|_{L_T^2L_x^2}^2+\left\|\partial_x u\right\|_{L_T^2L_x^2}).\tag{2.3}$$

Proof. By Lemma1 and Sobolev Embedding Theorem $H_x^1(\mathrm{R})\to L_x^\infty(\mathrm{R})$, we have

$$\begin{aligned}&\left\|\textstyle\int_0^t e^{\varepsilon(t-t')\partial_x^2}(-\frac{b}{2}\partial_x(1-\partial_x^2)^{-1}(u^2)-\frac{1}{2}\partial_x(u^2)-k\partial_x u)(x,t')dt'\right\|_{L_T^\infty L_x^2}\\&\le\frac{b}{2}\sup_{t\in[0,T]}\textstyle\int_0^t\left\|e^{\varepsilon(t-t')\partial_x^2}(-\partial_x(1-\partial_x^2)^{-1}(u^2)(x,t')\right\|_{L_x^2}dt'+\frac{1}{2}\sup_{t\in[0,T]}\int_0^t\left\|e^{\varepsilon(t-t')\partial_x^2}\partial_x(u^2)(x,t')\right\|_{L_x^2}dt'\\&+k\sup_{t\in[0,T]}\textstyle\int_0^t\left\|e^{\varepsilon(t-t')\partial_x^2}(\partial_x u)(x,t')\right\|_{L_x^2}dt'\\&\le\frac{b}{2}\textstyle\int_0^T\left\|(-\partial_x(1-\partial_x^2)^{-1}(u^2)\right\|_{L_x^2}dt+c\int_0^T\left\|u\partial_x u\right\|_{L_x^2}+\left\|\partial_x u\right\|_{L_x^2}dt\\&\le c(\left\|u\right\|_{L_T^2L_x^2}^2+\left\|\partial_x u\right\|_{L_T^2L_x^2}^2+\left\|\partial_x u\right\|_{L_T^2L_x^2}).\end{aligned}$$

Similarly, (2.2) is as follows

$$\begin{aligned}&\left\|\textstyle\int_0^t e^{\varepsilon(t-t')\partial_x^2}(\frac{b-3}{2}\partial_x(1-\partial_x^2)^{-1}(\partial_x u)^2)(x,t')dt'\right\|_{L_T^\infty L_x^2}\\&\le\frac{b-3}{2}\sup_{t\in[0,T]}\textstyle\int_0^t\left\|e^{\varepsilon(t-t')\partial_x^2}(\partial_x(1-\partial_x^2)^{-1}(\partial_x u)^2)(x,t')\right\|dt'_{L_T^\infty L_x^2}\le c\int_0^T\left\|\partial_x(1-\partial_x^2)^{-1}(\partial_x u)^2\right\|_{L_x^2}dt\\&\le c\textstyle\int_o^T\left\|\partial_x u\right\|_{L_x^2}^2dt=c\left\|\partial_x u\right\|_{L_T^2L_x^2}^2.\end{aligned}$$

From (2.1) and (2.2) we can get (2.3). ■

Lemma 2 *This lemma proved the inequality of $f(x,t)$.*

$$\left\|\int_0^t e^{\varepsilon(t-t')\partial_x^2}f(x,t')dt'\right\|_{L_T^2L_x^2}\le CT^{1/2}(\left\|u\right\|_{L_T^2L_x^2}^2+\left\|\partial_x u\right\|_{L_T^2L_x^2}^2+\left\|\partial_x u\right\|_{L_T^2L_x^2})$$

Proof.

$$\begin{aligned}&\left\|\textstyle\int_0^t e^{\varepsilon(t-t')\partial_x^2}f(x,t')dt'\right\|_{L_T^2L_x^2}\le(\int_o^T\left\|\int_{\mathrm{R}}(\int_0^t e^{\varepsilon(t-t')\partial_x^2}f(x,t'))dt')^2dx\right\|_{L_T^\infty}dt)^{1/2}\\&\le T^{1/2}\left\|\textstyle\int_0^t e^{\varepsilon(t-t')\partial_x^2}f(x,t')dt'\right\|_{L_T^\infty L_x^2}\\&\le T^{1/2}C(\left\|u\right\|_{L_T^2L_x^2}^2+\left\|\partial_x u\right\|_{L_T^2L_x^2}^2+\left\|\partial_x u\right\|_{L_T^2L_x^2}).\end{aligned}$$

■ The result follows from Lemma 1.

To investigate the estimates in H^1, we need the following lemma.

Lemma 3 *(see [20])For any* $u_0 \in L^2(\mathrm{R}), \varepsilon > 0$ *and* $\delta > 0$*, there exists* $T = T(u_0, \varepsilon) > 0$*, such that*

$$\left\| \partial_x e^{\varepsilon t \partial_x^2} u_0 \right\|_{L_T^2 L_x^2} = \left(\int_0^T \int\limits_{\mathrm{R}} \left| \partial_x e^{\varepsilon t \partial_x^2} u_0 \right|^2 dx dt \right)^{1/2} \leq \delta.$$

Lemma 4

$$\|f\|_{L_T^1 L_x^2} \leq C(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2})$$

Proof.

$$\begin{aligned}
&\|f\|_{L_T^1 L_x^2} = \left\| \tfrac{b-3}{2} \partial_x (1-\partial_x^2)^{-1}((\partial_x u)^2) - \tfrac{b}{2} \partial_x (1-\partial_x^2)^{-1}(u^2) - \tfrac{1}{2}\partial_x (u)^2 - k \partial_x u \right\|_{L_T^1 L_x^2} \\
&\leq \int_0^T \left\| \tfrac{b-3}{2} \partial_x (1-\partial_x^2)^{-1}((\partial_x u)^2) - \tfrac{b}{2} \partial_x (1-\partial_x^2)^{-1}(u^2) - \tfrac{1}{2}\partial_x (u)^2 - k \partial_x u \right\|_{L_x^2} dt \\
&\leq \tfrac{b}{2} \left(\int_0^T \|\partial_x u\|_{L_x^2}^2 dt + \int_0^T \|u\|_{L_x^2}^2 dt \right) + C' \int_0^T \|\partial_x u\|_{L_x^2}^2 dt + k \int_0^T \|\partial_x u\|_{L_x^2} dt \\
&\leq C(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}).
\end{aligned}$$

■

Then we have some estimates for v.

Lemma 5

$$\|v\|_{L_T^\infty L_x^2} \leq C(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}),$$

$$\|\partial_x v\|_{L_T^\infty L_x^2} \leq C \varepsilon^{-1/2} (\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2})^2.$$

Proof. see [20] ■

Lemma 6

$$\|\partial_x u\|_{L_T^2 L_x^2} \leq \delta + C \frac{1}{\varepsilon^{1/2}} (\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}).$$

Proof. From Lemma 3-5 ,we have

$$\begin{aligned}
&\|\partial_x u\|_{L_T^2 L_x^2} = \left\| \partial_x e^{\varepsilon t \partial_x^2} u_0 + \partial_x \int_0^t e^{\varepsilon (t-t') \partial_x^2} f(x,t') dt' \right\|_{L_T^2 L_x^2} \\
&\leq \left\| \partial_x e^{\varepsilon t \partial_x^2} u_0 \right\|_{L_T^2 L_x^2} + \left\| \partial_x \int_0^t e^{\varepsilon (t-t') \partial_x^2} f(x,t') dt' \right\|_{L_T^2 L_x^2} \\
&\leq \delta + \tfrac{1}{\varepsilon^{1/2}} \|f\|_{L_T^1 L_x^2} \\
&\leq \delta + C \tfrac{1}{\varepsilon^{1/2}} (\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}).
\end{aligned}$$

■

3 Local result: proof of Theorem 2

Let

$$\begin{gathered}
X_a^T = \{ u \in C([0,T) : L^2(\mathrm{R})) \cap C((0,T); H^1(\mathrm{R})) : \\
|||u||| = \left\| u - e^{\varepsilon t \partial_x^2} u_0 \right\|_{L_T^\infty L_x^2} + \|\partial_x u\|_{L_T^2 L_x^2} + \|u\|_{L_T^2 L_x^2} \leq a \}
\end{gathered}$$

and define the mapping $\phi : X_a^T \to X_a^T$ by

$$\begin{gathered}
\phi(u) = e^{\varepsilon t \partial_x^2} u_0 + \\
\int_0^t e^{\varepsilon (t-t') \partial_x^2} \left(\tfrac{b-3}{2} \partial_x (1-\partial_x^2)^{-1} (\partial_x u)^2 - \tfrac{b}{2} \partial_x (1-\partial_x^2)^{-1} (u^2) - \tfrac{1}{2} \partial_x (u)^2 - k \partial_x u \right)(x,t') dt'
\end{gathered} . \tag{3.1}$$

Theorem 3 *For any* $\varepsilon > 0$*, there exist* $T = T_\varepsilon > 0$ *and* $a > 0$ *such that* $\phi(X_a^T) \subseteq X_a^T$*. In addition,* $\phi : X_a^T \to X_a^T$ *is a contraction mapping.*

Proof. We first need to show that the map is well defined for some appropriate a and T .Let $u \in X_a^T$. We have

$$|||\phi u||| = \left\|\phi u - e^{\varepsilon t\partial_x^2}u_0\right\|_{L_T^\infty L_x^2} + \|\partial_x(\phi u)\|_{L_T^2L_x^2} + \|\phi u\|_{L_T^2L_x^2}\,. \tag{3.2}$$

Consider the terms in (3.2) one by one. From Lemma1, the first term in (3.2) can be estimated as follow:

$$\begin{aligned}
&\left\|\phi(u) - e^{\varepsilon t\partial_x^2}u_0\right\|_{L_T^\infty L_x^2}\\
&= \left\|\int_0^t e^{\varepsilon(t-t')\partial_x^2} f(u,\partial_x u)(x,t')dt'\right\|_{L_T^\infty L_x^2}\\
&= \left\|u(x,t) - e^{\varepsilon t\partial_x^2}u_0\right\|_{L_T^\infty L_x^2}\\
&\le C(\|u\|_{L_T^2L_x^2}^2 + \|\partial_x u\|_{L_T^2L_x^2}^2 + \|\partial_x u\|_{L_T^2L_x^2})\\
&\le C\,|||u|||^2
\end{aligned} \tag{3.3}$$

From Lemma 2, the second term in (3.2) can be estimated as follow:

$$\begin{aligned}
&\|\phi u\|_{L_T^2L_x^2}\\
&= \left\|e^{\varepsilon t\partial_x^2}u_0 + \int_0^t e^{\varepsilon(t-t')\partial_x^2} f(u,\partial_x u)(x,t')dt'\right\|_{L_T^2L_x^2}\\
&\le \left\|e^{\varepsilon t\partial_x^2}u_0\right\|_{L_T^2L_x^2} + \left\|\int_0^t e^{\varepsilon(t-t')\partial_x^2} f(u,\partial_x u)(x,t')dt'\right\|_{L_T^2L_x^2}\\
&\le T^{1/2}\|u_0\|_{L_x^2} + CT^{1/2}\,|||u|||^2\,;
\end{aligned} \tag{3.4}$$

From Lemma 6, the third term in (3.2) can be estimated as follow:

$$\begin{aligned}
&\|\partial_x(\phi u)\|_{L_T^2L_x^2}\\
&= \left\|\partial_x e^{\varepsilon t\partial_x^2}u_0 + \partial_x\int_0^t e^{\varepsilon(t-t')\partial_x^2} f(u,\partial_x u)(x,t')dt'\right\|_{L_T^2L_x^2}\\
&\le \delta + \frac{C}{\varepsilon^{1/2}}(\|u\|_{L_T^2L_x^2}^2 + \|\partial_x u\|_{L_T^2L_x^2}^2 + \|\partial_x u\|_{L_T^2L_x^2}) \le \delta + \frac{C}{\varepsilon^{1/2}}\,|||u|||\,.
\end{aligned} \tag{3.5}$$

Combine (3.2)-(3.5) we have that

$$\begin{aligned}
&|||\phi u|||\\
&= \left\|\phi u - e^{\varepsilon t\partial_x^2}u_0\right\|_{L_T^\infty L_x^2} + \|\partial_x(\phi u)\|_{L_T^2L_x^2} + \|\phi u\|_{L_T^2L_x^2}\\
&\le \delta + T^{1/2}\|u_0\|_{L_x^2} + C(1+T^{1/2})(|||u|||^2 + \frac{1}{\varepsilon^{1/2}}\,|||u|||)\\
&\le \delta + T^{1/2}\|u_0\|_{L_x^2} + C(1+T^{1/2})(1+\frac{1}{\varepsilon^{1/2}})a^2.
\end{aligned}$$

With appropriate values of δ, a, T, we are able to have that $|||\phi u||| \le a$, i.e, $\phi : X_a^T \to X_a^T$ is well defined.

With a similar argument, we can show that $\phi : X_a^T \to X_a^T$ is a contraction mapping:

$$|||\phi(u) - \phi(v)||| \le C'\,|||u - v|||$$

where $C' = C'(T, a, \varepsilon, \|u\|_{L_T^2L_x^2}, \|v\|_{L_T^2L_x^2}, \|\partial_x u\|_{L_T^2L_x^2}, \|\partial_x v\|_{L_T^2L_x^2})$ can be chosen as $0 < C' < 1$ with appropriate values of T and a. ■

Theorem 2 is merely Theorem 3 with a standard uniqueness argument.

4 Proof of Theorem 1

To prove Theorem 1, we need only to establish some priori estimates. To do so, we apply the operator $(1-\partial_x^2)$ to (1.4),

$$(1-\partial_x^2)\partial_t u-\varepsilon(1-\partial_x^2)\partial_x^2 u=\frac{b-3}{2}\partial_x((\partial_x u)^2)-\frac{b}{2}\partial_x(u^2)-\frac{1}{2}(1-\partial_x^2)\partial_x(u^2)-k(1-\partial_x^2)\partial_x u \quad (4.1)$$

Perform the standard energy estimate on (4.1), Multiply u to the equation and integrate with respect to x over R, we have that

$$\begin{aligned}
&\int u(1-\partial_x^2)\partial_t u dx-\varepsilon\int u(1-\partial_x^2)\partial_x^2 u dx\\
&=\frac{b}{2}\int u\partial_x(\partial_x u)^2 dx-\frac{b-3}{2}\int u\partial_x(u^2)dx-\frac{1}{2}\int u(1-\partial_x^2)\partial_x(u^2)dx-k\int u(1-\partial_x^2)\partial_x u dx\\
&=b\int u\partial_x u\partial_x^2 u dx-(b-3)\int u^2\partial_x u dx-\int u(1-\partial_x^2)u\partial_x u dx-k\int u(1-\partial_x^2)\partial_x u dx\\
&=b\int u\partial_x u\partial_x^2 u dx-(b-3)\int u^2\partial_x u dx-\int u^2\partial_x u dx+\int u\partial_x^2(u\partial_x u)dx-k(\int u\partial_x u dx-\int u\partial_x^3 u dx)
\end{aligned}$$

From the above proving we get one of the conversation laws is boundary. Thus we have an a priori estimate. Together with the local theory and the standard extension argument, we have a global solution for (1.4).

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References

[1] R.Camassa and D.HolmAn integrable shallow water equation with peaked solitions. *Phy.Rev.Letters.*71:1661-1664(1993)

[2] Dangping Ding, Lixin Tian: The study of solution of dissipative Camassa-Holm equation on total space.*International Journal of Nonlinear Science.* 1(1):37-42(2006)

[3] Tian,L.,Song,X.: New peaked solitary wave solutions of the generalized Camassa-Holm equation.*Chaos,Solitons and Fractals.*19(3):621-637(2004)

[4] Tian,L.,Yin,J.: New Compacton solutions and solitary solutions of fully nonlinear generalized Camassa-Holm equation.*Chaos,Solitons and Fractals.*20(4):289-299(2004)

[5] V.O.Vakhnenko, E.J.Parkes: Periodic and solitary -wave solutions of the Degasperis-Procesi Equation.*Chaos Solitions Fractals.* 201:059-073(2004

[6] A.Degasperis, D.D.Holm, A.N.W.Hone: A new integral equation with peakon Solutions. *Theoretical and Mathematical Physics.* 133:1463-1474(2002)

[7] H.R.Dullin, G.A.Gottwald, D.D.Holm: Camassa-Holm, Korteweg-de Veris-5 and other asymptotically equivalent equations for shallow water waves. *Fluid Dynamics Research.*33:73-79(2003)

[8] V.O.Vakhnenko , E.J.Parkes: Periodic and solitary-wave solutions of the Degasperis-Procesi equation. *Chaos Solitons Fractals.*20:1059-1073(2004)

[9] Z.Yin: On the Cauchy problem for an integrable equation with peakon solutions. *Illinois J.Math.*.47:649-666(2003)

[10] Z.Yin: Global weak solutions to a new periodic integrable equation with peakon solutions. *J.Funct.Anal.*.212:182-194(2004)

[11] Z.Yin: Global solutions to a new integrable equation with peakons. *Indiana Univ.Math. J.*.53:1189-1210(2004)

[12] J.Lenells: Traveling wave solutions of the Degasperis-Procesiequation.*J.Math.Anal.Appl.*.306:72-82(2005)

[13] G.M.Coclite, K.H.Karlsen, N.H.Risebro: Numerical schemes for computing discontinuous solutions of the Degasperis-Procesi equation. *J. Funct. Anal.*.6:1-23(2006)

[14] H.Lundmark: Formation and dynamics of shock waves in the Degasperis-Procesi equation. *The Royal Swedish Academy of Science. preprint.*

[15] Y.Matsuno:Multisolition solutions of the Degasperis-Procesi equation and their peakon limit. *Invere Problems*. 211:553-570(2005)

[16] G.M.Colite,K.H.Karlsen,H.Holden: On the Well-posedness of the Degasperis-Procesi equation. *Journal of Functional Analysis.* 233:60-91(2005)

[17] Lixin Tian, Xiuming Li: On the well-posedness problem for the generalized Degasperis- Procesi equation. *International Journal of Nonlinear Science.*1(2):67-76(2006)

[18] Lixin Tian, Meijie Ni: Blow-up Phenomena for the Periodic Degasperis- Procesi equation with Strong Dispersive Term. *International Journal of Nonlinear Science.*1(2):177-182(2006)

[19] Lixin Tian, Qing Shi: Boundary control on viscosity Dullin-Gottwald-Holm equation. *International Journal of Nonlinear Science.* 4(1):67-75(2007)

[20] Wee Keong Lim: On the global well-posedness for the viscous Camassa -Holm equation. *Journal of Mathematical analysis and Applications.*326:432-442(2004)