

New Explicit and Exact Solutions to the MKdV Equation

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Abstract: By introducing a transformation and applying the trial function method, abundant explicit and exact solutions including the solitary wave solutions, the singular travelling wave solutions, the triangle function periodic wave solutions, and etc. for the mKdV equation, are presented.

Keywords: trial function method; mKdV equation; explicit and exact solution

1 Introduction

In recent years, nonlinear wave equations (NWEs) have attracted considerable attention, partly due to their occurrence in many domains of science, in physics as well as in chemistry and biology, and partly owing to the interesting features and abundant varieties of properties of their solutions, especially thanks to the availability of computer symbolic system like Maple and Mathematica, which allow us to perform complicated and tedious algebraic calculations on a computer, as well as help us to seek new explicit and exact solutions of NWEs. Therefore, a vast variety of powerful and effective methods have been developed recently, such as the homogeneous balance method [1, 2], the hyperbolic tangent function expansion method [3–5], the trial function method [6–10], the sine-cosine method [11], the Jacobi elliptic function expansion method [12, 13], the superposition method [14], the auxiliary ordinary differential equation method [15–17], and so on. However, there is no general rule for solving NWEs. As a result, it is still a very significant task to search for more powerful and efficient methods to solve NWEs.

In the present paper, by introducing a transformation and utilizing the trial function method, we successfully find a series of explicit and exact solutions of the mKdV equation.

2 Solutions to the mKdV equation

The celebrated mKdV equation reads

$$\frac{\partial u}{\partial t} + \alpha u^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0 \quad (1)$$

which arises in many different fields, such as shallow water model, plasma science, biophysics and so on.

In order to solve Eq. (1) easily, here we introduce a transformation of the following form

$$u = u_0 + \frac{\partial v}{\partial x}, \quad v = v(y), \quad y = y(x, t) \quad (2)$$

where u_0 is a constant to be determined later, and $v(y)$ and $y(x, t)$ are two trial functions.

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To begin with, let us determine the trial function $y(x, t)$. As is commonly known, the solutions of NWEs should contain the factor $(kx - \omega t)$, i.e. $k(x - ct)$. As a consequence, we choose directly the trial function as the following form

$$y = e^{k(x-ct)} \tag{3}$$

where k and c are the wave number and wave speed, respectively. As for the other trial function $v(y)$, by our careful and repeated considerations, lastly we select it as the following form

$$v(y) = a \ln(b + y) \tag{4}$$

where a and b are constants to be determined later.

In view of Eqs. (2), (3) and (4), it is an easy matter to derive that

$$u = u_0 + \frac{\partial v}{\partial x} = u_0 + \frac{aky}{b + y} \tag{5}$$

$$\frac{\partial u}{\partial t} = -\frac{abck^2y}{(b + y)^2} \tag{6}$$

$$\frac{\partial u}{\partial x} = \frac{abk^2y}{(b + y)^2} \tag{7}$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{abk^4y(b^2 - 4by + y^2)}{(b + y)^4} \tag{8}$$

Here it is quite interesting that our ansatz (5) is in coincidence with the Backlund transformation of the mKdV equation given in Ref. [18] if we let $u_1(x, t) = 0$ and $a = 0$ in Eq. (14) of Ref. [18].

Plugging Eqs. (5)-(8) into Eq. (1) and collecting the coefficients of powers of y , then setting each of the obtained coefficients to zero, lead to a set of algebraic equations with respect to the unknown constants a, b and u_0 as follows

$$ab^3k^2(\alpha u_0^2 + k^2\beta - c) = 0 \tag{9}$$

$$2ab^2k^2(ak\alpha u_0 + \alpha u_0^2 - 2k^2\beta - c) = 0 \tag{10}$$

$$abk^2(2ak\alpha u_0 + \alpha u_0^2 + a^2k^2\alpha + k^2\beta^2 - c) = 0 \tag{11}$$

After solving the above system of algebraic equations by Mathematica, we get the following results.

Case 1: $c\alpha > 0, \alpha\beta < 0, c\beta < 0$

$$a = \mp \sqrt{\frac{-6\beta}{\alpha}}, u_0 = \pm \sqrt{\frac{3c}{\alpha}}, k = \pm \sqrt{\frac{-2c}{\beta}}, b = \text{an arbitrary constant} \tag{12}$$

Inserting Eq. (12) into Eq. (5) and considering Eq. (3) simultaneously, we obtain the general travelling wave solution to the mKdV equation (1) as follows

$$u_1 = \pm \sqrt{\frac{3c}{\alpha}} \mp 2\sqrt{\frac{3c}{\alpha}} \frac{e^{\pm \sqrt{\frac{-2c}{\beta}}(x-ct)}}{b + e^{\pm \sqrt{\frac{-2c}{\beta}}(x-ct)}} \tag{13}$$

If choosing $b = 1$ in Eq. (13) and making use of the following identity

$$\frac{e^{\pm 2x}}{e^{\pm 2x} + 1} = \frac{1}{2}(1 \pm \tanh x) \tag{14}$$

then we obtain the kink-type solitary wave solution to the mKdV equation (1) in the following form

$$u_2 = \pm \sqrt{\frac{3c}{\alpha}} \tanh \sqrt{\frac{-c}{2\beta}}(x - ct) \tag{15}$$

Similarly, if choosing $b = -1$ in Eq. (13) and taking advantage of the following identity

$$\frac{e^{\pm 2x}}{e^{\pm 2x-1}} = \frac{1}{2}(1 \pm \coth x) \quad (16)$$

then we acquire the singular travelling wave solution of the mKdV equation (1) as follows

$$u_3 = \pm \sqrt{\frac{3c}{\alpha}} \coth \sqrt{\frac{-c}{2\beta}}(x - ct) \quad (17)$$

Case 2: $c\alpha > 0, \alpha\beta < 0, c\beta > 0$

$$a = \mp \sqrt{\frac{-6\beta}{\alpha}}, u_0 = \pm i \sqrt{\frac{-3c}{\alpha}}, k = \pm i \sqrt{\frac{2c}{\beta}}, b = \text{an arbitrary constant} \quad (18)$$

where $i = \sqrt{-1}$.

Substituting Eq. (18) into Eq. (5) and considering Eq. (3) simultaneously, we get the following solution to the mKdV equation (1)

$$u_4 = \pm i \sqrt{\frac{-3c}{\alpha}} \mp 2i \sqrt{\frac{-3c}{\alpha}} \frac{e^{\pm i \sqrt{\frac{2c}{\beta}}(x-ct)}}{b + e^{\pm i \sqrt{\frac{2c}{\beta}}(x-ct)}} \quad (19)$$

If choosing $b = 1$ in Eq. (19) and making use of the former identity (14) and the following identity

$$\tanh(ix) = i \tan x \quad (20)$$

then Eq. (19) can be converted into

$$u_5 = \pm \sqrt{\frac{-3c}{\alpha}} \tan \sqrt{\frac{c}{2\beta}}(x - ct) \quad (21)$$

which is the triangle function periodic wave solution to the mKdV equation (1).

Similarly, if choosing $b = -1$ in Eq. (19) and making use of the preceding identity (16) and the following identity

$$\coth(ix) = -i \cot x \quad (22)$$

then Eq. (19) can be transformed to

$$u_6 = \pm \sqrt{\frac{-3c}{\alpha}} \cot \sqrt{\frac{c}{2\beta}}(x - ct) \quad (23)$$

Case 3: $c\alpha < 0, \alpha\beta > 0, c\beta < 0$

$$a = \mp i \sqrt{\frac{6\beta}{\alpha}}, u_0 = \pm i \sqrt{\frac{-3c}{\alpha}}, k = \pm \sqrt{\frac{-2c}{\beta}}, b = \text{an arbitrary constant} \quad (24)$$

Putting Eq. (24) into Eq. (5) and considering Eq. (3) simultaneously, we acquire the following solution of the mKdV equation (1)

$$u_7 = \mp i \sqrt{\frac{-3c}{\alpha}} \pm 2i \sqrt{\frac{-3c}{\alpha}} \frac{e^{\pm \sqrt{\frac{-2c}{\beta}}(x-ct)}}{b + e^{\pm \sqrt{\frac{-2c}{\beta}}(x-ct)}} \quad (25)$$

If taking $b = 1$ in Eq. (25) and making use of the previous identity (14), then Eq. (25) can be simplified as

$$u_8 = \pm i \sqrt{\frac{-3c}{\alpha}} \tanh \sqrt{\frac{-c}{2\beta}}(x - ct) \quad (26)$$

which is a complex line solution to the mKdV equation (1).

Similarly, if taking $b = -1$ in Eq. (25) and making use of the previous identity (16) simultaneously, then Eq. (25) can be reduced into

$$u_9 = \pm i \sqrt{\frac{-3c}{\alpha}} \coth \sqrt{\frac{-c}{2\beta}} (x - ct) \quad (27)$$

Case 4: $c\alpha > 0, \alpha\beta > 0, c\beta > 0$

$$a = \pm i \sqrt{\frac{6\beta}{\alpha}}, u_0 = \pm \sqrt{\frac{3c}{\alpha}}, k = \pm i \sqrt{\frac{2c}{\beta}}, b = \text{an arbitrary constant} \quad (28)$$

Plugging Eq.(28) into Eq. (5) and considering Eq. (3) simultaneously, we get the following solution for the mKdV equation (1)

$$u_{10} = \pm \sqrt{\frac{3c}{\alpha}} \mp 2 \sqrt{\frac{3c}{\alpha}} \frac{e^{\pm i \sqrt{\frac{2c}{\beta}} (x-ct)}}{b + e^{\pm i \sqrt{\frac{2c}{\beta}} (x-ct)}} \quad (29)$$

If taking $b = 1$ in Eq. (29) and making use of the previous two equalities (14) and (20), then Eq. (29) can be rewritten as

$$u_{11} = \pm i \sqrt{\frac{3c}{\alpha}} \tan \sqrt{\frac{c}{2\beta}} (x - ct) \quad (30)$$

which is a complex line solution to the mKdV equation (1).

Similarly, if taking $b = -1$ in Eq. (29) and making use of the previous two equalities (16) and (22), then Eq. (29) can be simplified as

$$u_{12} = \pm i \sqrt{\frac{3c}{\alpha}} \cot \sqrt{\frac{c}{2\beta}} (x - ct) \quad (31)$$

Apparently, the solutions u_2, u_3, u_5 and u_6 are not only similar to those given in Ref. [18] but also the same as those obtained in Ref. [19]. The rest of solutions, however, are the new ones for the mKdV equation which can not be found in literature to our knowledge. Finally, it should be remarked that we have verified all solutions (especially new ones) we obtained by putting them back into the original equation with the aid of Mathematica.

3 Conclusions

In summary, by introducing a transformation and utilizing the trial function approach, many types of explicit and exact solutions for the mKdV equation are successfully obtained, such as the solitary wave solutions, the singular travelling wave solutions, and the triangle function-type periodic wave solutions and so on, some of which are the first reported results to our knowledge.

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