

Riesz Multiwavelet of Multiplicity r and Multiresolution Analysis

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 (Received 8 February 2008, accepted 28 March 2008)

Abstract: In this paper, we obtain some properties that Riesz multiwavelets and the corresponding scaling functions should satisfy in order that the Riesz multiwavelets be associated with the multiresolution analysis (MRA). They are given in terms of the low/high-pass filters and in terms of the Fourier transform by using the newly obtained necessary and sufficient condition for the sum of two shift-invariant subspaces to be closed. The properties are used to improve the characterizations of Riesz multiwavelets associated with MRA previously obtained by some of the authors.

Key words: multiresolution analysis; Riesz multiwavelet; Lebesgue measure

1 Introduction

Wavelet is an applied subject developed in early nineteen eighties, which is an amelioration and development of traditional Fourier transformation. It has widely application in some side, such as image compression, data procession, filtration of signal etc [4,7]. Generally, multiresolution analysis is a better means to construct an wavelet. In this paper, we give the equivalent statements of Riesz multiwavelet of multiplicity r and the corresponding scaling functions associated with multiresolution analysis. They are given in terms of the low/high-pass filters and in terms of the Fourier transform. The properties are used to improve the characterizations of Riesz multiwavelets associated with MRA previously obtained by some of the authors.

Definition 1 [1] ψ is said to be a orthogonal wavelet of $L^2(\mathbb{R})$, If $\{\psi_{j,k} = D^j T_k \psi\}_{j,k \in \mathbb{Z}}$ is an orthonormal bases for $L^2(\mathbb{R})$, where $D : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is the unitary dyadic dilation operator defined by $Df(x) := 2^{\frac{1}{2}} f(2x)$; where $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is the unitary Translation operator defined by $T_r f(x) := f(x - r)$.

The set of all wavelet in $L^2(\mathbb{R})$ can be written $OW(L^2(\mathbb{R}))$. By [2, Theorem 3.21, Theorem 3.13], orthogonal wavelet is wavelet.

Definition 2 [3] $\Psi = (\psi_1, \psi_2, \dots, \psi_r)^T \in (L^2(\mathbb{R}))^r$ is said to be an orthogonal wavelet of multiplicity r in $L^2(\mathbb{R})$, if $\{T_n \Psi = T_k \psi_i, i = 1, 2, \dots, r\}_{n \in \mathbb{Z}}$ is an orthonormal bases for V_0 .

Let X, Y be two closed subspaces of a Hilbert space \mathcal{H} , then we say that X and Y are complementary subspaces if $X + Y = \mathcal{H}$ and $X \cap Y = \{0\}$. In this case, we write $\mathcal{H} = X \dot{+} Y$. If $X_i, i \in \mathbb{Z}$ is a closed subspace of Hilbert space \mathcal{H} such that $X_i \cap X_j = \{0\} (i \neq j)$ and $f \in \mathcal{H}$ can be written as norm convergent series $f = \sum_{i \in \mathbb{Z}} x_i, x_i \in X_i$, then we write $\mathcal{H} = \dot{+}_{i \in \mathbb{Z}} X_i$. We reserve the notation \oplus for the orthogonal sum.

Let $\Psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ be an orthogonal multiwavelet of multiplicity r in X , define

$$W_n^k = \overline{\text{span}\{D^n T_m \psi_k : m \in \mathbb{Z}\}}, \quad V_n^k = \bigoplus_{j=-\infty}^{n-1} W_j^k = \dots \oplus W_{n-3}^k \oplus W_{n-2}^k \oplus W_{n-1}^k,$$

$X_k = \overline{\bigcup_{n \in \mathbb{Z}} D^n V_0^k}, 1 \leq k \leq r$, then the following propositions are hold:

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Proposition 3 Under the above definition, we have:

- (1) $W_i^k \perp W_j^k (i \neq j, 1 \leq k \leq r)$;
- (2) $X_k = \bigoplus_{j>-\infty}^{+\infty} W_j^k (1 \leq k \leq r)$;
- (3) $V_{n+1}^k = V_n^k \oplus W_n^k (\forall n \in \mathbb{Z}, 1 \leq k \leq r)$;
- (4) $L^2(\mathbb{R}) = \bigoplus_{k=1}^r X_k$.

Forever, $\{V_n^k\}_{n \in \mathbb{Z}}$ be a family of closed subspaces of Hilbert space $X_k (1 \leq k \leq r)$, and it has the following properties:

- (1) $V_{n-1}^k \subset V_n^k (\forall n \in \mathbb{Z}, 1 \leq k \leq r)$;
- (2) $\bigcup_{n \in \mathbb{Z}} V_n^k = X_k$;
- (3) $\bigcap_{n \in \mathbb{Z}} V_n^k = \{0\}$;
- (4) $V_{n+1}^k = DV_n^k (\forall n \in \mathbb{Z}, 1 \leq k \leq r)$.

If function $\Psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ be a Riesz wavelet of multiplicity r in X , under the same notation, we have the Proposition similar to Proposition 3, only difference between them is the latter is orthogonal sum of closed subspaces and the former is sum of closed subspaces. From above analysis, each orthogonal (Riesz) wavelet can lead to orthogonal (sum decompose), and at the same time, it can creates an increasing closed subspaces that can satisfies the definition of multiresolution analysis(MRA), so there has an important question—which is first, MRA or wavelet? Similarly the notation of R. A. Zalik[5] and Bownik[6], we give the following conception. Function $\Phi = (\phi_1, \phi_2, \dots, \phi_r)^T$, satisfy $\phi_i \in V_0, 1 \leq i \leq r$, such that $\{T_k \phi_i : k \in \mathbb{Z}, 1 \leq i \leq r\}$ be Riesz base for V_0 , we say that this wavelet is associated with MRA; A wavelet ψ is obtained by an MRA, if there exists an MRA $\{V_j : j \in \mathbb{Z}\}$, such that $\psi \in V_1$. Notice that if a wavelet is associated with an MRA, then it is obviously obtained by the MRA; Recently, Bownik[6] showed that the converse also holds. Hence, we have that a Riesz wavelet ψ is associated with an MRA if and only if it is obtained by an MRA. Because Riesz wavelet of multiplicity r have better compatibility, in the second part, we obtain some properties that Riesz multiwavelets and the corresponding scaling functions should satisfy in order that the Riesz multiwavelets be associated with the multiresolution analysis(MRA). They are given in terms of the low/high-pass filters and in terms of the Fourier transform by using the newly obtained necessary and sufficient condition for the sum of two shift-invariant subspaces to be closed. The properties are used to improved the characterizations of Riesz multiwavelets associated with MRA previously obtained by some of the authors. Some related articles can be see references[8-10].

2 Some equivalent conditions of Riesz multiwavelets

Theorem 4 A Riesz multiwavelet $\Psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ of multiplicity r is associated with an MRA if and only if there exist two functions $P(x), Q(x) \in L^2_{r \times r}(\mathbb{R})$, called the low, high-pass filters, respectively, and $\Phi = (\phi_1, \phi_2, \dots, \phi_r)^T$ such that:

- (1) $\{T_k \phi_i : k \in \mathbb{Z}, 1 \leq i \leq r\}$ is a Riesz basis for V_0 ;

$$(2) \quad \hat{\Phi}(2x) = P(x)\hat{\Phi}(x) \quad \text{for a.e. } x \in \mathbb{R}; \quad (a)$$

$$(3) \quad \hat{\Psi}(2x) = Q(x)\hat{\Phi}(x) \quad \text{for a.e. } x \in \mathbb{R}; \quad (b)$$

Proof. Let Riesz multiwavelet $\Psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ of multiplicity r is associated with MRA, then by the definition of MRA, we can obtained property (1), and function Φ, Ψ have the following properties:

$$\Phi(t) = 2 \sum_{n \in \mathbb{Z}} P_n \Phi(2t - n), \quad \Psi(t) = 2 \sum_{n \in \mathbb{Z}} Q_n \Phi(2t - n), \quad (c)$$

where $(P_n)_{n \in \mathbb{Z}}, (Q_n)_{n \in \mathbb{Z}}$ be the matrix series of order r and its elements in $l^2(\mathbb{Z})$, define this set by $l^2_{r \times r}(\mathbb{Z})$. Take the Fourier transformation of equation (c), then (1),(2) follows.

Define $V_j = \overline{\text{span}}\{D^j T_k \phi_i : j, k \in \mathbb{Z}, 1 \leq i \leq r\}$, then we get the subspaces V_j satisfies (M1),(M4),(M5) of the definition of MRA, notice every Riesz base is an frame, by[3], we know (M3) is right; Similarly, take

the Fourier transformation of equation (b), $\Psi(t) = 2 \sum_{n \in \mathbb{Z}} Q_n \Phi(2t - n)$, where $(Q_n)_{n \in \mathbb{Z}} \in l^2_{r \times r}(\mathbb{Z})$. Since $\{D^j T_k \psi_i : j, k \in \mathbb{Z}, 1 \leq i \leq r\}$ be Riesz base for $L^2(\mathbb{R})$, therefore, $L^2(\mathbb{R}) = \overline{\text{span}}\{D^j T_k \psi_i : j, k \in \mathbb{Z}, 1 \leq i \leq r\}$, hence (M2) is qualified. This completes the proof. ■

Definition 5 [1] We say that A is essentially contained in B , if $A \setminus B$ has Lebesgue measure 0, We say that A is essentially equal to B , if both $A \setminus B$ and $B \setminus A$ has Lebesgue measure 0.

The support of $f(t)$ will be denoted by $\text{supp}\{f(t)\}$. With this notation we have:

Theorem 6 Suppose $\{D^j T_k \psi_i : j, k \in \mathbb{Z}, 1 \leq i \leq r\}$ be a Riesz base of $L^2(\mathbb{R})$, let $r(\omega) := (r_1(\omega), \dots, r_r(\omega))^T$, where $r_i(\omega) := \sqrt{\sum_{k \in \mathbb{Z}} |\hat{\psi}_i(\omega + 4k\pi)|^2}$. Then the following statements are equivalent:

- (1) Ψ is obtained by an MRA and $\text{supp}\{\hat{\Psi}(2\omega)\}$ is essentially equal to $\text{supp}\{\hat{\Psi}(\omega)\}$;
- (2) Ψ is obtained by an MRA and $\text{supp}\{\hat{\Psi}(2\omega)\}$ is essentially contained in $\text{supp}\{\hat{\Psi}(\omega)\}$;
- (3) There exists a 4π -periodic function $g(\omega) \in L^2_{r \times r}(\mathbb{R})$, such that $g(\omega)$ is invertible a.e. such that

$$\hat{\Psi}(2\omega) = g(\omega)\hat{\Psi}(\omega) \quad (d)$$

a.e. $\omega \in \mathbb{R}$;

(4) $r_i(\omega) > 0, 1 \leq i \leq r$ a.e. and There exists a 4π -periodic function $g(\omega) \in L^2_{r \times r}(\mathbb{R})$, such that (d) is satisfied a.e. $\omega \in \mathbb{R}$.

Proof. Clearly (1) \Rightarrow (2).

(2) \Rightarrow (3) Assume (2) is satisfied, let A and B be Riesz upper bounds and Riesz lower bound for $\{D^j T_k \psi_i : j, k \in \mathbb{Z}, 1 \leq i \leq r\}$, respectively, and let $c := \{c_k : k \in \mathbb{Z}\} \in l^2(\mathbb{Z})$. From the definition of Riesz base, we readily see that

$$A\|c\|_{l^2}^2 \leq \left\| \sum_{i=1}^r \sum_{k \in \mathbb{Z}^+} c_k T_k \psi_i \right\|^2 \leq B\|c\|_{l^2}^2,$$

and from [2, Theorem 3.24], we also obtain :

$$A \leq \sum_{i=1}^r \sum_{k \in \mathbb{Z}} |\hat{\psi}_i(\omega + 2k\pi)|^2 \leq B \quad \text{a.e. } \omega \in \mathbb{R} \quad (e)$$

Assume Ψ is obtained by an MRA, and let Φ and $P(x), Q(x)$ be functions that satisfy the properties (1), (2) described in Theorem 1, applying the identities in (a), (b), and again use (e), we deduce that for almost every $\omega \in \mathbb{R}$, such that $Q(\omega)$ is invertible, and have the following equation:

$$\hat{\Psi}(2\omega) = Q(\omega)\hat{\Psi}(\omega) = Q(\omega)P\left(\frac{\omega}{2}\right)\hat{\Psi}\left(\frac{\omega}{2}\right) = Q(\omega)P\left(\frac{\omega}{2}\right)Q\left(\frac{\omega}{2}\right)^{-1}\hat{\Psi}(\omega).$$

If N_1 denotes the set of $N_1 := \{\omega : Q(\frac{\omega}{2}) \text{ is not invertible, a.e. } \omega \in \mathbb{R}\}$, from Theorem 1(b), we see that $\hat{\Psi}(\omega)$ must vanish in N_1 , Applying the hypotheses, we therefore conclude that $\hat{\psi}(2\omega) = 0$ a.e. in N_1 . If N_2 denotes the set ω such that $Q(\frac{\omega}{2})$ is invertible and either $Q(\omega)$ is not invertible or $P(\frac{\omega}{2})$ is not invertible, a.e. $\omega \in \mathbb{R}$. By (2) we see that $\hat{\Psi}(2\omega)$ must vanish a.e. in N_2 . Therefore, if

$$g(\omega) := \begin{cases} 1, & \text{if } \omega \in N_1 \cup N_2; \\ Q(\omega)P\left(\frac{\omega}{2}\right)Q\left(\frac{\omega}{2}\right)^{-1}, & \text{otherwise.} \end{cases}$$

we see that (3) is satisfied. Clearly $g(\omega)$ is 4π -periodic.

Assume let (3) is satisfied, to prove (4), let

$$g(\omega) = (g_{i,j}(\omega))_{r \times r}, m(\omega) := \min_{1 \leq i, j \leq r} \{|g_{i,j}(\omega)|, |g_{i,j}(\omega + 2\pi)|\}.$$

then $m(\omega) > 0$, and apply (d) and (e), we conclude that for almost every $\omega \in \mathbb{R}$,

$$\begin{aligned} (r_i(\omega))^2 &= \sum_{k \in \mathbb{Z}} |\hat{\psi}_i(\omega + 4k\pi)|^2 \\ &= \sum_{k \in \mathbb{Z}} \left| \sum_{j=1}^r g_{ji}\left(\frac{\omega}{2} + 2k\pi\right) \hat{\psi}_j\left(\frac{\omega}{2} + 2k\pi\right) \right|^2 \\ &\geq m\left(\frac{\omega}{2}\right)^2 \sum_{k \in \mathbb{Z}} \left| \sum_{j=1}^r \hat{\psi}_j\left(\frac{\omega}{2} + 2k\pi\right) \right|^2 \\ &\geq m\left(\frac{\omega}{2}\right)^2 \sum_{k \in \mathbb{Z}} |\hat{\psi}_i\left(\frac{\omega}{2} + 2k\pi\right)|^2 \\ &\geq m\left(\frac{\omega}{2}\right)^2 A \\ &> 0. \end{aligned}$$

Assume now that (4) holds, next we will show other conditions (1),(2),(3) of Theorem 3 are satisfied. First we prove (1).

Let $q_i(\omega) := r_i(2\omega)$, and let

$$\hat{\phi}_i(\omega) := \begin{cases} \hat{\psi}_i(2\omega)q_i(\omega)^{-1}, & q_i(\omega) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Assume $q_i(\omega) \neq 0$, then $q_i(\omega + 2k\pi) = q_i(\omega) \neq 0$, therefore

$$\hat{\phi}_i(\omega + 2k\pi) = \hat{\psi}_i(2\omega + 4k\pi)q_i(\omega)^{-1}.$$

since $q_i(\omega) \neq 0$, a.e. This implies that $\sum_{k \in \mathbb{Z}} |\hat{\phi}_i(\omega + 2k\pi)|^2 = 1, 1 \leq i \leq r$. And at the same time, we have

$$\begin{aligned} \int_{\mathbb{R}} |\hat{\phi}_i(\omega)|^2 d\omega &= \sum_{k \in \mathbb{Z}} \int_{2k\pi}^{2(k+1)\pi} |\hat{\phi}_i(\omega)|^2 d\omega \\ &= \int_0^{2\pi} \sum_{k \in \mathbb{Z}} |\hat{\phi}_i(\omega + 2k\pi)|^2 d\omega \\ &= 2\pi. \end{aligned}$$

Hence $\hat{\phi}_i \in L^2(\mathbb{R}), 1 \leq i \leq r$. Let $\phi_i := \hat{\phi}_i$, therefore $\phi_i \in L^2(\mathbb{R}), 1 \leq i \leq r$, by [2, Theorem 3.24], we know that, $T_n \phi_i$ is Riesz base for its close linear span $V_0^i := \overline{\text{span}}\{T_n \phi_i : 1 \leq i \leq r\}$. Since $D^j T_k \psi_i$ be Riesz base for $L^2(\mathbb{R})$ and by definition (1.1), we get: $V_i \cap V_j = \{0\}$, and $L^2(\mathbb{R}) = \dot{\sum}_{k \in \mathbb{Z}} \dot{\sum}_{1 \leq i \leq r} V_k^i$, then by Theorem 1, $\Psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ is associated with an MRA; By (4) $\text{supp}\{\hat{\Psi}(2\omega)\}$ is essentially contained in $\text{supp}\{\hat{\Psi}(\omega)\}$, therefore Theorem (2), (3) follows. i.e. there exists an function $g(x) \neq 0$, s.t. $\omega \in \mathbb{R}$, a.e. hence (4) holds. This implies that $\text{supp}\{\hat{\Psi}(2\omega)\}$ is essentially equal to $\text{supp}\{\hat{\Psi}(\omega)\}$. ■

Corollary 7 Let $\{\Psi_{j,k} : j, k \in \mathbb{Z}\} \in L^2(\mathbb{R})$ and $\text{supp}\hat{\Psi}(2\omega)$ is essentially contained in Riesz base of $\text{supp}\hat{\Psi}(\omega)$, $r(\omega), \hat{\Phi}(\omega)$ are defined as above, let $\{c_k : k \in \mathbb{Z}\}$ be the series of Fourier coefficient of $r(\frac{\omega}{2})$. For each $j \in \mathbb{Z}$, define $V_j := \overline{\text{span}}\{\Phi_{j,k} : k \in \mathbb{Z}\}$, then $(\{V_j : j \in \mathbb{Z}, \Phi\})$ be an MRA, $\Psi \in L^2(\mathbb{R})$, and satisfy the following equation:

$$\Psi(t) = \sum_{k \in \mathbb{Z}} Q(\omega) \Phi(2\omega - k).$$

Theorem 8 Let $\{\Psi_{j,k} : j, k \in \mathbb{Z}\}$ be Riesz base for $L^2(\mathbb{R})$, and $\hat{\Psi}(\omega) \neq 0$, a.e. $\omega \in \mathbb{R}$. $g(\omega) = (g_{i,j}(\omega))_{r \times r}$ be defined as Theorem 3, then $g_{i,j}(\omega)$ can not be a scalar matrix a.e. $\omega \in \mathbb{R}$.

Proof. Since $\{\Psi_{j,k} : j, k \in \mathbb{Z}\}$ is a Riesz base of $L^2(\mathbb{R})$, the support of function Φ must have positive Lebesgue measure. Suppose $g_{i,j}(\omega)$ can be a scalar α , a.e. then $\alpha \neq 0$, a.e. and $\omega \in D$, then for every $k \in \mathbb{R}, \hat{\Phi}(2^k \omega) \neq 0, g(2^k \omega) \neq 0, \omega \in D$,

$$\hat{\Phi}(2^n \omega) = \left(\prod_{k=0}^{n-1} g(2^k \omega) \right) \hat{\Phi}(\omega), \quad n \in \mathbb{Z}^+.$$

and

$$\hat{\Psi}(2^{-n} \omega) = \hat{\Phi}(\omega) \prod_{k=0}^n g(2^{-k} \omega)^{-1}, \quad n \in \mathbb{Z}^+. \tag{i}$$

We deduce that

$$\hat{\Psi}(2^n \omega) = \alpha^n \hat{\Psi}(\omega) \quad \omega \in D, n \in \mathbb{Z}.$$

then for every $\omega \in D$, a.e.

$$\sum_{j \in \mathbb{Z}} |\hat{\Psi}(2^j \omega)|^2 = |\Psi(\hat{\omega})|^2 \sum_{j \in \mathbb{Z}} |\alpha|^{2j} = +\infty.$$

Because the set D have positive Lebesgue measure, and every Riesz base must be a frame, this contradicts[2, Theorem 3.21]. ■

Theorem 9 Suppose $\Psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ is a Riesz multiwavelet of multiplicity r , under the same notations as above, the following assertions are equivalent:

- (1) Ψ is associated with an MRA;
- (2) V_0 is shift-invariant and $\dim(\overline{\text{span}}\{\hat{\Phi}(2^j(x + 2k\pi))\}_{k \in \mathbb{Z}}) = r$ for a.e. $x \in \mathbb{R}$;
- (3) V_0 is shift-invariant and $\sum_{i=1}^r \sum_{j \geq 1} \sum_{k \in \mathbb{Z}} |\hat{\psi}_i(2^j(x + 2\pi k))|^2 > 0$ for a.e. $x \in \mathbb{R}$;
- (4) There exist $P(x), Q(x) \in \tilde{L}_{r \times r}^2[-\pi, \pi]$, $\Phi(x) = (\phi_1, \phi_2, \dots, \phi_r)^T \in (L^2(\mathbb{R}))^r$, such that $\{T^k \phi_i : k \in \mathbb{Z}, 1 \leq i \leq r\}$ be Riesz bases for subspace V_0 , and $\hat{\Phi}(2x) = P(x)\hat{\Phi}(x)$, $\hat{\Psi}(2x) = P(x)\hat{\Psi}(x)$ for a.e. $x \in \mathbb{R}$;
- (5) $\{T^n \psi_i : n \in \mathbb{Z}, 1 \leq i \leq r\}$ is a Riesz basis for subspace $V_1 \cap V_0^\perp$;
- (6) $\{T^n \phi_i : n \in \mathbb{Z}, 1 \leq i \leq r\}$ is a Riesz basis for $\overline{\text{span}}\{T^n \phi_i : n \in \mathbb{Z}, 1 \leq i \leq r\} = V_0$.

Proof. (1) \implies (2) Since the association of Ψ with an MRA implies exist $\Phi = (\phi_1, \phi_2, \dots, \phi_r)^T$, such that $\{T^n \phi_k : 1 \leq k \leq r, n \in \mathbb{Z}\}$ be Riesz bases for V_0^k , and at the same time, $V_0 = \bigoplus_{k=1}^r V_0^k$, that is, $V_0 = S(\{\Phi\}) = S(\{\phi_1\}) \oplus S(\{\phi_2\}) \oplus \dots \oplus S(\{\phi_r\})$. $\phi_k \in L^2(\mathbb{R}), 1 \leq k \leq r$, so V_0 is shift-invariant, and a.e. $x \in \mathbb{R}$,

$$\begin{aligned} & \dim(\overline{\text{span}}\{\hat{\Phi}(2^j(x + 2k\pi))\}_{k \in \mathbb{Z}}) \\ &= \dim(S(\Phi)(x)) \\ &= \dim(S(\phi_1)(x) \oplus S(\phi_2)(x) \oplus \dots \oplus S(\phi_r)(x)) \\ &= \dim(\overline{\text{span}}\{\hat{\phi}_1(2^j(x + 2k\pi))\}_{k \in \mathbb{Z}}) + \dots + \dim(\overline{\text{span}}\{\hat{\phi}_r(2^j(x + 2k\pi))\}_{k \in \mathbb{Z}}) \\ &= r. \end{aligned}$$

(1) \implies (3) If Ψ be a Riesz multiwavelet of multiplicity r , then by analysis of Proposition 3, $L^2(\mathbb{R}) = \dot{\bigoplus}_{i=1}^r X_i$, where every ψ_i is Riesz wavelet of closed subspace X_i , by[5], $\sum_{j=1}^{+\infty} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j(x + 2k\pi))|^2 > 0$ for a.e. $x \in \mathbb{R}$, hence $\sum_{j \geq 1} \sum_{k \in \mathbb{Z}} |\hat{\phi}_i(2^j(x + 2\pi k))|^2 > 0 (1 \leq i \leq r)$ for a.e. $x \in \mathbb{R}$, then $\sum_{i=1}^r \sum_{j \geq 1} \sum_{k \in \mathbb{Z}} |\hat{\psi}_i(2^j(x + 2\pi k))|^2 > 0$ for a.e. $x \in \mathbb{R}$.

(2) \implies (3) Trivial.

(3) \implies (1) Suppose V_0 is shift-invariant and $\sum_{i=1}^r \sum_{j \geq 1} \sum_{k \in \mathbb{Z}} |\hat{\psi}_i(2^j(x + 2\pi k))|^2 > 0$ for a.e. $x \in \mathbb{R}$, by definition of function Ψ , we have $\psi_i \neq 0, 1 \leq i \leq k$, then $\sum_{j \geq 1} \sum_{k \in \mathbb{Z}} |\hat{\psi}_i(2^j(x + 2\pi k))|^2 > 0$ for a.e. $x \in \mathbb{R}$, where $V_0 = \dot{\bigoplus}_{k=1}^r V_0^k$. by [4], there exist $\phi_k \in V_0^k$, such that the shift of ϕ_k is a Riesz basis for V_0^k , since $V_0 = \dot{\bigoplus}_{k=1}^r V_0^k$, hence $\{T^n \phi_i : n \in \mathbb{Z}, 1 \leq i \leq r\}$ is a Riesz basis for V_0 , therefore, Ψ is associated with an MRA.

(1) \implies (5) Suppose Ψ be a Riesz multiwavelet of multiplicity r , then by analysis of Proposition 3, $V_1 \cap V_0^\perp$; $L^2(\mathbb{R}) = \dot{\bigoplus}_{i=1}^r X_i$, where every ψ_i is Riesz wavelet of closed subspace X_i , i.e. $\{T^n \psi_i : n \in \mathbb{Z}\}$ is a Riesz basis for subspace W_0^i . Notice that $V_1 \cap V_0^\perp = \dot{\bigoplus}_{i=1}^r W_0^i$, therefore, $\{T^n \psi_i : n \in \mathbb{Z}, 1 \leq i \leq r\}$ is a Riesz basis for subspace $V_1 \cap V_0^\perp$.

(5) \implies (6) This follows from Proposition 3.

The equivalence of conditions (1), (4) and (6) are given at the Theorem 1 . ■

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