ACADIMIC
World Academic Union

ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.6(2008) No.2,pp.114-117

Stability Results for a Class of Elliptic Problems

G.A. Afrouzi [*], Z. Sadeghi
Department of Mathematics, Faculty of Basic Sciences,
University of Mazandaran, Babolsar, Iran

(Received 20 September 2007, accepted 29 January 2008)

Abstract: We prove stability/instability of positive stationary solutions to certain classes of elliptic problems of the form

$$\begin{cases} -\triangle\varphi(u(x)) = \lambda a(x)f(u(x)) & x \in \Omega, \\ B_\alpha u(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $\triangle$ denotes the Laplacian operator, Ω is a bounded and regular domain in $R^n, (n \geq 1)$ having smooth boundary $B_\alpha u(x) = \alpha h(x)u(x) + (1-\alpha)\frac{\partial u}{\partial n}$ where $\alpha \in [0,1]$ is a constant and $h : \partial\Omega \to R^+$ is a smooth function with $h \equiv 1$ when $\alpha = 1$, $\lambda > 0$, and $\varphi(u) \in R^+$ and f are smooth functions with $\varphi(0) = 0$. We also assume that weight $a(x) : \Omega \to R$ satisfies either $a(x) > 0$ or $a(x) < 0$ for all $x \in \Omega$.

Key words: Stability results; elliptic problems; positive stationary solutions
AMS subject classification:35J60, 35B30, 35B40

1 Introduction

Consider the elliptic boundary value problem

$$-\triangle\varphi(u(x)) = \lambda a(x)f(u(x)) \qquad x \in \Omega, \tag{1}$$

$$B_\alpha u(x) = 0 \qquad x \in \partial\Omega, \tag{2}$$

where $\lambda > 0$ is a parameter, Ω is a bounded and regular domain in R^n having smooth boundary $B_\alpha u(x) = \alpha h(x)u(x) + (1-\alpha)\frac{\partial u}{\partial n}$ where $\alpha \in [0,1]$ is a constant and $h : \partial\Omega \to R^+$ is a smooth function with $h \equiv 1$ when $\alpha = 1$, i.e; the boundary condition may be of Dirichlet($u = 0$), Neumann ($\frac{\partial u}{\partial n} = 0$) or mixed type (robin boundary condition), and $\varphi(u) \in R^+$ is a smooth function with $\varphi(0) = 0$, and weight function $a(x) : \Omega \to R$ satisfies either $a(x) > 0$ or $a(x) < 0$ for all $x \in \Omega$. We shall assume throughout that smooth function f satisfies either of the hypotheses (F) or (G) below:

(F) (i) $f : [0,\infty) \to R$ and $\frac{f(u)}{\varphi(u)}$ is strictly decreasing, or

(ii) $f : [0,\infty) \to R$ and $\frac{f(u)}{\varphi(u)}$ is strictly increasing

(G) (i) $f : [0,\infty) \to R^+$ such that $f(0) = 0$ and $f(u), \varphi(u)$ are respectively strictly concave and convex for all $u \in R^+$, or

(ii) $f : [0,\infty) \to R^+$ such that $f(0) = 0$ and $f(u), \varphi(u)$ are respectively strictly convex and concave for all $u \in R^+$.

In order to remind, we say that u is a positive solution of (1-2) if u is a classical solution with $u(x) \in R$ for all $x \in \bar{\Omega}$ and $u(x) > 0$ for all $x \in \Omega$, and also call a function g strictly convex(or concave) if $g'' > 0 (g'' < 0)$, respectively, and not constant zero on any subinterval.

---

[*] **Corresponding author**. *E-mail address*: afrouzi@umz.ac.ir

IJNS.2008.10.15/170

Equations of the form (1-2) enter a wide class of models in applied sciences, as they describe stationary nonlinear diffusion processes. Stability in the case of linear diffusion ($\varphi(u) = u$) was studied by several authors(refer [2],[3],[5] and [6]). The aim of the present paper is to extent Theorem 1 of [6] and Lemma 1 for $p = 2$ of [4]. Also different works have been made studying existence of solutions for this type of problems for systems case(see [1],[7]), Passo and Mottoni in [7] produce stability by a detailed analysis of the associated parabolic problem, but we by following definition prove stability of solutions.

Definition 1 *We call a positive stationary solution to (1-2) a stable solution if the first eigenvalue* λ_1 *of the linearized equation associated with (1-2), viz,*

$$-\triangle(\varphi'(u)v) - \lambda a(x)f'(u)v(x) = \lambda v(x) \qquad x \in \Omega, \tag{3}$$

$$B_\alpha v(x) = 0 \qquad x \in \partial\Omega, \tag{4}$$

is positive. Otherwise u called unstable. Note that Equation (3) obtained from the formal derivative of the operator Δ *on* $\varphi(u)$.

Our main results are the following:

Theorem 2 *Let f satisfies (F), and assume that the first eigenfunction of (3-4) can be chosen nonnegative. If* $a(x) > 0(a(x) < 0)$ *then every positive stationary solution of (1-2) is stable(unstable) in the case (i) and unstable(stable) in (ii).*

Theorem 3 *Let f satisfies (G), and assume that the first eigenfunction of (3-4) can be chosen nonnegative. If* $a(x) > 0(a(x) < 0)$ *then every positive stationary solution of (1-2) is stable(unstable) in the case (i) and unstable(stable) in (ii).*

Addition to the above results, we generalize this results to the following problem

$$\begin{cases} -\triangle\varphi(u(x)) = \lambda f(x, u(x)) & x \in \Omega, \\ B_\alpha u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{I}$$

where Ω, φ are as defined before, and $f : \Omega \times R^+ \to R$ is a smooth function.

Theorem 4 *(i) Let* $\frac{f(x,u)}{\varphi(u)}$ *is strictly decreasing for all fixed* $x \in \Omega$*, then every positive stationary solution of (I) is stable.*

(ii) Let $\frac{f(x,u)}{\varphi(u)}$ *is strictly increasing for all fixed* $x \in \Omega$*, then every positive stationary solution of (I) is unstable.*

Theorem 5 *Let* $f(x, u) \in R^+$*. Then*

(i)if $u \to f(x, u), u \to \varphi(u)$ *be respectively strictly convex and concave, and* $f(x, 0) = 0$ *for all fixed* $x \in \Omega$*, then every positive stationary solution of (I) is unstable.*

(ii)if $u \to f(x, u), u \to \varphi(u)$ *be respectively strictly concave and convex, and* $f(x, 0) = 0$ *for all fixed* $x \in \Omega$*, then every positive stationary solution of (I) is stable.*

2 Proofs

Before presenting the proof of our Theorems, we introduce the following lemma:

Lemma 6 *For any positive stationary solution of (1-2) and corresponding eigenpair* $(\lambda_1, \psi(x))$ *of (3-4), there holds*

$$(-\lambda_1)\int_\Omega \varphi(u)\psi(x)dx = \lambda\int_\Omega a(x)\psi(x)[f'(u)\varphi(u) - \varphi'(u)f(u)]dx.$$

Proof. Suppose u_0 be a positive stationary solution of (1-2). The Linearized equation associated with (1-2) about u_0 consists of

$$-\triangle(\varphi'(u_0)\psi) - \lambda a(x)f'(u_0)\psi(x) = \lambda\psi(x) \qquad x \in \Omega, \tag{5}$$

$$B_\alpha\psi(x) = 0 \qquad x \in \partial\Omega.$$

Where $(\lambda_1, \psi(x))$ denotes respectively first eigenvalue and corresponding eigenfunction that $\psi(x) > 0$ in Ω. We multiply (1) by $\varphi'(u_0)\psi$ and integrate over Ω to obtain

$$-\int_\Omega \varphi'(u_0)\psi(x)\triangle\varphi(u_0)dx - \int_\Omega \lambda a(x)\psi(x)\varphi'(u_0)f(u_0)dx = 0$$

or

$$\int_\Omega \nabla(\varphi'(u_0)\psi(x))\nabla\varphi(u_0)dx - \int_{\partial\Omega} \varphi'(u_0)\psi(x)\frac{\partial\varphi(u_0)}{\partial n}ds - \int_\Omega \lambda a(x)\psi(x)\varphi'(u_0)f(u_0)dx = 0, \tag{6}$$

similarly multiply (5) by $\varphi(u_0)$ and integrate over Ω to finally conclude that

$$\int_\Omega \nabla(\varphi(u_0))\nabla(\varphi'(u_0)\psi(x))dx - \int_{\partial\Omega} \varphi(u_0)\frac{\partial(\varphi'(u_0)\psi(x))}{\partial n}ds - \int_\Omega \lambda a(x)\psi(x)f'(u_0)\varphi(u_0)dx$$

$$= \lambda_1 \int_\Omega \varphi(u_0)\psi(x)dx. \tag{7}$$

Now by subtracting (6), (7), we have

$$(-\lambda_1)\int_\Omega \varphi(u_0)\psi(x)dx = \int_{\partial\Omega}[\varphi(u_0)\frac{\partial(\varphi'(u_0)\psi(x))}{\partial n} - \varphi'(u_0)\psi(x)\frac{\partial\varphi(u_0)}{\partial n}]ds$$

$$+\lambda\int_\Omega a(x)\psi(x)[f'(u_0)\varphi(u_0) - \varphi'(u_0)f(u_0)]dx. \tag{8}$$

It is obvious when $\alpha = 1$ then $B_\alpha u_0 = u_0 = 0$ and thus $\varphi(u_0) = \varphi(0) = 0$ for all $s \in \partial\Omega$, and because of ψ is corresponding eigenfunction, $\psi = 0$ for all $s \in \partial\Omega$, so we have

$$\int_{\partial\Omega}[\varphi(u_0)\frac{\partial(\varphi'(u_0)\psi(x))}{\partial n} - \varphi'(u_0)\psi(x)\frac{\partial\varphi(u_0)}{\partial n}]ds = 0,$$

and however when $\alpha \neq 1$, then

$$\int_{\partial\Omega}[\varphi(u_0)\frac{\partial(\varphi'(u_0)\psi(x))}{\partial n} - \varphi'(u_0)\psi(x)\frac{\partial\varphi(u_0)}{\partial n}]ds = \int_{\partial\Omega}\frac{\alpha h\psi(s)\varphi'(u_0)}{(1-\alpha)}\{\varphi(u_0) - \varphi(u_0)\}ds = 0,$$

therefore (8) gives

$$(-\lambda_1)\int_\Omega \varphi(u_0)\psi(x)dx = \lambda\int_\Omega a(x)\psi(x)[f'(u_0)\varphi(u_0) - \varphi'(u_0)f(u_0)]dx.$$

■

Proof of Theorem 2 :

Proof. We will give the proof only in (i) case. Second case is quite similar to first case. Because of $\frac{f(u)}{\varphi(u)}$ is strictly decreasing, we conclude

$$0 > (\frac{f(u)}{\varphi(u)})'(u_0) = \frac{f'(u_0)\varphi(u_0) - \varphi'(u_0)f(u_0)}{(\varphi(u_0))^2} \quad for \ u_0 > 0,$$

hence $f'(u_0)\varphi(u_0) - \varphi'(u_0)f(u_0)$ is negative, and thus if $a(x) > 0(< 0)$ then by Lemma 6, we obtain

$$(-\lambda_1)\int_\Omega \varphi(u_0)\psi(x)dx < 0(> 0).$$

It yields that $\lambda_1 > 0 (< 0)$. Therefore every positive stationary solution of (1-2) is stable(unstable). ■

Proof of Theorem 3 :
Proof. Similarly we will give the proof only in (i) case. Let $l(u_0) = f'(u_0)\varphi(u_0) - \varphi'(u_0)f(u_0)$ for all $u_0 \in [0, +\infty)$. Then

$$l'(u_0) = f''(u_0)\varphi(u_0) - \varphi''(u_0)f(u_0) < 0 \quad for\ u_0 > 0,$$

and so

$$l(u_0) < l(0) = 0 \quad for\ u_0 > 0.$$

Therefore for all $u_0 \in R^+$, $f'(u_0)\varphi(u_0) - \varphi'(u_0)f(u_0)$ is negative. Thus if $a(x) > 0 (< 0)$ then $\lambda_1 > 0 (< 0)$. This completes the proof. ■

Proof of Theorem 4 :
Proof. Proceeding as the proof Lemma 6, we can infer that

$$(-\lambda_1) \int_\Omega \varphi(u_0)\psi(x)dx = \lambda \int_\Omega \psi(x)[f_u(x, u_0)\varphi(u_0) - \varphi'(u_0)f(x, u_0)]dx.$$

and thus by hypothesis, the integrand on the right-hand side is negative in the case (i) and positive in (ii). This proves that every positive stationary solution of (I) is stable in (i) and unstable in (ii). ■

Proof of Theorem 5 :
Proof. Similarly following the proof Lemma 6, and with introducing $l(x, u_0) = \varphi(u_0)\partial_u f(x, u_0) - \varphi'(u_0)f(x, u_0)$, we obtain the desired sign of the first eigenvalue λ_1 via the appropriate analogue of Theorem 3. ■

References

[1] A. Canada, J.L. Gamez: Elliptic systems with nonlinear diffusion in population dynamics. *Differential Equations and Dynamical Systems*. 3(2): 189-204(1995)

[2] A. Tertikas: Stability and instability of positive solutions of semipositone problems. *Proceedings of the American Mathematical Society*. 114(4): 1035-1040 (1992)

[3] C. Maya, and R. Shivaji: Instability of nonnegative solution for a class of semilinear elliptic boundary value problems. *Journal of Computational and Applied Mathematics.* 88:125-128(1998)

[4] J. Karatson, P.L. Simon: On the linearized Stability of positive solutions of quasilinear problems with p-convex or p-concave nonlinearity. *Nonlinear Analysis*. 47: 4513-4520(2001)

[5] J. Karatson, P.L. Simon: On the stability properties of nonnegative solutions of semilinear problems with convex or concave nonlinearity. *Journal of Computational and Applied Mathematics*. 131: 497-501(2001)

[6] K.J. Brown, P. Hess: Stability and uniqueness of positive solutions for a semi-linear elliptic boundary value problem. *Differential and Integral Equations*. 3(2): 201-207(1990)

[7] R. Dal Passo, P. de Mottoni:Some existence, uniqueness and stability results for a class of semilinear degenerate elliptic systems. *Bolletino U.M.I. Anal. Funzionale e Appl.* VI: 203-231(1984)