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Noncontinuous Wave Solutions for a New Sort of Double Sine-Gordon Equation

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Abstract: In this paper we research the exact solutions for a sort of new Sine-Gordon equation. By using the tanh method and a variable separated ordinary difference method, we get some new exact travelling wave solutions, among which there are two new noncontinuous wave solutions.

Key words: double Sine-Gordon equation; noncontinuous wave solution; tanh method; variable separated ODE method

1 Introduction

The double sine-Gordon equation is a significant equation. It has such applications in physics as nonlinear optics, Josephson array, ferromagnetic material, charge density waves, and liquid helium and so on, which offers simple and ideal models. Double sine-Gordon has been used to model several physical systems like the spin waves in super fluid, self-induced transparency in accounting degeneracy of atomic levels, electromagnetic waves propagating in semiconductor quantum super lattices, some features of the propagation of resonant ultra-short optical pulse through degenerate media, non-linear excitations in a compressible chain of *XY* dipoles under conditions of piezo electric coupling macromolecules etc. Popov[1] presented a perturbation theory for the double sine-Gordon equation and obtained a system of differential equations that showed the solitary parameters modification under the influence of the perturbation. Recently, more exact solutions of double sine-Gordon equation have been obtained by using F-expansion method[2]. The Jacobi-sn and Jacobi-cn function solutions to double sine-Gordon equation were obtained by using the Jacobi elliptic function expansion with symbolic computation[3]. Yu and Tian[4] studied exact solitary wave solutions of approximate fully nonlinear double sine-Gordon equation by using ansatz method.

The study of solitons is an important part in the solitary wave theory. Rosenau and Hyman[6] studied the generalized nonlinear dispersive equation $K(m,n)$ and obtained solitary wave solutions with compact support in it which they were called compacton. Tian and Yin[7] introduced a fifth-order$K(m,n)$ equation with nonlinear dispersion to obtain compacton solutions. Camassa and Holm[8] found peakon solutions, which have discontinuous first-order derivative at the wave peaks.

The extended tanh method[9] is used to derive abundant solitary wave solutions of nonlinear wave equations. The obtained solutions include solitons and kinks solutions. The extended tanh method presents a wider applicability for handling nonlinear wave equations. A variety of powerful methods, such as bilinear tranformation[10,11], and homogeneous balance method[12].

In this paper we study the solitary wave solutions of a new type of nonlinear double sine-Gordon equations, lots of new solitary waves are obtained. By improved some classical methods of finding solutions of nonlinear equations, we study solutions of nonlinear wave equation with complex nonlinear terms (double sine-Gordon equation) and find abundant solitary wave solutions(kink solution, anti-kink solution, periodic wave solution) and a kind of new discontinuous solution. Then we prove that the discontinuous solution

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is discontinuous solitary wave solution by the conservation equation theory. The fully nonlinear approximate double sine-Gordon equation is investigated and obtained compacton solution, peakon solution, multi-compacton, multi-peakon solution and discontinuous solitary wave solution.

2 Noncontinuous solitary wave solution of the double Sine-Gordon equation

In this paper, we will investigate a new noncontinuous method of the new sort of double sine-Gordon equation.

$$(u)_{tt} - k(u)_{xx} + 2\alpha\sin(2u) + \beta\sin(4u) = 0 \tag{2.1}$$

We give the definitions of conservation law equation and noncontinuous solution according to the reference [5].

Definition 2.1 *If the equation set*

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{k} A_i(u, x_1, ..., x_k, t)\frac{\partial u}{\partial x_i} = g(u, x_1, ..., x_k, t) \tag{2.2}$$

can be transformed into the following form

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{k} \frac{\partial f_i(u, x_1, ...x_{k,} t)}{\partial x_i} = g(u, x_1, ..., x_k, t) \tag{2.3}$$

(2.2) is called conservation law equation. If $g(u, x_1, ..., x_k, t)$=0, (2.2) is called the homogeneous conservation law equation. Where u is an unknown vector function,

u=$(u_1, ..., u_n)^T$,T denotes transposition and gis a known vector function,

$g = (g_1, ...g_n)^T$, $A_i$is a known matrix function $(i = 1, \cdots, k)$, $A_i = \begin{bmatrix} a_{11}^{(i)} & a_{1n}^{(i)} \\ a_{n1}^{(i)} & a_{nn}^{(i)} \end{bmatrix}$;x_i and t are called spatial and time variables, respectively.

Now, we rewrite the double sine-Gordon equation to the form of conservation law equation. Let $u_x = v, u_t = w$,so $v_t = w_x = u_{xt}$. The double sine-Gordon equation can be rewritten to the following equation set

$$\begin{cases} w_t - kv_x + 2\alpha\sin(2u) + \beta\sin(4u) = 0 \\ v_t = w_x \\ u_t = w \end{cases} \tag{2.4}$$

Namely, the conservation law equation of double sine-Gordon equation is as follows

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}_t - \begin{bmatrix} 0 \\ w \\ v \end{bmatrix}_x = \begin{bmatrix} w \\ 0 \\ -2\alpha\sin(2u) - \beta\sin(4u) \end{bmatrix} \tag{2.5}$$

where $g(u, x, t) = \begin{bmatrix} w \\ 0 \\ -2\alpha\sin(2u) - \beta\sin(4u) \end{bmatrix}$,$f(u, x, t) = \begin{bmatrix} 0 \\ w \\ v \end{bmatrix}$.

Definition 2.2 *Let u be a weak solution of*

$$\frac{\partial u}{\partial t} + \frac{\partial f(u, x, t)}{\partial x} = g(u, x, t) \tag{2.6}$$

except for the first kind of discontinuity on finite curves, it preserves continuous one-order partial derivative in other definition region. Then u can be called a noncontinuous solution.

Proposition 2.3 *The discontinuous vectors function $u(x, t)$ is a noncontinuous solution of (2.6), if and only if:*

(1) In the smooth parts, it is the classical solution of equation .

(2) It satisfies $[u]_s = f$ on discontinuous lines, where $\frac{dx}{dt} = s$ denotes the slope.

Now we study the exact solutions to double sine-Gordon equation with the tanh method. For Eq.(2.1), let $\xi = x - ct$, in which c is the wave speed. Then Eq.(2.1) can be converted to be

$$(c^2 - k)u'' + 2 * \alpha \sin(2u) + \beta \sin(4u) = 0 \tag{2.7}$$

Set $v = e^{i2u}, i = \sqrt{-1}$, then

$$u = \frac{1}{2i} \ln v \tag{2.8}$$

and we find

$$\begin{aligned} \sin(2u) &= \tfrac{v - v^{-1}}{2i}, \sin(4u) = {(v^2 - v^{-2})}/{2i} \\ \cos(2u) &= \tfrac{v + v^{-1}}{2} \end{aligned} \tag{2.9}$$

This also gives

$$u = \frac{1}{2} \arccos[\frac{v + v^{-1}}{2}] \tag{2.10}$$

Substituting the transformations (2.8) and (2.9) into (2.7) , we have the ordinary difference equation

$$2\beta v^4 + 4v^3 - 4v - 2\beta + 2(c^2 - k)vv'' - 2(c^2 - k)(v')^2 = 0 \tag{2.11}$$

We can set the solution form as follows:

$$v = a_0 + a_1 Y \tag{2.12}$$

where a_0, a_1 are unknown parameters, $Y = \tanh[\mu(x - ct)]$. Substitute (2.12) into Eq. (2.11), collect the coefficients of each power of Y and let the coefficients of Y^j $(j = 0, 1, \cdots)$ equal to zero. Using symbolic computation program Maple, we obtain

$$a_0 = -\frac{\alpha}{\beta} \quad , a_1 = \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \quad , \quad \alpha > \beta, \quad \mu = \sqrt{\frac{(\alpha^2 - \beta^2)}{\beta(k - c^2)}}, \alpha > \beta, k > c^2,$$

where c is the wave speed. We have:

$$v(x,t) = -\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \tanh[\mu(x - ct)], \alpha > \beta \tag{2.13}$$

$$v(x,t) = -\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \coth[\mu(x - ct)], \alpha > \beta \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.10), we obtain the exact solutions to Eq.(2.1)

$$\begin{aligned} u(x,t) &= \tfrac{1}{2} \arccos\{\tfrac{1}{2}(-\tfrac{\alpha}{\beta} + \tfrac{\sqrt{\alpha^2-\beta^2}}{\beta} \tanh[\mu(x - ct)] \\ &+ \tfrac{1}{-\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2-\beta^2}}{\beta} \tanh[\mu(x-ct)]})\} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} u(x,t) &= \tfrac{1}{2} \arccos\{\tfrac{1}{2}(-\tfrac{\alpha}{\beta} + \tfrac{\sqrt{\alpha^2-\beta^2}}{\beta} \coth[\mu(x - ct)] \\ &+ \tfrac{1}{-\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2-\beta^2}}{\beta} \coth[\mu(x-ct)]})\} \end{aligned} \tag{2.16}$$

On the other hand, for $\alpha < \beta$,

$$\begin{aligned} u(x,t) &= \tfrac{1}{2} \arccos\{\tfrac{1}{2}(-\tfrac{\alpha}{\beta} - \tfrac{\sqrt{\alpha^2-\beta^2}}{\beta} \tanh[\mu(x - ct)] \\ &+ \tfrac{1}{-\frac{\alpha}{\beta} - \frac{\sqrt{\alpha^2-\beta^2}}{\beta} \tanh[\mu(x-ct)]})\} \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} u(x,t) &= \tfrac{1}{2} \arccos\{\tfrac{1}{2}(-\tfrac{\alpha}{\beta} - \tfrac{\sqrt{\alpha^2-\beta^2}}{\beta} \coth[\mu(x - ct)] \\ &+ \tfrac{1}{-\frac{\alpha}{\beta} - \frac{\sqrt{\alpha^2-\beta^2}}{\beta} \coth[\mu(x-ct)]})\} \end{aligned} \tag{2.18}$$

where $\mu = \sqrt{\frac{2(\alpha^2-\beta^2)}{2\beta(k-c^2)}}, \alpha > \beta, k > c^2,$

It is noticed that the exact double Sine-Gordon equation is based on $n = 1$.

In what follows we will employ a variable separated ordinary difference equation method to formally derive a new type of traveling wave solutions to the Eq.(2.1) by using the variable $\xi = x - ct$, which is equivalent to the following form

$$u'' + \frac{2\alpha}{c^2-k}\sin(2u) + \frac{\beta}{c^2-k}\sin(4u) = 0 \tag{2.19}$$

Next, we assume that

$$u'(\xi) = \frac{du}{d\xi} = a + b\cos(2u) \tag{2.20}$$

where a, b are parameters determined later. We have

$$u''(\xi) + 2ab\sin(2u) + b^2\sin(4u) = 0 \tag{2.21}$$

Comparing (2.21) with (2.19), we obtain

$$2ab = \frac{2\alpha}{c^2-k}, 2b^2 = \frac{2\beta}{c^2-k} \tag{2.22}$$

So

$$a = \frac{2\alpha}{\sqrt{4\beta(c^2-k)}}, b = \sqrt{\frac{\beta}{c^2-k}} \tag{2.23}$$

We write the Eq. (2.20) as follows

$$\frac{1}{a + b\cos(2u)}du = d\xi \tag{2.24}$$

Integrating both sides of (2.24), we find the following solutions

$$\frac{1}{\sqrt{a^2-b^2}}\arctan(\sqrt{\frac{a-b}{a+b}}\tan(u)) = \xi + \xi_0 (a > b) \tag{2.25}$$

$$\frac{1}{\sqrt{a^2-b^2}}\arctan(\sqrt{\frac{b-a}{a+b}}\tan(u)) = \xi + \xi_0 (a < b) \tag{2.26}$$

$$\frac{1}{\sqrt{a^2-b^2}}arc\coth(\sqrt{\frac{b-a}{a+b}}\tan(u)) = \xi + \xi_0 (a < b) \tag{2.27}$$

where ξ_0 is an integral constant. The exact solutions are given as follows

$$u(x,t) = \arctan\{\sqrt{\tfrac{\alpha+\beta}{\alpha-\beta}}\tan(\sqrt{\tfrac{\alpha^2-\beta^2}{\beta(c^2-k)}}(x-ct)+\xi_0))\}(\alpha > \beta) \tag{2.28}$$

and

$$u(x,t) = \arctan\{\sqrt{\tfrac{\alpha+\beta}{\beta-\alpha}}\tanh(\sqrt{\tfrac{\beta^2-\alpha^2}{\beta(c^2-k)}}(x-ct)+\xi_0))\}(\alpha < \beta) \tag{2.29}$$

and

$$u(x,t) = \arctan\{\sqrt{\tfrac{\beta-\alpha}{\beta+\alpha}}\tanh(\sqrt{\tfrac{\beta^2-\alpha^2}{\beta(c^2-k)}}(x-ct)+\xi_0))\}(\alpha < \beta) \tag{2.30}$$

and

$$u(x,t) = \arctan\{\sqrt{\tfrac{\beta-\alpha}{\beta+\alpha}}\coth(\sqrt{\tfrac{\beta^2-\alpha^2}{\beta(c^2-k)}}(x-ct)+\xi_0))\}(\alpha < \beta) \tag{2.31}$$

Taken $\alpha = 3, \beta = 1, c = 2, k = 1, t = 1, \xi_0 = 0$, Fig.1 shows the graphics of the solution (2.28). Taken $\alpha = 1, \beta = 3, c = 2, k = 1, t = 1, \xi_0 = 0$, Fig.2 shows the graphics of the solution (2.29). Taken $\alpha = 1, \beta = 3, c = 2, k = 1, t = 1, \xi_0 = 0$, Fig.3 shows the graphics of the solution (2.30). Taken $\alpha = 1, \beta = 3, c = 2, k = 1, t = 1, \xi_0 = 0$, Fig.4 shows the graphics of the solution (2.31).

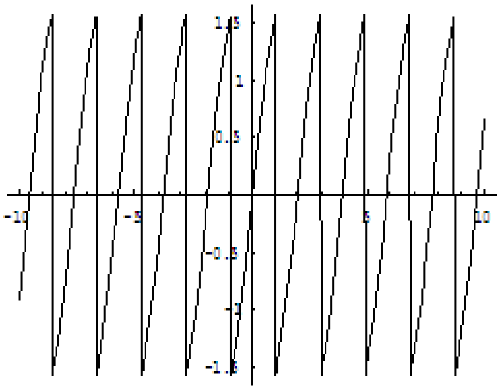

Figure 1: Noncontinuous waves

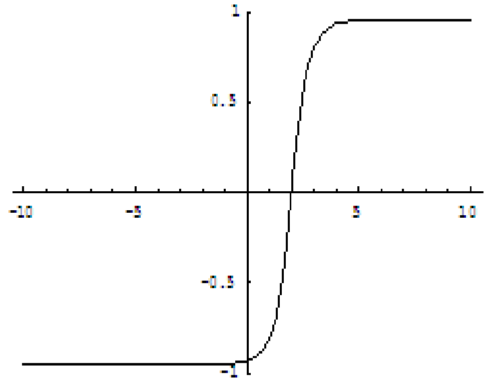

Figure 2: Kink wave

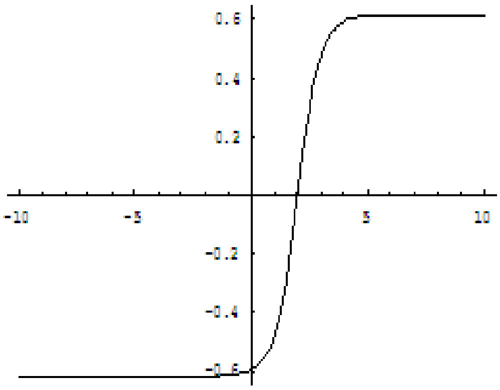

Figure 3: Kink wave

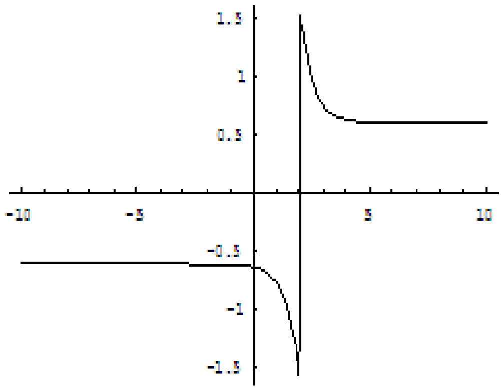

Figure 4: Noncontinuous wave

But if $a = b$, (2.27) is converted to the following form

$$\frac{1}{1+\cos(2u)}du = ad\xi \tag{2.32}$$

Integrating both sides of (2.32), we find the following solutions

$$\frac{1}{2}\tan(u) = a(\xi + \xi_0)$$

So we get

$$u(x,t) = \arctan[2a(\xi + \xi_0)] \tag{2.33}$$

or

$$u(x,t) = \arctan(\frac{4\alpha}{\sqrt{c^2-k}}((x-ct)+\xi_0)) \tag{2.34}$$

Taken $a = b = 1$, ξ_0=0, Fig.5 shows the graphics of the solution (2.33). Taken $\alpha = 1, c = 2, k = 1, t = 1, \xi_0 = 0$, Fig.6 shows the graphics of the solution (2.34).

If $a = -b$, (2.24) is converted to the following form

$$\frac{1}{1-\cos(2u)}du = ad\xi \tag{2.35}$$

Integrating both sides of (2. 35), we find the following solutions

$$-\frac{1}{2}\cot(u) = a(\xi + \xi_0) \tag{2.36}$$

So we get

$$u(x,t) = \arctan(-2a(\xi + \xi_0)) \tag{2.37}$$

which is equivalent to

$$u(x,t) = arc\cot(-\frac{4\alpha}{\sqrt{c^2-k}}((x-ct)+\xi_0)) \tag{2.38}$$

Taken $a = -b = 1$,ξ_0=0, Fig.7 shows the graphics of the solution (2.37).

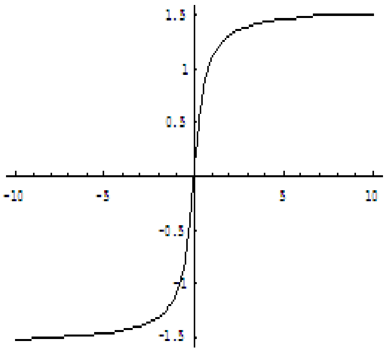

Figure 5: Kink wave

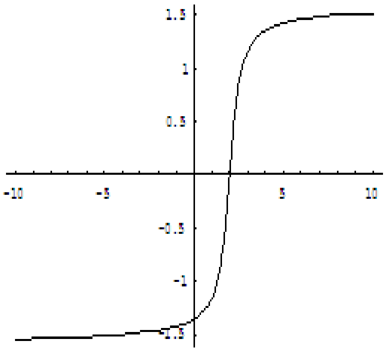

Figure 6: Kink wave

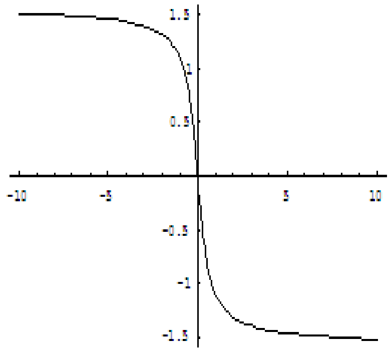

Figure 7: Anti-kink wave

Taken $\alpha = 1, c = 2, k = 1, t = 1, \xi_0 = 0$, Fig.8 shows the graphics of the solution (2.38). It is noticed that

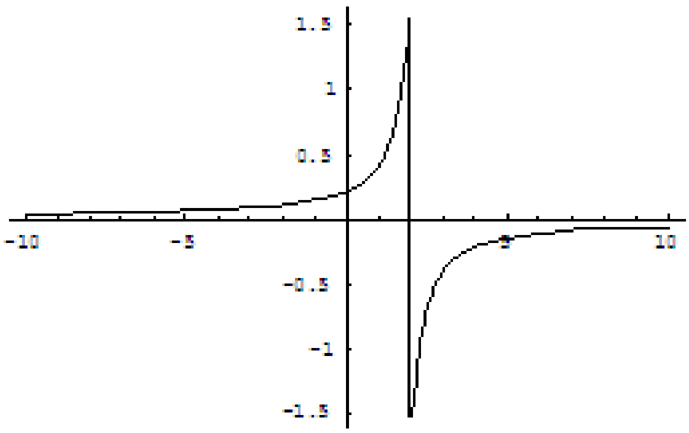

Figure 8: Noncontinuous wave

all the above figures are also based on $n = 1$.

Then we prove that the noncontinuous solutions obtained from the double sine-Gordon equation are noncontinuous solitary wave solutions.

Corollary 2.4 *The following solution of the double sine-Gordon equation*

$$u(x,t) = \arctan\{\sqrt{\frac{\beta-\alpha}{\beta+\alpha}}\coth(\sqrt{\frac{\beta^2-\alpha^2}{\beta(c^2-k)}}(x-ct)+\xi_0))\}(\alpha<\beta)$$

is a noncontinuous solution.

Proof. It is easy to know that

(1) the above solution is the classical solution of the double sine-Gordon equation in the smooth parts.

(2) the discontinuous line is $\xi = -\sqrt{\frac{8\beta(c^2-1)}{4\beta^2-\alpha^2}}\xi_0$.

We will prove that the expression $[u*]\, s = [f]$ holds on discontinuous lines. Here,

$$u* \equiv \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad s = \frac{dx}{dt} = c, \quad f = \begin{bmatrix} 0 \\ w \\ v \end{bmatrix}$$

(i) $[u] = u\,(0^+, t) - u\,(0^-, t) = 0.$

(ii) $v = u_x = u_\xi = \dfrac{\sqrt{\frac{\beta-\alpha}{\alpha+\beta}\sqrt{\frac{\beta^2-\alpha^2}{\beta(c^2-k)}}}\ \ 1-\coth^2\ \sqrt{\frac{\beta^2-\alpha^2}{\beta(c^2-k)}}(x-ct)+\xi_0}{1+\ \sqrt{\frac{\beta-\alpha}{\alpha+\beta}}\coth\ \sqrt{\frac{\beta^2-\alpha^2}{\beta(c^2-k)}}(x-ct)+\xi_0\ \ ^2}$, $\therefore [v] = 0.$

(iii) $w = u_t = -cu_\xi = \dfrac{-c\sqrt{\frac{\beta-\alpha}{\alpha+\beta}\sqrt{\frac{\beta^2-\alpha^2}{\beta(c^2-k)}}}\ \ 1-\coth^2\ \sqrt{\frac{\beta^2-\alpha^2}{\beta(c^2-k)}}(x-ct)+\xi_0}{1+\ \sqrt{\frac{\beta-\alpha}{\alpha+\beta}}\coth\ \sqrt{\frac{\beta^2-\alpha^2}{\beta(c^2-k)}}(x-ct)+\xi_0\ \ ^2}$, $\therefore [w] = 0$

Therefore, $[u*] = 0$, $[f] = 0$. Namely, case(2) holds. We complete the proof of the corollary. ■

We can also prove that the solution of double sine-Gordon equation,

$$u(x,t) = arc\cot\left(-\frac{4\alpha}{\sqrt{c^2-k}}((x-ct)+\xi_0)\right)$$

is also the noncontinuous solution.

In summary, through Tanh method and the variable separation method, we changed the partial differential equation into an easy form of ordinary differential equation. By Maple, we obtained abundant and exact solutions. We also got a new sort of noncontinuous solution and proved that the solution of double sine-Gordon equation is also the noncontinuous solution.

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