

A Class of Geom/Geom/1 Discrete-time Queueing System with Negative Customers

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(Received 18 July 2007, accepted 12 October 2007)

Abstract: Negative arrivals are widely used as a control mechanism in many telecommunication and computer networks. This paper analyses a discrete-time single-server queue with geometrical arrival of both positive and negative customers and they may arrive at the same time, negative customers can be serviced as if positive customers. There are two cases: negative customers remove positive customers from the end of the queue in which the customer currently being serviced can and cannot be killed by a negative customer. When negative customers arrive, there is no customer, negative customers can be serviced. The paper carried out the associated stationary distribution. For example, the probability generating function of the number of customers in the waiting line and the steady-state distribution of the waiting line size.

Keywords: discrete-time queue; negative customers; RCE-immune and immune servicing killing policies

1 Introduction

During the last decade, there has been an increasing interest in queueing systems with negative arrivals. Negative customers systems were introduced by Gelenbe[1]. Many continuous-time queueing models with negative arrivals have been discussed during the last years, but the analysis of discrete-time queueing models has received considerable attention in the scientific literature over the past years in view of its applicability in the study of many computer and communication systems in which time is slotted, for instance ATM (Asynchronous Transfer Mode) and BISDN (Broadband Integrated Services Digital Networks). Negative customers can be serviced in Zhu[2], in base of this, this paper researched a class of Geom/Geom/1 discrete-time queueing system with negative customers.

2 The mathematical model

We consider a discrete-time queueing system where the time axis is divided into equal intervals and all queueing activities take place at the slot boundaries. Two types of customers, positive and negative, arrive according to geometrical arrival processes with probabilities p and q and may arrive at the same time[3]. We will discuss the model for the early arrival system policy[5] and negative customers as the same as positive customers. If a positive customer upon arrival finds the server idle, it immediately enters the service station; otherwise, it removes a negative customer according to a killing policy. Negative customers arrival in the same way. There are two killing policies: The RCE-immune servicing killing policy and the RCE-immune servicing killing policy.

Service times are independent and geometrically distributed with the probabilities $\bar{s}^{\pm} = 1 - s^{\pm}$, where s^{\pm} are the probabilities positive and negative customers do not conclude their services in a slot.

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In order to avoid trivial cases, we assume $0 < p < 1, 0 < q < 1, 0 < \bar{s}^{\pm} < 1$

At time m^+ (the moment immediately after the m th slot), the system can be described by the process $X_m = X_m^+ + X_m^-$, which represents the number of customers in the system, $X_m^+ (\geq 0)$ is the number of positive customers and $X_m^- (\leq 0)$ is the number of negative customers.

It can be readily proven that $\{X_m, m \in N\}$ is the one-dimensional Markov chain of our queueing system and its states space is $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. Our main purpose is to find the stationary distribution $\pi_k = \lim_{m \rightarrow \infty} p[X_m = k] \quad (k = 0, \pm 1, \pm 2, \dots)$ of the Markov chain $\{X_m, m \in N\}$.

In order to resolve the Kolmogorov equations of the distribution π_k , we need to introduce the auxiliary generating function $\Phi(z) = \Phi^+(z) + \Phi^-(z) = \sum_{k=1}^{\infty} \pi_k z^{k-1} + \sum_{k=-\infty}^{-1} \pi_k z^{k-1}, |z| \leq 1$. Let us note that the probability generating function of the system size is given by $\Psi(z) = \pi_0 + z\Phi(z)$. The waiting line size is noted N_q and the system size (all customers) is noted N .

3 The RCE-immune servicing killing policy

We assume the end of queue is killed by the arrival, if there is no customer, the arrival enters the service station.

If $k, k' \geq 0$, the one-step transition probabilities $p_{k',k} = P[X_{m+1} = k | X_m = k']$ are given by the formulae:

$$\begin{aligned} p_{0,0} &= \bar{p} + pq & p_{1,0} &= \bar{s}^+ \bar{p} + \bar{s}^+ pq + s^+ \bar{p}q & p_{2,0} &= \bar{s}^+ \bar{p}q & p_{0,1} &= p\bar{q}, \\ p_{1,1} &= \bar{s}^+ p\bar{q} + s^+ \bar{p}\bar{q} + s^+ pq & p_{2,1} &= \bar{s}^+ \bar{p}\bar{q} + \bar{s}^+ pq + s^+ \bar{p}q & p_{3,1} &= \bar{s}^+ \bar{p}q \\ p_{k-1,k} &= s^+ p\bar{q}k \geq 2 & p_{k,k} &= \bar{s}^+ p\bar{q} + s^+ \bar{p}\bar{q} + s^+ pqk \geq 2 \\ p_{k+1,k} &= \bar{s}^+ \bar{p}\bar{q} + \bar{s}^+ pq + s^+ \bar{p}qk \geq 2 & p_{k+2,k} &= \bar{s}^+ \bar{p}qk \geq 2 \end{aligned}$$

If $k, k' \leq 0$,

$$\begin{aligned} p_{0,0} &= \bar{q} + pqp_{-1,0} = \bar{s}^- \bar{q} + \bar{s}^- pq + s^- p\bar{q}p_{-2,0} = \bar{s}^- p\bar{q}p_{0,-1} = \bar{p}q \\ p_{-1,-1} &= \bar{s}^- \bar{p}\bar{q} + s^- \bar{p}\bar{q} + s^- pqp_{-2,-1} = \bar{s}^- \bar{p}\bar{q} + \bar{s}^- pq + s^- p\bar{q}p_{-3,-1} = \bar{s}^- p\bar{q} \\ p_{k+1,k} &= s^- \bar{p}qk \leq -2p_{k,k} = \bar{s}^- \bar{p}\bar{q} + s^- \bar{p}\bar{q} + s^- pqk \leq -2 \\ p_{k-1,k} &= \bar{s}^- \bar{p}\bar{q} + \bar{s}^- pq + s^- p\bar{q}k \leq -2p_{k-2,k} = \bar{s}^- p\bar{q}k \leq -2 \end{aligned}$$

where $\bar{p} = 1 - p, \bar{q} = 1 - q$.

First we research the case of $k, k' \geq 0$, The Kolmogorov equations for the distribution π_k are

$$\pi_0 = (\bar{p} + pq)\pi_0 + (\bar{s}^+ \bar{p} + \bar{s}^+ pq + s^+ \bar{p}q)\pi_1 + \bar{s}^+ \bar{p}q\pi_2 \quad (1)$$

$$\pi_1 = p\bar{q}\pi_0 + (\bar{s}^+ p\bar{q} + s^+ \bar{p}\bar{q} + s^+ pq)\pi_1 + (\bar{s}^+ \bar{p}\bar{q} + \bar{s}^+ pq + s^+ \bar{p}q)\pi_2 + \bar{s}^+ \bar{p}q\pi_3 \quad (2)$$

$$\pi_k = s^+ p\bar{q}\pi_{k-1} + (\bar{s}^+ p\bar{q} + s^+ \bar{p}\bar{q} + s^+ pq)\pi_k + (\bar{s}^+ \bar{p}\bar{q} + \bar{s}^+ pq + s^+ \bar{p}q)\pi_{k+1} + \bar{s}^+ \bar{p}q\pi_{k+2}, k \geq 2 \quad (3)$$

and the normalization condition is $\sum_{k=0}^{\infty} \pi_k = 1$.

Multiplying Eq.(3) by z^{k-1} and summing over k leads to

$$\begin{aligned} [z^2 - (\bar{s} + sz)(\bar{p} + pz)(q + \bar{q}z)]\Phi^+(z) &= -[s^+ \bar{p}(q + \bar{q}z) + qz(s^+ p + s^+ \bar{p})]z\pi_2 + \\ &\{ (1 - \bar{s}^+ p\bar{q})z^2 - (\bar{s}^+ + sz)[\bar{p}\bar{q}z + q(\bar{p} + pz)]\}\pi_1 - \bar{s}^+ \bar{p}qz^2\pi_3 \end{aligned}$$

Substituting Eqs.(1) and (2) into the above equation yields

$$\begin{aligned} [s^+ p\bar{q}z^2 - (1 - s^+ \bar{p}\bar{q} - s^+ pq - p\bar{q})z - \bar{s}^+ \bar{p}q]\Phi^+(z) &= \\ -\bar{s}^+ \bar{p}qz\pi_2 - [(\bar{s}^+ \bar{p}\bar{q} + \bar{s}^+ q + s^+ \bar{p}q)z + \bar{s}^+ \bar{p}q]\pi_1 \end{aligned} \quad (4)$$

Let us observe that the polynomial in the left-hand side of Eq.(4) has two roots z_1^* and z_2^* , which satisfy the conditions $-1 < z_1^* < 0, z_2^* > 0$,

Letting $z = z_1^*$ in Eq.(4), we get

$$\pi_2 = \frac{(\bar{s}^+ \bar{p} \bar{q} + \bar{s}^+ q + s^+ \bar{p} q) z_1^* + \bar{s}^+ \bar{p} q}{-\bar{s}^+ \bar{p} q z_2^*} \pi_1$$

and substituting this expression into Eq.(4), we obtain

$$\Phi^+(z) = \frac{z_2^* \pi_1}{z_2^* - z} = \pi_1, \quad \sum_{k=0}^{\infty} \left(\frac{z}{z_2^*}\right)^k = \pi_1 + \pi_1 \frac{z}{z_2^*} + \dots = \pi_1 + \pi_2 z + \dots$$

Equalizing the coefficient of z on both sides of this equation, we have $\pi_1 = z_2^* \pi_2 = (z_2^*)^2 \pi_3 = \dots = (z_2^*)^{k-1} \pi_k$, Combining this result together with Eq.(1) gives

$$\pi_1 = \frac{p \bar{q} z_2^* \pi_0}{(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q}$$

and consequently

$$\Phi^+(z) = \frac{p \bar{q} (z_2^*)^2 \pi_0}{(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q} \frac{1}{z_2^* - z}$$

From the normalization condition $\pi_0 + \Phi^+(1) = 1$, we find the unknown constant

$$\pi_0 = \frac{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q] (z_2^* - 1)}{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q] (z_2^* - 1) + p \bar{q} (z_2^*)^2}$$

Then

$$\Phi^+(z) = \frac{p \bar{q} (z_2^*)^2}{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q] (z_2^* - 1) + p \bar{q} (z_2^*)^2} \frac{z_2^* - 1}{z_2^* - z}$$

If $k, k' \leq 0$, we can observe the unknown constant in the same way:

$$\pi_{-1} = \frac{\bar{p} q z_{-2}^* \pi_0}{(\bar{s}^- \bar{q} + \bar{s}^- p q + s^- \bar{p} \bar{q}) z_{-2}^* + \bar{s}^- \bar{p} \bar{q}}$$

$$\Phi^-(z) = \frac{\bar{p} q (z_{-2}^*)^2}{[(\bar{s}^- \bar{q} + \bar{s}^- p q + s^- \bar{p} \bar{q}) z_{-2}^* + \bar{s}^- \bar{p} \bar{q}] (z_{-2}^* - 1) + \bar{p} q (z_{-2}^*)^2} \frac{z_{-2}^* - 1}{z_{-2}^* - z}$$

where z_{-2}^* is one root of the polynomial $s^- \bar{p} \bar{q} z^2 - (1 - s^- \bar{p} \bar{q} - s^- p q - \bar{p} q) z - s^- \bar{p} \bar{q} = 0$.

Corollary 1 The generating function of the stationary distribution of the chain is given by

$$\Psi(z) = \pi_0 + z \Phi(z) = \frac{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q] (z_2^* - z) + p \bar{q} (z_2^*)^2 z}{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q] (z_2^* - 1) + p \bar{q} (z_2^*)^2} \frac{z_2^* - 1}{z_2^* - z} + \frac{\bar{p} q (z_{-2}^*)^2 z}{[(\bar{s}^- \bar{q} + \bar{s}^- p q + s^- \bar{p} \bar{q}) z_{-2}^* + \bar{s}^- \bar{p} \bar{q}] (z_{-2}^* - 1) + \bar{p} q (z_{-2}^*)^2} \frac{z_{-2}^* - 1}{z_{-2}^* - z}$$

Corollary 2 The probability generating function of the number of customers in the waiting line is given by

$$\phi(z) = \pi_0 + \Phi(z) = \frac{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q] (z_2^* - z) + p \bar{q} (z_2^*)^2 z}{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q] (z_2^* - 1) + p \bar{q} (z_2^*)^2} \frac{z_2^* - 1}{z_2^* - z} + \frac{\bar{p} q (z_{-2}^*)^2 z}{[(\bar{s}^- \bar{q} + \bar{s}^- p q + s^- \bar{p} \bar{q}) z_{-2}^* + \bar{s}^- \bar{p} \bar{q}] (z_{-2}^* - 1) + \bar{p} q (z_{-2}^*)^2} \frac{z_{-2}^* - 1}{z_{-2}^* - z}$$

Corollary 3 The steady-state distribution of the waiting line size is given by

$$P[N_q = 0] = \pi_0^+ + \pi_1 + \pi_{-1} = \frac{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q + p \bar{q}) z_2^* + \bar{s}^+ \bar{p} q](z_2^* - 1)}{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q](z_2^* - 1) + p \bar{q} (z_2^*)^2} + \frac{\bar{p} q z_{-2}^* \pi_0}{(\bar{s}^- \bar{q} + \bar{s}^- p q + s^- p \bar{q}) z_{-2}^* + \bar{s}^- p \bar{q}}$$

If $k \geq 1$,

$$P[N_q = k] = \pi_{k+1} = \frac{p \bar{q} (z_2^*)^{1-k} (z_2^* - 1)}{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q](z_2^* - 1) + p \bar{q} (z_2^*)^2}$$

If $k \leq -1$,

$$P[N_q = k] = \pi_{k-1} = \frac{\bar{p} q (z_{-2}^*)^{3-k} (z_{-2}^* - 1)}{[(\bar{s}^- \bar{q} + \bar{s}^- p q + s^- p \bar{q}) z_{-2}^* + \bar{s}^- p \bar{q}](z_{-2}^* - 1) + \bar{p} q (z_{-2}^*)^2}$$

Corollary 4 The steady-state distribution of the system size is given by

$$P[N = 0] = \pi_0 = \frac{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q](z_2^* - 1)}{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q](z_2^* - 1) + p \bar{q} (z_2^*)^2}$$

If $k \geq 1$,

$$P[N = k] = \pi_k = \frac{p \bar{q} (z_2^*)^{2-k} (z_2^* - 1)}{[(\bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q) z_2^* + \bar{s}^+ \bar{p} q](z_2^* - 1) + p \bar{q} (z_2^*)^2}$$

If $k \leq -1$,

$$P[N = k] = \pi_k = \frac{\bar{p} q (z_{-2}^*)^{2-k} (z_{-2}^* - 1)}{[(\bar{s}^- \bar{q} + \bar{s}^- p q + s^- p \bar{q}) z_{-2}^* + \bar{s}^- p \bar{q}](z_{-2}^* - 1) + \bar{p} q (z_{-2}^*)^2}$$

4 The RCE-immune servicing killing policy

The analysis of the preceding section assumes that the last customer in the system will be killed by a arrival whether or not it is actually receiving service at the instant of the negative arrival. In this section, only customers waiting to commence service will be killed, if there is no customers, the arrival accept service.

If $k \geq 0$, The Kolmogorov equations for the distribution π_k are

$$\pi_0 = \bar{p} \pi_0 + \bar{s}^+ \bar{p} \pi_1$$

$$\pi_1 = p \pi_0 + \bar{s}^+ p + s^+ \bar{p} + s^+ p q \pi_1 + \bar{s}^+ \bar{p} + \bar{s}^+ p q + s^+ \bar{p} q \pi_2 + \bar{s}^+ \bar{p} q \pi_3$$

$$\pi_k = s^+ p \bar{q} \pi_{k-1} + \bar{s}^+ p \bar{q} + s^+ \bar{p} \bar{q} + s^+ p q \pi_k + \bar{s}^+ \bar{p} \bar{q} + \bar{s}^+ p q + s^+ \bar{p} q \pi_{k+1} + \bar{s}^+ \bar{p} q \pi_{k+2}, k \geq 2$$

If $k \leq 0$, The Kolmogorov equations for the distribution π_k are

$$\pi_0 = \bar{q} \pi_0 + \bar{s}^- \bar{q} \pi_{-1} \pi_{-1} = q \pi_0 + \bar{s}^- q + s^- \bar{q} + s^- p q \pi_{-1} + \bar{s}^- \bar{q} + \bar{s}^- p q + s^- p \bar{q} \pi_{-2} + \bar{s}^- p \bar{q} \pi_{-3}$$

$$\pi_k = s^- \bar{p} q \pi_{k+1} + \bar{s}^- \bar{p} q + s^- \bar{p} \bar{q} + s^- p q \pi_k + \bar{s}^- \bar{p} \bar{q} + \bar{s}^- p q + s^- p \bar{q} \pi_{k-1} + \bar{s}^- p \bar{q} \pi_{k-2}, k \leq -2$$

The solutions of equations are acquired as the same way as the preceding section.

Corollary 5 The probability generating function of the number of customers in the waiting line is given by

$$\phi(z) = \pi_0 + \Phi(z) = \frac{\bar{s}^+ \bar{p} (z_2^* - z) + p z_2^* z_2^* - 1}{\bar{s}^+ \bar{p} (z_2^* - 1) + p z_2^* z_2^* - z} + \frac{q z_{-2}^*}{\bar{s}^- \bar{q} (z_{-2}^* - 1) + q z_{-2}^* z_{-2}^* - z}$$

Corollary 6 The steady-state distribution of the waiting line size is given by

$$P[N_q = 0] = \pi_0 + \pi_1 + \pi_{-1} = \frac{(\bar{s}^+ \bar{p} + p)(z_2^* - 1)}{\bar{s}^+ \bar{p}(z_2^* - 1) + pz_2^*} + \frac{q(z_{-2}^* - 1)}{\bar{s}^- \bar{q}(z_{-2}^* - 1) + qz_{-2}^*}$$

If $k \geq 1$,

$$P[N_q = k] = \pi_{k+1} = \frac{p(z_2^* - 1)}{\bar{s}^+ \bar{p}(z_2^* - 1) + pz_2^*} \frac{1}{(z_2^*)^k}$$

If $k \leq -1$,

$$P[N_q = k] = \pi_{k-1} = \frac{q(z_{-2}^* - 1)}{\bar{s}^- \bar{q}(z_{-2}^* - 1) + qz_{-2}^*} \frac{1}{(z_{-2}^*)^{k-2}}$$

Corollary 7 The steady-state distribution of the system size is given by

$$P[N = 0] = \pi_0 = \frac{\bar{s}^+ \bar{p}(z_2^* - 1)}{\bar{s}^+ \bar{p}(z_2^* - 1) + pz_2^*}$$

If $k \geq 1$,

$$P[N = k] = \pi_k = \frac{p(z_2^* - 1)}{\bar{s}^+ \bar{p}(z_2^* - 1) + pz_2^*} \frac{1}{(z_2^*)^{k-1}}$$

If $k \leq -1$,

$$P[N = k] = \pi_k = \frac{q(z_{-2}^* - 1)}{\bar{s}^- \bar{q}(z_{-2}^* - 1) + qz_{-2}^*} \frac{1}{(z_{-2}^*)^{k-1}}$$

Corollary 8 The factorial moments of k th order for the waiting line size are given by

$$\alpha_k = E[N_q(N_q - 1)(N_q - 2) \dots (N_q - k + 1)] = \frac{pz_2^*}{\bar{s}^+ \bar{p}(z_2^* - 1) + pz_2^*} \frac{k!}{(z_2^* - 1)^k}, k \geq 1$$

Corollary 9 The factorial moments of k th order for the system size are given by

$$\beta_k = E[N(N - 1)(N - 2) \dots (N - k + 1)] = \frac{p(z_2^*)^2}{\bar{s}^+ \bar{p}(z_2^* - 1) + pz_2^*} \frac{k!}{(z_2^* - 1)^k}, k \geq 1$$

Remark 10 Let us note that $\beta_k = \alpha_k z_2^*$, $k \geq 1$

5 Conclusion

This paper researches a class of Geom/Geom/1 according to two kinds of killing policies and acquires the probability generating function of the number of customers in the waiting line and the generating function of the stationary distribution of the chain. The discrete-time Geom/Geom/1 queue with negative customers and disasters may be discussed in the future.

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