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The Quantization Dimension and other Dimensions of Probability Measures

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Abstract: In this paper, we study the lower and the upper quantization dimensions of probability measures and their equivalent definitions. Then we prove that the upper quantization dimension is always between the packing and the upper box-counting dimension, whereas the lower quantization dimension is between the Hausdorff and the lower box-counting dimension. Finally, we give a concrete probability measure which all of its quantization dimension and other dimensions agree.

Keywords: quantization dimension; Hausdorff dimension; packing dimension; box-counting dimension; self-similar set

1 Introduction

Many authors[1-3] have studied fractal properties of measures. In this paper, we study the quantization dimension of probability measures. The quantization problem consists in studying the quantization error induced by the approximation of a given probability measure with discrete probability measures of finite supports. This problem originated in information theory and some engineering technology. Its history goes back to the 1940s[4]. Graf and Luschgy studied this problem systematically and gave a general mathematical treatment of it[5]. Two important objects in the quantization theory are the quantization coefficient and the quantization dimension. Let μ be a probability measure on $\mathbb{R}^d$ and let $0 < l < \infty$. The n th quantization error of μ of order l is defined by

$$V_{n,l}(\mu) = \inf\{\int \min_{a\in\alpha} \|x-a\|^l d\mu(x) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \le n\}.$$

We define the upper and lower quantization dimension of μ of order l , $\underline{\dim}_Q^l(\mu)$ and $\overline{\dim}_Q^l(\mu)$ as the unique real numbers $\underline{s}$ and $\overline{s} \in [0,\infty]$ respectively, satisfying

$$\liminf_{n\to\infty} n^{\frac{l}{s}} V_{n,l}(\mu) = \begin{cases} \infty, & \text{for} \quad s < \underline{s}\,, \\ 0, & \text{for} \quad s > \underline{s}\,, \end{cases}$$

$$\limsup_{n\to\infty} n^{\frac{l}{s}} V_{n,l}(\mu) = \begin{cases} \infty, & \text{for} \quad s < \overline{s}\,, \\ 0, & \text{for} \quad s > \overline{s}\,. \end{cases}$$

If $\underline{\dim}_Q^l(\mu) = \overline{\dim}_Q^l(\mu)$ we call the common value the quantization dimension of μ of order l and denote it by $\dim_Q^l(\mu)$. The problem of determining the quantization dimension function for a general probability is open. However, Graf and Luschgy have determined a formula for an iterated function system using a finite number of contracting similarities $\{f_1, \cdots, f_N\}$ on $\mathbb{R}^d$ satisfying the OSC (abbreviation Open Set Condition) and give a probability vector $\{\mu_1, \cdots, \mu_N\}$. The measure μ satisfies:

$$\mu = \Sigma_{i=1}^N \mu_i \mu \circ f_i^{-1}.$$

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They show that $D_l := \dim_Q^l(\mu)$ satisfies

$$\sum_{i=1}^{N}(\mu_i c_i^l)^{\frac{D_l}{l+D_l}} = 1 \,, \tag{1}$$

where c_i is the contraction coefficient for the map f_i[6]. The above result was extended by Lindsay and Mauldin to the F-conformal measures associated with finitely many conformal maps[7]. Zhu Sanguo extended this result to certain Cantor-like sets under a hereditary condition[8]. P ötzelberger [9] has shown that for distributions with bounded support

$$\dim_H^*(\mu) \le \underline{\dim}_Q^2(\mu) \le \underline{\dim}_B^*(\mu),$$

$$\dim_P^*(\mu) \le \overline{\dim}_Q^2(\mu) \le \overline{\dim}_B^*(\mu),$$

where $\dim_H^*(\mu)$, $\dim_P^*(\mu)$, $\underline{\dim}_B^*(\mu)$, $\overline{\dim}_B^*(\mu)$ denote the Hausdorff, packing, lower and upper box-counting dimension of μ, respectively. In this paper, we mainly study the quantization dimension's equivalent definition of order l and compare it to other dimensions of probability measures μ .

2 Definitions and notations

Let a finite Borel measure ν on $\mathbb{R}^d$ and a set E be given. $\nu_{|E}$ and $\nu_{\upharpoonright E}$ denote the restriction of ν to E and the conditional distribution of ν given E, i.e. $\nu_{|E}(A) = \nu(A \cap E)$ and for $0 < \nu(E) < \infty$, $\nu_{\upharpoonright E}(A) = \nu(A \cap E)/\nu(E)$. $B(x, r)$ denotes the open ball with center x and radius r.

Let μ and ν be finite measures on $\mathbb{R}^d$. We say that μ is absolutely continuous with respect to ν if for each set $A \subset \mathbb{R}^d$, $\mu(A) = 0$ implies $\nu(A) = 0$, which is denoted by $\nu \ll \mu$.

The upper Hausdorff dimension $\dim_H^*(\mu)$ and the upper packing dimension $\dim_P^*(\mu)$ of probability measure μ are defined by

$$\dim_H^*(\mu) = \inf\{\dim_H(E)|\ E \subseteq \mathbb{R}^d,\ \text{Borel and}\ \mu(E) = 1\},$$

$$\dim_P^*(\mu) = \inf\{\dim_P(E)|\ E \subseteq \mathbb{R}^d,\ \text{Borel and}\ \mu(E) = 1\}.$$

Analogously we define the lower and the upper box-counting dimension by

$$\underline{\dim}_B^*(\mu) = \inf\{\underline{\dim}_B(E)|E \subseteq \mathbb{R}^d,\ \text{Borel and}\ \mu(E) = 1\}, \tag{2}$$

$$\overline{\dim}_B^*(\mu) = \inf\{\overline{\dim}_B(E)|E \subseteq \mathbb{R}^d,\ \text{Borel and}\ \mu(E) = 1\}. \tag{3}$$

Note that for $s = \dim_H^*(\mu)$ (and analogously for $\dim_P^*(\mu)$, $\overline{\dim}_B^*(\mu)$ and $\underline{\dim}_B^*(\mu)$) a Borel set E exists with $\mu(E) = 1$ and $\dim_H(E) = s$. Upper and lower local dimensions provide an alternative definition of $\dim_H^*(\mu)$ and $\dim_P^*(\mu)$[5],

$$\dim_H^*(\mu) = \inf\{s|\underline{\dim}_{\text{loc}}(\mu, x) \le s \quad \mu - a.e.\},$$

$$\dim_P^*(\mu) = \inf\{s|\overline{\dim}_{\text{loc}}(\mu, x) \le s \quad \mu - a.e.\}. \tag{4}$$

3 Main results

Lemma 3.1 [10, 11]*If for all* $\alpha > s$ *sets* E_i *exist, such that* $E = \cup_{i=1}^{\infty} E_i$ *and* $\overline{\dim}_B(E_i) < \alpha$*, then* $\dim_P(E) \le s$.

Lemma 3.2 [10, 11] $A \subseteq \{x|\overline{\dim}_{loc}(\mu, x) \ge \alpha\}$ *and* $\mu(A) > 0$ *implies* $\dim_P(A) \ge \alpha$.

Theorem 3.3 *Equivalent definitions of the lower and the upper quantization dimensions are as following.*

$$\underline{\dim}^l_Q(\mu) = \liminf_{n\to\infty} \frac{l\log n}{-\log V_{n,l}(\mu)},$$

$$\overline{\dim}^l_Q(\mu) = \limsup_{n\to\infty} \frac{l\log n}{-\log V_{n,l}(\mu)}.$$

Proof. Let $\underline{s} = \liminf_{n\to\infty} \frac{l\log n}{-\log V_{n,l}(\mu)}$. For $s > \underline{s}$, consider $s^* \in (\underline{s}\ ,\ s)$. We choose a sequence $n_k \to \infty, k \to \infty$, such that

$$\lim_{k\to\infty} \frac{l\log n_k}{-\log V_{n_k,l}(\mu)} = \underline{s} < s^*.$$

Then $\exists k_0 \in \mathbb{N},\ \ \forall k > k_0$, such that

$$\frac{l\log n_k}{-\log V_{n_k,l}(\mu)} < s^* \Longleftrightarrow \log n_k^l < \log\big(\frac{1}{V_{n_k,l}(\mu)}\big)^{s^*} \Longleftrightarrow n_k^l (V_{n_k,l}(\mu))^{s^*} < 1.$$

Thus

$$\lim_{k\to\infty} n_k^{\frac{l}{s^*}} V_{n_k,l}(\mu) \leq 1,$$

and

$$\liminf_{n\to\infty} n^{\frac{l}{s}} V_{n,l}(\mu) \leq \lim_{k\to\infty} n_k^{\frac{l}{s}} V_{n_k,l}(\mu) = \lim_{k\to\infty} n_k^{\frac{l}{s^*}} V_{n_k,l}(\mu) n_k^{\frac{l(s^*-s)}{ss^*}} = 0\ .$$

For $s < \underline{s}$, consider $s^* \in (s\ ,\ \underline{s})$ and determine a number $N(s^*)$, such that for any $n > N(s^*)$, the following inequality is true.

$$s^* < \frac{l\log n}{-\log V_{n,l}(\mu)} \Longleftrightarrow 1 < n^l (V_{n,l}(\mu))^{s^*}$$

Then

$$\lim_{n\to\infty} n^{\frac{l}{s^*}} V_{n,l}(\mu) \geq 1$$

and

$$\liminf_{n\to\infty} n^{\frac{l}{s}} V_{n,l}(\mu) = \lim_{n\to\infty} n^{\frac{l}{s^*}} V_{n,l}(\mu) n^{\frac{l(s^*-s)}{ss^*}} = \infty.$$

Therefore

$$\underline{\dim}^l_Q(\mu) = \underline{s} = \liminf_{n\to\infty} \frac{l\log n}{-\log V_{n,l}(\mu)}.$$

$\overline{\dim}^l_Q(\mu) = \limsup_{n\to\infty} \frac{l\log n}{-\log V_{n,l}(\mu)}$ can be proved the same way. ■

Theorem 3.4 *Let μ be a probability distribution on $[0,1]^d$. Then*

(i) $\dim^*_H(\mu) \leq \underline{\dim}^l_Q(\mu) \leq \underline{\dim}^*_B(\mu)$.

(ii) $\dim^*_P(\mu) \leq \overline{\dim}^l_Q(\mu) \leq \overline{\dim}^*_B(\mu)$.

(iii)If $\dim^*_H(\mu) = \overline{\dim}^*_B(\mu)$, *then* $\underline{\dim}^l_Q(\mu) = \overline{\dim}^l_Q(\mu) = \dim^*_H(\mu)$.

(iv) Let $E \in [0,1]^d$ be a Borel s-set, i.e. $0 < \mathcal{H}^s(E) < \infty$, and $\mathcal{H}^s_{|E} \ll \mu$. Then

$$\liminf_{n\to\infty} n^{\frac{l}{s}} V_{n,l}(\mu) > 0.$$

(v) Let $E \subseteq [0,1]^d$ be a Borel set of exact upper box-counting dimensions with $\mu(E) = 1$. Then

$$\limsup_{n\to\infty} n^{\frac{l}{s}} V_{n,l}(\mu) < \infty.$$

Proof. (i) First, we prove $\underline{\dim}_Q^l(\mu) \leq \underline{\dim}_B^*(\mu)$. It is sufficient to show $V_{n,l}(\mu) = O(\frac{1}{n^{l/\alpha}})$ for all $\alpha > \underline{\dim}_B^*(\mu)$, which implies $\liminf_{n\to\infty} n^{\frac{l}{s}} V_{n,l}(\mu) = 0$ for $\alpha > \underline{\dim}_B^*(\mu)$. Let $s = \underline{\dim}_B^*(\mu)$ and $\alpha > s$. A Borel set E exists with $\mu(E) = 1$ and $\underline{\dim}_B(E) < (s+\alpha)/2$. Let $r_0 > 0$ such that $N_K(E,r) \leq r^{-\alpha}$ for all $r \in (0, r_0]$. $N_K(E,r)$ denotes the covering number of E by r-meshed cubes, i.e. the minimal number of r-meshed cubes required to cover E. Let $n \geq r_0^{-\alpha}$ and $r = n^{-1/\alpha}$. Note that $\widetilde{n} := N_K(E,r) \leq n$. Let $\{S_i | i \leq \widetilde{n}\}$ be a cover of E with r-meshed cubes. $O_{\widetilde{n}} = \{a_1, \cdots, a_{\widetilde{n}}\}$ denotes the set of centers of cubes. $x \in S_i$ implies $\|x - a_i\|^l \leq (\frac{\sqrt{d}}{2} r)^l$ and

$$V_{n,l}(\mu) \leq V_{\widetilde{n},l}(\mu) \leq \sum_{i=1}^{\widetilde{n}} \mu(S_i)(\frac{\sqrt{d}}{2} r)^l = n^{-l/\alpha}(\frac{\sqrt{d}}{2})^l,$$

so $\underline{\dim}_Q^l(\mu) \leq \alpha$. By the arbitrariness of α, we have $\underline{\dim}_Q^l(\mu) \leq \underline{\dim}_B^*(\mu)$.

$\overline{\dim}_Q^l(\mu) \leq \overline{\dim}_B^*(\mu)$ can be proved the same way.

Next, we prove $\dim_H^*(\mu) \leq \underline{\dim}_Q^l(\mu)$. Let $s = \dim_H^*(\mu)$ and $\alpha < s$. A Borel set E exists with $\mu(E) > 1$ and $\dim_H(E) > \alpha$. (4) implies that for μ-a.e. $x \in E$ and $r_x > 0$ exists, such that for all convex sets U, diam$(U) \leq r_x$ and $x \in U$ imply $\mu(U) \leq$ diam$(U)^\alpha$. But then a Borel set $E_0 \subseteq E$ with $\mu(E_0) > 0$ and r_0 exist, such that for $x \in E_0$ and all convex sets U with diam$(U) \leq r_0$ and $x \in U$, $\mu(U) \leq$ diam$(U)^\alpha$ holds. Let $B_1, \cdots, B_n$ be an optimal partition for $\mu_0 := \mu_{\restriction E_0}$ with conditional means $a_1, \cdots, a_n$. Let $2r_i = r_0 \wedge (\mu(B_i \cap E_0)/2)^{1/\alpha}$, $J_1 = \{i | r_i = \mu(B_i \cap E_0)/2)^{1/\alpha}/2\}$ and $J_2 = \{i | r_i = r_0/2\}$. Then

$$\begin{aligned}
\mu_0(B_i \setminus B(a_i, r_i)) \quad & \geq \frac{1}{\mu(E_0)}(\mu(B_i \cap E_0) - \mu(B(a_i, r_i) \cap E_0)) \\
& \geq \frac{1}{\mu(E_0)}(\mu(B_i \cap E_0) - (2r_i)^\alpha) \\
& \geq \frac{1}{\mu(E_0)}(\mu(B_i \cap E_0) - \mu(B_i \cap E_0)/2) \\
& = \frac{\mu_0(B_i)}{2}.
\end{aligned}$$

Thus

$$\begin{aligned}
n^{l/\alpha} V_{n,l}(\mu) \quad & \geq \mu(E_0) n^{l/\alpha} V_{n,l}(\mu_0) \\
& = \mu(E_0) n^{l/\alpha} \sum_{i=1}^n \int_{B_i} \|x - a_i\|^l d\mu_0(x) \\
& \geq \mu(E_0) n^{l/\alpha} \sum_{i=1}^n \int_{B_i \setminus B(a_i, r_i)} \|x - a_i\|^l d\mu_0(x) \\
& \geq \mu(E_0) n^{l/\alpha} \sum_{i=1}^n r_i^l \mu_0(B_i \setminus B(a_i, r_i)) \\
& \geq \frac{\mu(E_0) n^{l/\alpha} r_0^l \mu_0(\cup_{i \in J_2} B_i)}{2 \times 2^l} + \frac{\mu(E_0)^{1+l/\alpha} n^{l/\alpha}}{2^{1+l+l/\alpha}} \sum_{i \in J_1} \mu_0(B_i)^{1+l/\alpha} .
\end{aligned}$$

If $\mu_0(\cup_{i \in J_2}) \geq \frac{1}{2}$, the first summand is at least $\frac{\mu(E_0) r_0^l}{4 \times 2^l}$. Otherwise, the second summand satisfies:

$$\frac{\mu(E_0)^{1+l/\alpha} n^{l/\alpha} \mu_0(\cup_{j \in J_1} B_j)^{1+l/\alpha}}{2^{1+l+l/\alpha}} \sum_{i \in J_1} (\frac{\mu_0(B_i)}{\mu_0(\cup_{j \in J_1} B_j)})^{1+l/\alpha}$$

$$\geq \frac{\mu(E_0)^{1+l/\alpha}}{2^{2+l+2l/\alpha}} (\frac{n}{|J_1|})^{l/\alpha} \geq \frac{\mu(E_0)^{1+l/\alpha}}{2^{2+l+2l/\alpha}} .$$

This concludes the proof of $\dim_H^*(\mu) \leq \underline{\dim}_Q^l(\mu)$.

(ii) To prove $\dim_P^*(\mu) \leq \overline{\dim}_Q^l(\mu)$, let $\alpha_0 = \overline{\dim}_Q^l(\mu)$ and $\alpha_1 = \dim_P^*(\mu)$. We assume $\alpha_0 < \alpha_1$ and deduce a contradiction. The assumption implies that for all $\alpha \in (\alpha_0, \alpha_1)$ and every sequence n_i with $n_i \to \infty$,

$$\lim_{i\to\infty} n_i^{l/\alpha} V_{n,l}(\mu) = 0.$$

Let $\beta > 0$ with $\alpha_0 < \frac{l(\alpha_1-\beta)}{l+\beta}$. An $\alpha_2 > \alpha_1$ exists, such that

$$\alpha_0 < \frac{l(\alpha_1 - \beta)}{l + \beta + \alpha_2 - \alpha_1} =: \gamma\ .$$

We denote $E_0 = \{x|\overline{\dim}_{loc}(\mu, x) > \alpha_1 - \beta/2\}$. The definition of $\dim_P^*$ implies $\mu(E_0) > 0$ and according to Lemma 3.2, $\dim_P(E_0) \geq \alpha_1 - \beta/2$ holds. For $x \in E_0$ and $\widetilde{r}_x > 0$ exists, such that for all cubes S with $x \in S$ of side-length at most $|S| \leq \widetilde{r}_x, \mu(S) \geq |S|^{\alpha_2}$ holds. Let $\{r_i\}$ be a sequence of positive numbers with $r_i \downarrow 0$ and $E_i = \{x \in E_0|\widetilde{r}_x \geq r_i\}$. Then $E_0 = \cup E_i$, we will prove that $\overline{\dim}_B(E_i) \leq \alpha_1 - \beta$. Lemma 3.1 implies $\dim_P(E_0) \leq \alpha_1 - \beta < \alpha_1 - \beta/2$. There are contradiction to above discussion, so $\dim_P^*(\mu) \leq \overline{\dim}_Q^l(\mu)$.

Fix i and choose an arbitrary positive and decreasing null-sequence $\{\delta_j\}$ with $\delta_1 < r_i$. Let

$$n_j = [\frac{1}{3^d 2} N_K(E_i, \delta_j)].$$

$\{n_j\}$ is unbounded. Let $O = \{a_1, \cdots, a_j\}$ be a set of n_j prototypes in $[0, 1]^d$. The $N_K(E_i, \delta_j)$ cubes, which have nonempty intersection with E_i consist of

(1) at most $n_j 3^d$ cubes that contain a point a_k, or which are neighboring such a cube,

(2) the remaining cubes.Their distance to the nearest prototype is at least δ_j.There are at least $N_K(E_i, \delta_j) - n_j 3^d$ such cubes. Let $\widetilde{E}$ denote the union of these cubes. Then

$$\mu(\widetilde{E}) \geq (N_K(E_i, \delta_j) - n_j 3^d)\delta_j^{\alpha_2} \geq \frac{\delta_j^{\alpha_2}}{2} N_K(E_i, \delta_j).$$

Let $\alpha \in (\alpha_0, r)$. Since $\alpha > \alpha_0 = \overline{\dim}_Q^l(\mu)$ a j_0 exists with $n_j^{\frac{l}{\alpha}} V_{n_j,l}(\mu) \leq 1$ for $j \geq j_0$. But then, for $j \geq j_0 (c_1, c_2$ are constants),

$$\begin{aligned} 1 \quad &\geq n_j^{\frac{l}{\alpha}} V_{n_j,l}(\mu) \geq n_j^{\frac{l}{\alpha}} \textstyle\int_{\tilde{E}} \min_{a\in O} \|x - a\|^l d\mu(x) \quad \geq n_j^{\frac{l}{\alpha}} \delta_j^l \frac{\delta_j^{\alpha_2}}{2} N_K(E_i, \delta_j) \\ &\geq c_1 N_K(E_i, \delta_j)^{1+\frac{l}{\alpha}} \delta_j^{l+\alpha_2}. \end{aligned}$$

Thus

$$N_K(E_i, \delta_j) \leq c_2 \delta_j^{-(l+\alpha_2)\frac{\alpha}{\alpha+l}}.$$

The sequence δ_j being arbitrary, implies

$$\begin{aligned} \overline{\dim}_B(E_i) \quad &\leq \frac{(l + \alpha_2)\alpha}{l + \alpha} \leq \frac{(l + \alpha_2)\gamma}{l + \gamma} \\ &= \frac{(l + \alpha_2)\frac{l(\alpha_1-\beta)}{l+\beta+\alpha_2-\alpha_1}}{l + \frac{l(\alpha_1-\beta)}{l+\beta+\alpha_2-\alpha_1}} \\ &= \frac{l(l + \alpha_2)(\alpha_1 - \beta)}{l^2 + l\alpha_2} \\ &= \alpha_1 - \beta. \end{aligned}$$

(iii) By (i) and (ii), we easily have (iii).

(iv) Let $E_0 \subseteq E$ be a Borel set which satisfies $\mu(E_0) > 0$, then there exist three positive constants $c_1,\ c_2,\ r_0$ with $r_0 < 1$, such that $\mu(x) \geq c_1 \mathcal{H}^s_{|E}(x)$ for $x \in E_0$ from $\mathcal{H}^s_{|E} \ll \mu$. Because of $0 < \mathcal{H}^s(E) <$

∞, we could have $\mathcal{H}^s(U \cap E) \le c_2|U|^s$ for $x \in E_0$ and all convex sets U with $x \in U$ and diam$(U) \le r_0$. In particular, for $x \in E_0$ and convex sets U with $x \in U$ of diameter at most r_0,

$$\mathcal{H}^s(U \cap E_0) \le c_2|U|^s.$$

We may choose $r_0 < (\frac{\mathcal{H}(E_0)}{c_2})^{\frac{1}{s}}$. Again, let $(B_1, \cdots, B_r)$ be an optimal partition for $\mu_0 := \mu_{\upharpoonright} E_0$ with prototypes $a_1, \cdots, a_n$. We define

$$c_3 = 1 - \frac{c_2 r_0^s}{\mathcal{H}(E_0)}$$

and

$$r_i = \big(\mathcal{H}^s(B_i \cap E_0)\frac{1-c_3}{c_2}\big)^{1/s}.$$

Then $2r_i \le r_0, c_3 > 0$ and

$$\begin{aligned}
\mu(E_0)\mu_0(B_i \setminus B(a_i, r_i)) \quad &\ge c_1\mathcal{H}^s((B_i \setminus B(a_i, r_i)) \cap E_0) \\
&\ge c_1(\mathcal{H}^s(B_i \cap E_0) - \mathcal{H}^s(B(a_i, r_i) \cap E_0) \\
&\ge c_1(\mathcal{H}^s(B_i \cap E_0) - c_2(2r_i)^s) \\
&= c_1(\mathcal{H}^s(B_i \cap E_0) - (1-c_3)\mathcal{H}^s(B_i \cap E_0) \\
&= c_1 c_3 \mathcal{H}^s(B_i \cap E_0).
\end{aligned}$$

Thus

$$\begin{aligned}
n^{l/s}V_{n,l}(\mu) \quad &\ge n^{l/s}\mu(E_0)V_{n,l}(\mu) \\
&= n^{l/s}\mu(E_0)\sum_{i=1}^n \int_{B_i} \|x - a_i\|^l d\mu_0(x) \\
&\ge n^{l/s}\mu(E_0)\sum_{i=1}^n \int_{B_i \setminus B(p_i, r_i)} r_i^l d\mu_0(x) \\
&\ge n^{l/s}\sum_{i=1}^n r_i^l c_1 c_3 \mathcal{H}^s(B_i \cap E_0) \\
&\ge n^{l/s}\frac{c_1c_3}{2^l}\big(\frac{1-c_3}{c_2}\big)^{l/s}\sum_{i=1}^n \mathcal{H}^s(B_i \cap E_0)^{1+l/s} \\
&=: c_4 n^{l/s}\sum_{i=1}^n \mathcal{H}^s(B_i \cap E_0)^{1+l/s} \\
&= c_4\Big[n^{\frac{l}{s}\cdot\frac{s}{s+l}}\big(\sum_{i=1}^n \mathcal{H}^s(B_i \cap E_0)^{\frac{s+l}{s}}\big)^{\frac{s}{s+l}}\Big]^{\frac{s+l}{s}} \\
&= c_4\Big[\big(\sum_{i=1}^n 1^{\frac{s+l}{l}}\big)^{\frac{l}{s+l}}\big(\sum_{i=1}^n \mathcal{H}^s(B_i \cap E_0)^{\frac{s+l}{s}}\big)^{\frac{s}{s+l}}\Big]^{\frac{s+l}{s}} \\
&\ge c_4\Big[\sum_{i=1}^n 1 \cdot \mathcal{H}^s(B_i \cap E_0)\Big]^{\frac{s+l}{s}} \quad \text{(using Hölder's inequality)} \\
&= c_4\mathcal{H}^s(E_0)^{1+l/s}.
\end{aligned}$$

(v) If E has exact upper box-counting dimension s, then $\overline{\dim}_B(E) = s$ and a constant C exists such that $N_K(E, r) \le Cr^{-s}$. In that case, we choose $r = (C/n)^{1/s}$, which leads to

$$V_{n,l}(\mu) \le \sum_{i=1}^{\widetilde{n}} \mu(S_i)\big(\frac{\sqrt{d}}{2}r\big)^l = \big(\frac{\sqrt{d}}{2}\big)^l\big(\frac{C}{n}\big)^{l/s},$$

where $\widetilde{n}$ and S_i denote in (i).

This finishes the proof of (v). ■

Example 3.5 *A self-similar set E satisfy OSC. The self-similar measure associated with $\{f_1, \cdots, f_N\}$ and a given probability vector $\{\mu_1, \cdots, \mu_N\}$ is the unique Borel probability measure satisfying $\mu = \Sigma_{i=1}^N \mu_i \mu \circ f_i^{-1}$. $c_1, \cdots, c_N$ are contraction ratios of $f_1, \cdots, f_N$. If $c_1 = \cdots = c_N = \frac{1}{m}$ ($m > 2, m \in \mathbb{R}$) and $\mu_1 = \cdots = \mu_N = \frac{1}{N}$. We have known*

$$\dim_H^*(\mu) = \frac{\sum_{i=1}^N \mu_i \log \mu_i}{\sum_{i=1}^N \mu_i \log c_i}.[12]$$

so $\dim_H^*(\mu) = \log_m N$. We also easy have $\dim_H(E) = \dim_B(E) = \log_m N$. From Theorem 3.4 , (2) and (3) we have

$$\dim_Q^l(\mu) = \underline{\dim}_Q^l(\mu) = \overline{\dim}_Q^l(\mu) = \dim_H^*(\mu).$$

In fact, $\dim_Q^l(\mu) = D_l$. From (1) we could have

$$\begin{aligned} N \times (\frac{1}{N} \times (\frac{1}{m})^l)^{\frac{D_l}{D_l+l}} = 1 \qquad & \Longrightarrow \frac{D_l}{D_l+l} \log(\frac{1}{N} \times (\frac{1}{m})^l) = \log \frac{1}{N} \\ & \Longrightarrow \frac{D_l}{D_l+l} = \log_{\frac{1}{N} \times (\frac{1}{m})^l} \frac{1}{N} \\ & \Longrightarrow D_l(1 - \log_{\frac{1}{N} \times (\frac{1}{m})^l} \frac{1}{N}) = l \log_{\frac{1}{N} \times (\frac{1}{m})^l} \frac{1}{N} \\ & \Longrightarrow D_l \log_{\frac{1}{N} \times (\frac{1}{m})^l} (\frac{1}{m})^l = l \log_{\frac{1}{N} \times (\frac{1}{m})^l} \frac{1}{N} \\ & \Longrightarrow D_l = \log_{(\frac{1}{m})^l} (\frac{1}{N})^l. \end{aligned}$$

So $\dim_Q^l(\mu) = D_l = \log_m N$.

In this case, from above discussion we have

$$\dim_H^*(\mu) = \dim_Q^l(\mu) = \dim_H(E) = \dim_B(E) = \log_m N.$$

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References

[1] Dehua Liu, Meifeng Dai: The Hausdorff measure of the attactor of an iterated function system with parameter. *International Journal of Nonlinear Science*. 3(2):150-154(2007)

[2] Ying Jiang, Meifeng Dai: Image measures and statical mechanical characterization for their dimensions. *International Journal of Nonlinear Science*. 3(1):32-39(2007)

[3] Ying Jiang, Meifeng Dai: Properties of distribution class and spectral class for a self-similar set. *International Journal of Nonlinear Science*. 4(3):186-192(2007)

[4] J.A. Bucklew, G.L. Wise: Multidimensional asymptotic quantization with rth power distortion measures. *IEEE Trans. Inform. Theory*. 28:239-247(1982)

[5] S. Graf, H. Luschgy: Foundations of quantization for probability distributions. *Lecture Notes in Math.(Springer-Verlag)*1730(2000)

[6] S. Graf, H. Luschgy: The quantization dimension of self-similar probabilities. *Math. Nachr*. 241:103-109(2002)

[7] L.J. Lindsay, R.D. Mauldin: Quantization dimension for conformal function system. *Nonlinearity* 15(1):189-199(2002)

[8] Sanguo. Zhu: Quantization dimension of probability measures supported on Cantor-like sets. *Journal of Mathematical Analysis and Applications*.338:742-750(2008)

[9] Pötzelberger Klaus: The quantization dimension of distributions. *Math.Proc. Cambridge philos*. Soc 131:507-519(2001)

[10] K. J. Falconer: Techniques in fractal geometry. *John Wiley.*(1997)

[11] C. Tricot: Two definitions of fractal dimensions. *Math. Proc. Phil. Soc.*. 91:57-74(1982)

[12] Ercai. Chen, K$\ddot{u}$pper. Tassilo: The pointwise dimension of self-similar measures in complete metric spaces. *Chaos, Solitons and Fractals*. 22:641-651(2004)