

## Reaction-Diffusion Systems and Stability Properties of Positive Solutions

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**Abstract:** This paper deals with the stability of positive solutions to the systems of the form

$$\begin{cases} -\Delta u_i(x) = f_i(u_1(x), u_2(x), \dots, u_n(x)) & x \in \Omega, \\ Bu_i(x) = 0 & x \in \partial\Omega, \end{cases}$$

where  $\Omega \subset R^N (N \geq 1)$  is a bounded domain with a smooth boundary  $Bz(x) = \alpha h(x)z + (1 - \alpha) \frac{\partial z}{\partial n}$  where  $\alpha \in [0, 1], h : \partial\Omega \rightarrow R^+$  with  $h = 1$  when  $\alpha = 1$ , and  $f_i : [0, +\infty)^n \rightarrow R$  are  $C^1$  functions for  $i = 1, \dots, n$ . In particular, we establish conditions for stability/instability when the system is cooperative and strictly coupled ( $\frac{\partial f_i}{\partial u_j} \geq 0, i \neq j, \sum_{j=1, j \neq i}^n (\frac{\partial f_i}{\partial u_j})^2 > 0$ ) and strictly coupled competitive ( $\frac{\partial f_i}{\partial u_j} \leq 0, i \neq j$ ). Then we apply our results to various examples.

**Keywords:** reaction-diffusion systems; stability properties; positive solutions  
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### 1 Introduction

We discuss stability of positive solutions  $u := (u_1, \dots, u_n)$  ( $u_i \geq 0$  and  $\sum_{i=1}^n u_i^2 > 0$ ) of systems of the form

$$-\Delta u_i(x) = f_i(u_1(x), u_2(x), \dots, u_n(x)) \quad x \in \Omega, \quad (1)$$

$$Bu_i(x) = 0 \quad x \in \partial\Omega, \quad (2)$$

for  $i = 1, \dots, n$ , where  $\Omega$  is a bounded region in  $R^N; N \geq 1$  with smooth boundary

$$Bz(x) = \alpha h(x)z + (1 - \alpha) \frac{\partial z}{\partial n}, \quad (3)$$

here  $B$  is a boundary operator and  $(\frac{\partial}{\partial n})$  denotes the outward conormal derivative,  $\alpha \in [0, 1], h : \partial\Omega \rightarrow R^+$  with  $h = 1$  when  $\alpha = 1$ , i.e; the boundary condition may be of Dirichlet ( $u = 0$  on  $\partial\Omega$ ), Neumann ( $\frac{\partial z}{\partial n} = 0$  on  $\partial\Omega$ ) or mixed type (robin boundary condition (for  $\alpha \neq 0, 1$ )). For  $i=1, \dots, n, f_i : [0, +\infty)^n \rightarrow R$  are  $C^1$  functions satisfying either

$$\frac{\partial f_i}{\partial u_j} \geq 0, i \neq j, j = 1, \dots, n, \sum_{j=1, j \neq i}^n (\frac{\partial f_i}{\partial u_j})^2 > 0 \quad (4)$$

(strictly coupled cooperative system) or

$$\frac{\partial f_i}{\partial u_j} \leq 0, i \neq j, j = 1, 2, n = 2 \quad (5)$$

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(strictly coupled competitive system).

One can think of a solution  $u := (u_1, \dots, u_n)$  of (1)-(2) as an equilibrium solution to the parabolic problem

$$\begin{cases} (v_i)_t - \Delta v_i = f_i(v_1, v_2, \dots, v_n) & \text{in } (0, \infty) \times \Omega, \\ Bv_i(t, x) = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ v_i(0, x) = k_i(x) & \text{in } \Omega, \end{cases} \quad (6)$$

for  $i=1, \dots, n$ . Hence, denoting by  $k(x) := (k_1(x), \dots, k_n(x))$  and the corresponding solution of (6) by  $v(t, x) := (v_1(t, x), \dots, v_n(t, x))$ , we say a positive solution  $u$  of (1)-(2) is stable (in the maximum norm) if for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that  $\|k - u\|_\infty < \delta$  implies  $\|v(t, x) - u\|_\infty < \epsilon$  for all  $t > 0$ . We say  $u$  is unstable if it is not stable.

As a reminder, if  $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_n)$  be any positive solution to system (1)-(2), then the linearized equation about  $\tilde{u}$  consists of

$$-\Delta(\phi_i(x)) - \sum_{j=1}^n \frac{\partial f_i}{\partial u_j}(\tilde{u})\phi_j(x) = \mu\phi_i(x), \quad x \in \Omega, \quad (7)$$

$$B\phi_i(x) = 0, \quad x \in \partial\Omega, \quad (8)$$

for  $i=1, \dots, n$ , where Equation (7) obtained from the formal derivative of the operator  $\Delta$ .

It follows from proposition (3.1) of [6] that there is a unique eigenvalue  $\mu_1$  with strictly positive eigenfunction  $\phi = (\phi_1, \dots, \phi_n)$  of (7)-(8) (i.e.,  $\phi_i > 0$  in  $\Omega$  for every  $i=1, \dots, n$ ). By proof same the proof of Theorem 1.1 and 1.2 of [3] for cooperative systems, one can show that if  $\mu_1 < 0$  then positive solution  $\tilde{u}$  of (1)-(2) is unstable, otherwise  $\tilde{u}$  is stable. Hence we by this fact establish our results.

A large number of works have been made extensively studying stability of solutions in the case when  $n = 1$  (see [1],[2],[4],[5]). In [3] authors studied case  $n > 1$  with only Dirichlet boundary condition. The purpose of this work is to extend their results to Eqs. (1)-(2), under certain conditions.

## 2 Main results

In this section, by analyzing linearized equation, we will prove the stability of positive solutions.

**Theorem 1** *Let (4) hold, and assume that for  $i=1, \dots, n$ ,*

$$\sum_{j=1}^n \tilde{u}_j(x) \frac{\partial f_j}{\partial u_i}(\tilde{u}) > (<) f_i(\tilde{u}), \quad (9)$$

*then any positive solution of Equations (1)-(2) is linearly unstable(stable).*

**Proof.** Let  $\tilde{u} := (\tilde{u}_1, \dots, \tilde{u}_n)$  be any positive solution to (1)-(2). For each  $i=1, \dots, n$ , multiplying (7) by  $-\tilde{u}_i$ , (1) by  $\phi_i$ , add integrating by parts over  $\Omega$  and add the two resulting expressions to get

$$\begin{aligned} & \int_{\Omega} [\tilde{u}_i(x)\Delta\phi_i - \phi_i(x)\Delta\tilde{u}_i]dx + \int_{\Omega} \sum_{j=1}^n \tilde{u}_j(x) \frac{\partial f_j}{\partial u_i}(\tilde{u})\phi_j(x)dx - \int_{\Omega} f_i(\tilde{u})\phi_i(x)dx \\ & = -\mu_1 \int_{\Omega} \tilde{u}_i(x)\phi_i(x)dx. \end{aligned} \quad (10)$$

But by Green identity

$$\int_{\Omega} [\tilde{u}_i(x)\Delta\phi_i - \phi_i(x)\Delta\tilde{u}_i]dx = \int_{\partial\Omega} [\tilde{u}_i(x) \frac{\partial\phi_i}{\partial n} - \phi_i(x) \frac{\partial\tilde{u}_i}{\partial n}]ds, \quad (11)$$

now observe that in (3) when  $\alpha = 1$  then  $B\tilde{u}_i = \tilde{u}_i = 0$  for all  $s \in \partial\Omega$  and because  $\phi_i$  are eigenfunctions, thus  $\phi_i = 0$  for all  $s \in \partial\Omega$ . Hence

$$\int_{\partial\Omega} [\tilde{u}_i(x) \frac{\partial\phi_i}{\partial n} - \phi_i(x) \frac{\partial\tilde{u}_i}{\partial n}]ds = 0, \quad (12)$$

and when  $\alpha \neq 1$ , we obtain

$$\int_{\partial\Omega} [\bar{u}_i(x) \frac{\partial \phi_i}{\partial n} - \phi_i(x) \frac{\partial \bar{u}_i}{\partial n}] ds = \int_{\partial\Omega} \frac{\alpha h \phi_i(s)}{(1-\alpha)} (\bar{u}_i - \bar{u}_i) ds = 0. \tag{13}$$

Thus using (12) and (13) in (11), (10) becomes

$$\int_{\Omega} \sum_{j=1}^n \bar{u}_i(x) \frac{\partial f_i}{\partial u_j}(\bar{u}) \phi_j(x) dx - \int_{\Omega} f_i(\bar{u}) \phi_i(x) dx = -\mu_1 \int_{\Omega} \bar{u}_i(x) \phi_i(x) dx, \tag{14}$$

now adding (14) over  $i=1, \dots, n$ , and then rearranging the terms, we have

$$\int_{\Omega} \sum_{i=1}^n \phi_i(x) \left( \sum_{j=1}^n u_j(x) \frac{\partial f_j}{\partial u_i} - f_i(\bar{u}) \right) dx = -\mu_1 \int_{\Omega} \sum_{i=1}^n \bar{u}_i(x) \phi_i(x) dx. \tag{15}$$

By hypothesis, for each  $i=1, \dots, n$ ,  $\bar{u}_i > 0$  and  $\phi_i > 0$  in  $\Omega$ , thus if (9) holds, then  $\mu_1 < 0 (> 0)$  and it show that every positive solution of (1) – (2) is linearly unstable(stable). ■

**Theorem 2** *Let (4) hold, and assume that for  $i=1, \dots, n$ ,  $f_i(u_1, \dots, u_n)/u_i$  with respect to  $u_i$  be strictly increasing, then any positive solution of Equations (1)-(2) is linearly unstable.*

**Proof.** Let  $\bar{u} := (\bar{u}_1, \dots, \bar{u}_n)$  be any positive solution to (1)-(2). The proof proceeds in the same way as for Theorem 1., instead of (15) we obtain

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \bar{u}_i(x) \frac{\partial f_i}{\partial u_j}(\bar{u}) \phi_j(x) dx + \int_{\Omega} \sum_{i=1}^n \phi_i(x) [\bar{u}_i(x) \frac{\partial f_i}{\partial u_i}(\bar{u}) - f_i(\bar{u})] dx \\ & = -\mu_1 \int_{\Omega} \sum_{i=1}^n \bar{u}_i(x) \phi_i(x) dx. \end{aligned} \tag{16}$$

On the other hand since  $f_i(u_1, \dots, u_n)/u_i$  with respect to  $u_i$  is strictly decreasing,

$$\left( \frac{f_i}{u_i} \right)_{u_i}(\bar{u}_i) = \frac{\bar{u}_i (f_i)_{u_i}(\bar{u}_i) - f_i(\bar{u}_i)}{\bar{u}_i^2} > 0 \tag{17}$$

(here  $(\frac{f_i}{u_i})_{u_i}$  denotes the partial derivative of  $\frac{f_i}{u_i}$  with respect to  $u_i$ ). Also  $\bar{u}_i > 0$ ,  $\phi_i > 0$  in  $\Omega$ , and by (4), we have

$$-\mu_1 \int_{\Omega} \sum_{i=1}^n \bar{u}_i(x) \phi_i(x) dx > 0.$$

So,  $\mu_1 < 0$ , hence any positive solution of Equations (1)-(2) is linearly unstable. ■

**Theorem 3** *Let (5) hold, and assume that for  $i=1, 2$*

$$\sum_{j=1}^2 (-1)^{i+j} \bar{u}_j(x) \frac{\partial f_j}{\partial u_i}(\bar{u}) > (<) f_i(\bar{u}), \tag{18}$$

*then any positive solution of Equations (1)-(2) is linearly unstable(stable).*

**Proof.** Let  $\bar{u} := (\bar{u}_1, \bar{u}_2)$  be any positive solution to (1)-(2). Let  $v_1 = u_1$  and  $v_2 = -u_2$  then

$$-\Delta v_1(x) = f_1(u_1, u_2) = f_1(v_1, -v_2) = F_1(v_1, v_2) \quad x \in \Omega, \tag{19}$$

$$-\Delta v_2(x) = -f_2(u_1, u_2) = -f_2(v_1, -v_2) = F_2(v_1, v_2) \quad x \in \Omega, \tag{20}$$

notice that  $\frac{\partial F_i}{\partial v_j} = -\frac{\partial f_i}{\partial u_j} > 0$  for  $j \neq i$ , thus Equation (19)-(20) is cooperative with respect to  $F_1$  and  $F_2$ . Now if let  $\bar{v}_1 = \bar{u}_1$  and  $\bar{v}_2 = -\bar{u}_2$  then the linearized equation about  $\bar{v} = (\bar{v}_1, \bar{v}_2)$  consists of

$$-\Delta(\phi_1(x)) - \frac{\partial F_1}{\partial v_1}(\bar{v}) \phi_1(x) - \frac{\partial F_1}{\partial v_2}(\bar{v}) \phi_2(x) = \mu \phi_1(x) \quad x \in \Omega, \tag{21}$$

$$-\Delta(\phi_2(x)) - \frac{\partial F_2}{\partial v_1}(\bar{v})\phi_1(x) - \frac{\partial F_2}{\partial v_2}(\bar{v})\phi_2(x) = \mu\phi_2(x) \quad x \in \Omega, \quad (22)$$

$$B\phi_1(x) = 0 = B\phi_2(x) \quad x \in \partial\Omega, \quad (23)$$

calculate  $-(19)\phi_1 + (21)\bar{v}_1 + (20)\phi_2 - (22)\bar{v}_2$  and integrate over  $\Omega$  and then simplify to get

$$\begin{aligned} \mu_1 \int_{\Omega} [\bar{v}_1\phi_1 - \bar{v}_2\phi_2] dx &= \int_{\Omega} \phi_1(x) [\bar{v}_2(x) \frac{\partial F_2}{\partial v_1}(\bar{v}) - \bar{v}_1(x) \frac{\partial F_1}{\partial v_1}(\bar{v}) + F_1(\bar{v})] dx \\ &+ \int_{\Omega} \phi_2(x) [-\bar{v}_1(x) \frac{\partial F_1}{\partial v_2}(\bar{v}) + \bar{v}_2(x) \frac{\partial F_2}{\partial v_2}(\bar{v}) - F_2(\bar{v})] dx. \end{aligned} \quad (24)$$

Now by hypothesis,  $\mu_1 < 0 (> 0)$  if terms on right-hand side be negative (positive), that is if (18) holds. Hence every positive solution of (1)-(2) is unstable (stable). ■

### 3 Examples

Consider following problem

$$\begin{cases} -\Delta u(x) = \lambda f(u(x), v(x)) & x \in \Omega, \\ Bu = 0 = Bv & x \in \partial\Omega, \end{cases} \quad (25)$$

where  $\lambda$  is a positive parameter.

**Example 1.** For semipositone system

$$\begin{cases} f(u, v) = u^p + uv^\alpha - \epsilon_1 \\ g(u, v) = v^q + vu^\beta - \epsilon_2 \end{cases}$$

where  $p, q > 1, \alpha, \beta > 0, \epsilon_1, \epsilon_2 > 0$ , clearly we have  $\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} > 0$ , and

$$\sum_{j=1}^2 u_j(x) \frac{\partial f_j}{\partial u_1} = u \frac{\partial f}{\partial u} + v \frac{\partial g}{\partial u} = pu^p + uv^\alpha + \beta v^2 u^{\beta-1} > f(u, v)$$

and

$$\sum_{j=1}^2 u_j(x) \frac{\partial f_j}{\partial u_2} = u \frac{\partial f}{\partial v} + v \frac{\partial g}{\partial v} = \alpha u^2 v^{\alpha-1} + qv^q + vu^\beta > g(u, v).$$

Thus (9) satisfied and all positive solutions of (25) are unstable for all  $\lambda > 0$ . Or one can infer unstable by Theorem 2, because

$$\left( \frac{f(u, v)}{u} \right)_u = (u^{p-1} + v^\alpha - \frac{\epsilon_1}{u})_u = (p-1)u^{p-2} + \frac{\epsilon_1}{u^2} > 0,$$

and

$$\left( \frac{g(u, v)}{v} \right)_v = (v^{q-1} + u^\beta - \frac{\epsilon_2}{v})_v = (q-1)v^{q-2} + \frac{\epsilon_2}{v^2} > 0.$$

**Example 2.** For positone system

$$\begin{cases} f(u, v) = (u+1)^p + (v+1)^q + 1 \\ g(u, v) = u(v+1)^{q-1} + 1 \end{cases}$$

where  $0 \leq p < 1, q < 1$ , clearly we have  $\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} > 0$ , then

$$\sum_{j=1}^2 u_j(x) \frac{\partial f_j}{\partial u_1} = u \frac{\partial f}{\partial u} + v \frac{\partial g}{\partial u} = pu(u+1)^{p-1} + v(v+1)^{q-1} < p(u+1)^p + (v+1)^q < f(u, v)$$

and

$$\sum_{j=1}^2 u_j(x) \frac{\partial f_j}{\partial u_2} = u \frac{\partial f}{\partial v} + v \frac{\partial g}{\partial v} = qu(v+1)^{q-1} + (q-1)uv(v+1)^{q-2} < (2q-1)u(v+1)^{q-1} < g(u, v)$$

Thus (9) satisfied and all positive solutions of (25) are stable for all  $\lambda > 0$ .

**Example 3.** For semipositone system

$$\begin{cases} f(u, v) = u^p - uv^\alpha - \epsilon_1 \\ g(u, v) = v^q - vu^\beta - \epsilon_2 \end{cases}$$

where  $p, q > 1, \alpha, \beta > 0, \epsilon_1, \epsilon_2 > 0$ , clearly we have  $\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} < 0$ , then

$$\sum_{j=1}^2 (-1)^{1+j} u_j(x) \frac{\partial f_j}{\partial u_1} = u \frac{\partial f}{\partial u} - v \frac{\partial g}{\partial u} = pu^p - uv^\alpha + \beta v^2 u^{\beta-1} > f(u, v)$$

and

$$\sum_{j=1}^2 (-1)^{2+j} u_j(x) \frac{\partial f_j}{\partial u_2} = -u \frac{\partial f}{\partial v} + v \frac{\partial g}{\partial v} = \alpha u^2 v^{\alpha-1} + qv^q - vu^\beta > g(u, v)$$

Thus (18) satisfied and all positive solutions of (25) are unstable for all  $\lambda > 0$ .

**Example 4.** For semipositone system

$$\begin{cases} f(u, v) = u^p + \frac{1}{(u+v+1)^\alpha} + \epsilon_1 \\ g(u, v) = v^q + \frac{1}{(u+v+1)^\alpha} + \epsilon_2 \end{cases}$$

where  $0 < p, q, \alpha < 1, \epsilon_1, \epsilon_2 > 0$ , clearly we have  $\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} < 0$ , then

$$\sum_{j=1}^2 (-1)^{1+j} u_j(x) \frac{\partial f_j}{\partial u_1} = u \frac{\partial f}{\partial u} - v \frac{\partial g}{\partial u} = pu^p - \frac{\alpha}{(u+v+1)^{\alpha+1}}(u-v) < f(u, v)$$

and

$$\sum_{j=1}^2 (-1)^{2+j} u_j(x) \frac{\partial f_j}{\partial u_2} = -u \frac{\partial f}{\partial v} + v \frac{\partial g}{\partial v} = qv^q - \frac{\alpha}{(u+v+1)^{\alpha+1}}(v-u) < g(u, v)$$

Thus (18) satisfied and all positive solutions of (25) are stable for all  $\lambda > 0$ .

**Example 5.** Consider system

$$\begin{cases} -\Delta u_i(x) = \lambda f_i(u_1, \dots, u_n) & x \in \Omega, \\ Bu_i = 0 & x \in \partial\Omega, \end{cases} \tag{26}$$

where  $f_i = u_1^{\alpha_{i1}} u_2^{\alpha_{i2}} \dots u_n^{\alpha_{in}}$ ,  $\alpha_{ik} > 1$ , for  $i, k = 1, \dots, n$ , and  $\lambda > 0$  is a parameter, clearly we have  $\frac{\partial f_i}{\partial u_j} = \alpha_{ij} u_j^{\alpha_{ij}-1} \prod_{k=1, k \neq j}^n u_k^{\alpha_{ik}} > 0$ , for  $i = 1, \dots, n; i \neq j$ , and

$$\begin{aligned} \sum_{j=1}^n u_j(x) \frac{\partial f_j}{\partial u_1} &= \alpha_{11} u_1^{\alpha_{11}} \prod_{k=1, k \neq 1}^n u_k^{\alpha_{1k}} + \dots + \alpha_{n1} u_n u_1^{\alpha_{n1}} \prod_{k=1, k \neq 1}^n u_k^{\alpha_{nk}} \\ &> \alpha_{11} u_1^{\alpha_{11}} \prod_{k=1, k \neq 1}^n u_k^{\alpha_{1k}} > f_1(u_1, \dots, u_n), \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^n u_j(x) \frac{\partial f_j}{\partial u_2} &= \alpha_{12} u_1 u_2^{\alpha_{12}-1} \prod_{k=1, k \neq 2}^n u_k^{\alpha_{1k}} + \alpha_{22} u_2^{\alpha_{22}} \prod_{k=1, k \neq 2}^n u_k^{\alpha_{2k}} + \dots + \alpha_{n2} u_n u_2^{\alpha_{n2}} \prod_{k=1, k \neq 2}^n u_k^{\alpha_{nk}} \\ &> \alpha_{22} u_2^{\alpha_{22}} \prod_{k=1, k \neq 2}^n u_k^{\alpha_{2k}} > f_2(u_1, \dots, u_n). \end{aligned}$$

Similarly for  $i = 3, \dots, n$ , we obtain  $\sum_{j=1}^n u_j(x) \frac{\partial f_j}{\partial u_i} > f_i(u_1, \dots, u_n)$ . Hence by Theorem 1., any positive stationary solution of Equation (26) is linearly unstable. Or since

$$\left( \frac{f_i(u_1, \dots, u_n)}{u_i} \right)_{u_i} = (\alpha_{ii} - 1) u_i^{\alpha_{ii}-2} \prod_{k=1, k \neq i}^n u_k^{\alpha_{ik}} > 0,$$

for  $i = 1, \dots, n$ , thus by Theorem 2, any positive stationary solution is linearly unstable.

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