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Oscillation of Some Second Order Damped Difference Equations

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Abstract: Sufficient conditions for the oscillation of solution of some second order damped difference equation are obtained. Some examples and their numerical solutions are given to illustrate our results.

Keywords:damped; difference equations; dscillation; non-oscillation

1 Introduction

In this paper, we consider the oscillation behavior of solutions of second order linear and nonlinear damped difference equations of the form

$$\Delta\left(r_n\,\Delta y_n\right)+p_n\,\Delta y_n+q_n y_{n+1}=0,\quad n=0,1,2,.... \tag{1}$$

$$\Delta\left(r_n\Delta y_n\right)\,+\,p_n\,\Delta y_n\,+q_n\,f(y_{n+1})=0, \tag{2}$$

$$\Delta(r_n\psi(y_n)\Delta y_n)+p_n\Delta y_n+q_n y_{n+1}=0, \tag{3}$$

$$\Delta(r_n\psi(y_n)\Delta y_n)+p_n\Delta y_n+q_n f(y_{n+1})=0, \tag{4}$$

where the following conditions are assumed to hold.

(a) $\{r_n\},\{p_n\},\{q_n\}$ are real sequences, and q_n is not identically zero for infinitely many values n.

(b) $f,\psi:R\to R$ are continuous for all $x\neq 0$.

(c) There exists a real valued function g such that $f(u)-f(v)=g(u,v)\,(u-v)$ for all $u\neq 0,\ v\neq 0$, and $g(u,v)\geq L>0$,

$$K^2=\sum_{n=M}^{\infty}(n+1)\,p_n^2<\infty;\ K>0, \tag{5}$$

$$\sum_{n=0}^{\infty}r_n^2<\infty, \tag{6}$$

$$\sum_{n=0}^{\infty}(n+1)\,q_n=\infty, \tag{7}$$

$$(n+1)\,p_n\leq r_n,\quad \text{for sufficiently large}\quad n \tag{8}$$

and

$$(n+1)p_n\leq r_n\psi(y_n)\quad \text{for sufficiently large}\quad n \tag{9}$$

A solution $\{y_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non-oscillatory, Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$.

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In the last decides, there has been an increasing interest in obtaining sufficient conditions for the oscillation or non-oscillation of solutions for difference equations [6–8] and their analogues differential equations [2–4, 9, 10]. In [7] sufficient conditions for the oscillation of all solutions of a damped difference equation of the form $\Delta^2 y_n + p_n \Delta y_n + q_n f(y_n + 1) = 0, \quad n = 0, 1, \cdots$, are obtained. No sign conditions on the sequences $\{p_n\}$ and $\{q_n\}$ are assumed. Examples are inserted in the text to illustrate the results. In the absence of damping, there are a great number of papers [4, 7].

2 Main Results

We need the following lemma [7]

Lemma 1 *Let $K_1(n, s, y)$ be defined on $N \times N \times R^+$, N= {0, 1, 2 . . . }, $R^+ = [0, \infty)$, such that for fixed n and s, the function K_1 is non-decreasing in y. Let $\{h_n\}$ be a given sequence and the sequences $\{y_n\}$ and $\{z_n\}$ be defined on N satisfying, for all $n \in N$,*

$$y_n \geq h_n + \sum_{s=0}^{n-1} K_1(n, s, y_s), \tag{10}$$

and

$$z_n = h_n + \sum_{s=0}^{n-1} K_1(n, s, z_s), \tag{11}$$

respectively. Then $z_n \leq y_n$ for all $n \in N$.

Now, we present our results.

Theorem 1 *In addition to (a) and (b), let (5), (6), (7) and (8) hold, and then all solutions of (1) are oscillatory.*

Proof. Suppose the contrary. Then we may assume that $\{y_n\}$ be a non-oscillatory solution of (1), then $y_n > 0$ (or $y_n < 0$) for all $n \geq M - 1$. Multiplying (1) by $\frac{n+1}{y_{n+1}}$, and summing from M to $n - 1$, we get

$$\frac{n\, r_n\, \Delta y_n}{y_n} - \sum_{s=M}^{n-1} \frac{r_s\, \Delta y_s}{y_{s+1}} + \sum_{s=M}^{n-1} \frac{s\, r_s\, (\Delta y_s)^2}{y_s\, y_{s+1}} + \sum_{s=m}^{n-1} \frac{(s+1)\, p_s\, \Delta y_s}{y_{s+1}} + \sum_{s=M}^{n-1} (s+1)\, q_s = \frac{M\, r_M\, \Delta y_M}{y_M} \quad , \tag{12}$$

Using Schwartz's inequality [5], we have

$$\sum_{s=M}^{n-1} \frac{(s+1)\, p_s\, \Delta y_s}{y_{s+1}} \leq K\Big(\sum_{s=M}^{n-1} \frac{(s+1)(\Delta y_s)^2}{y_{s+1}^2}\Big)^{\frac{1}{2}}, \tag{13}$$

and

$$\sum_{s=M}^{n-1} \frac{r_s\, \Delta y_s}{y_{s+1}} \leq \Big(\sum_{s=M}^{n-1} r_s^2\Big)^{\frac{1}{2}} \Big(\sum_{s=M}^{n-1} \frac{(\Delta y_s)^2}{y_{s+1}^2}\Big)^{\frac{1}{2}}. \tag{14}$$

From (12), (13) and (14), we obtain

$$\begin{aligned} \frac{n\, r_n\, \Delta y_n}{y_n} - \Big(\sum_{s=M}^{n-1} r_s^2\Big)^{\frac{1}{2}} \Big(\sum_{s=M}^{n-1} \frac{(\Delta y_s)^2}{y_{s+1}^2}\Big)^{\frac{1}{2}} + \sum_{s=M}^{n-1} \frac{s\, r_s\, (\Delta y_s)^2}{y_s y_{s+1}} - K\Big(\sum_{s=M}^{n-1} \frac{(s+1)(\Delta y_s)^2}{y_{s+1}^2}\Big)^{\frac{1}{2}} \\ + \sum_{s=M}^{n-1} (s+1) q_s = \frac{M\, r_M\, \Delta y_M}{y_M}. \end{aligned} \tag{15}$$

Now, we use (5), (6) and (7) in (15), we observe that $\frac{n\, r_n\, \Delta y_n}{y_n} \rightarrow -\infty$ as $n \rightarrow \infty$.

Hence, there exist $M_1 \geq M$ such that

$$\Delta y_n < 0 , \quad for \;\; n \geq M_1, \tag{16}$$

rewrite (12) as

$$\begin{aligned} &\frac{n\, r_n\, \Delta y_n}{y_n} + \sum_{s=M_1}^{n-1} \frac{s\, r_s\, (\Delta y_s)^2}{y_s y_{s+1}} = \frac{M\, r_M\, \Delta y_M}{y_M} + \sum_{s=M}^{M_1-1} \frac{(r_s-(s+1)p_s)}{y_{s+1}} \Delta y_s \\ &+ \sum_{s=M_1}^{n-1} \frac{(r_s-(s+1)p_s)}{y_{s+1}} \Delta y_s - \sum_{s=M}^{M_1-1} \frac{s\, r_s\, (\Delta y_s)^2}{y_s y_{s+1}} - \sum_{s=M}^{n-1} (s+1)\, q_s \end{aligned} . \tag{17}$$

From (7), (8), (16) and (17), there exists an integer $M_2 \geq M_1$, such that

$$\frac{n\, r_n\, \Delta y_n}{y_n} + \sum_{s=M_1}^{n-1} \frac{s\, r_s\, (\Delta y_s)^2}{y_s y_{s+1}} \leq -m, m \geq M_2,$$

where m is positive constant.

Let $w_n = -n\, \Delta y_n$, then $w_n \geq m \frac{y_n}{r_n} + \sum_{s=M_2}^{n-1} \frac{y_n\, r_s\, (-\Delta y_s)}{r_n\, y_s\, y_{s+1}}\, w_s \,;\;\; n \geq M_2$, also, let

$$v_n = m \frac{y_n}{r_n} + \sum_{s=M_2}^{n-1} \frac{y_n\, r_s\, (-\Delta y_s)}{r_n\, y_s\, y_{s+1}}\, v_s \tag{18}$$

Applying Lemma 1, we have $w_n \geq v_n$. From (18), we have

$$\frac{r_n\, v_n}{y_n} = m + \sum_{s=M_2}^{n-1} \frac{r_s (-\Delta y_s)\, v_s}{y_s\, y_{s+1}}, \tag{19}$$

applying the operator Δ in the last equation, we get $\Delta\,(r_n\, v_n) = 0$.

Therefor $w_n \geq v_n = m\, y_{M_2}\,,\; for \;\, n \geq M_2$, hence $\Delta\, y_n \leq -\frac{m}{n} \frac{y_{M_2}}{r_n}$.

Summing the last inequality from M_2 to $n-1$, we obtain

$$y_n \leq y_{M_2} - m\, y_{M_2} \sum_{s=M_2}^{n-1} \frac{1}{s\, r_s}, \tag{20}$$

which yields a contradiction. The proof is similar for the case $y_n < 0$. This completes the proof of the theorem. ■

Example 1 *Consider the damped difference equation*

$$\Delta(\frac{1}{2n+1} \Delta y_n) - \frac{1}{(n+1)(2n+1)} \Delta y_n + \frac{2n+3}{n+1} y_{n+1} = 0. \tag{E$_1$}$$

All assumption condition of Theorem 1 are satisfied, hence, every solution of (E_1) is oscillatory. We find the numerical solution of this damped difference equation as in the following Fig. 1, which illustrate our theorem.

Remark 1 *Any solution of the damped difference equation*

$$\Delta(n(n+1)\Delta y_n) + n(n+2)\Delta y_n + (n+2) y_{n+1} = 0, \tag{E$_2$}$$

are non-oscillatory as we show from our theorem and the following Fig. 2 show the numerical solution of the damped difference equation E_2.

Following the same steps of the proof of Theorem 1, we obtain the following theorems.

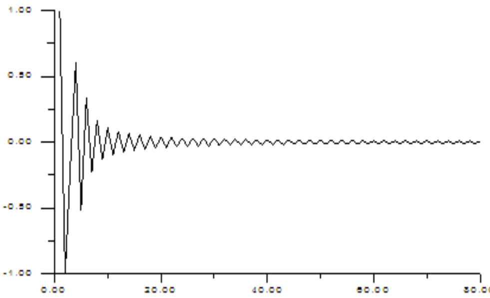

Figure 1: Numerical solution of the damped difference equation E_1

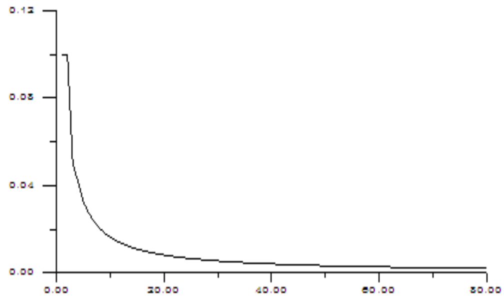

Figure 2: Numerical solution of the damped difference equation E_2

Theorem 2 *In addition to (a), (b), (c), let (5), (6), (7) and (8) be satisfied, and then all solutions of (2) are oscillatory.*

Example 2 *Consider the damped difference equation*

$$\Delta(\frac{2}{n}\Delta y_n) + \frac{1}{n(n+1)}\Delta y_n + \frac{2(4n+1)}{n(n+1)}y_n^3 = 0. \tag{E$_3$}$$

All assumption conditions of Theorem 2 are satisfied, then, all solutions of (E_3) are oscillatory. Our numerical solution of the damped difference equation (E_3) is shown in the following Table 1.

Table 1: Numerical solution of the damped difference equation E_3

N	3	4	5	6	7
y_n	3.35E-4	-1.01E-3	3.00E-3	-8.7E-3	2.48E-2
N	8	9	10	11	12
y_n	-7.00E-2	1.96E-1	-5.4E-1	1.50	-4.15

Theorem 3 *In addition to (a) and (b), let (5), (6), (7) and (9) hold , and then all solutions of (3) are oscillatory.*

Example 3 *Consider the damped difference equation*

$$\Delta(\frac{1}{n^2}y_n^2\Delta y_n) - \frac{1}{n(2n+1)}\Delta y_n + \frac{4n^2+5n+1}{n}y_{n+1} = 0, \tag{E$_4$}$$

all solutions of (E_4) are oscillatory. Our numerical solutions found that all solutions are oscillatory, and in the following table one of them.

Table 2: Numerical solution of the damped difference equation E_4

N	1	2	5	10	15
y_n	1.0	-2.0	-113.36	185.94	-533.27
N	20	25	35	113	120
y_n	683.84	-670.09	13805.3	-51670.	-96056.0

Theorem 4 *In addition to (a), (b) and (c), let (5), (6), (7) and (9) hold, then all solutions of (4) are oscillatory.*

Now, in the following we consider linear damped difference equations, simple sufficient conditions for the oscillatory of the solutions are given. We recall that a nontrivial solution of difference equation is said to be oscillatory, if for every $N > 0$, there exists an $n \geq N$, such that $y_n y_{n+1} \leq 0$, and otherwise it is said to be non-oscillatory. We consider the oscillation behavior of the solutions of the linear difference equations of the form

$$\Delta y_n + q_n y_n = 0, \tag{21}$$

$$\Delta y_n + q_n y_{n+1} = 0, \tag{22}$$

$$\Delta^2 y_n + p_n \Delta y_n + q_n y_n = 0, \tag{23}$$

$$\Delta^2 y_n + p_n \Delta y_n + q_n y_{n+1} = 0, \tag{24}$$

$$\Delta(r_n \Delta y_n) + p_n \Delta y_n + q_n y_n = 0, \tag{25}$$

and

$$\Delta(r_n \Delta y_n) + p_n \Delta y_n + q_n y_{n+1} = 0, \tag{26}$$

where the following conditions are assumed to hold

(d) $\{q_n\}$ is real sequence for all values of n.

(e) $\{p_n\}$ is real sequence for all values of n.

(f) $\{r_n\}$ is of positive values for any n.

$$q_n \geq 1\,, \tag{27}$$

$$\frac{1}{1+q_n} \leq 0\,, \tag{28}$$

$$p_n \geq 2\,, \tag{29}$$

$$1 + q_n > p_n, \tag{30}$$

$$p_n < 1\,, \tag{31}$$

$$p_n + q_n > 2, \tag{32}$$

$$p_n > r_n + 2\,r_{n+1} > r_n + r_{n+1}\,, \tag{33}$$

$$r_n + q_n > p_n, \tag{34}$$

$$r_n > p_n, \tag{35}$$

and

$$p_n + q_n > r_n + r_{n+1}. \tag{36}$$

Now, we present our results in the following.

Theorem 5 *In addition to (d), assume that (27) is satisfied, then all solutions of (21) are oscillatory, and one solution of (21) is*

$$y_n = y_0 \prod_{s=0}^{n-1} (1 - q_s)\,, \tag{37}$$

Proof. From (21), we get

$$y_{n+1} = (1 - q_n) y_n. \tag{38}$$

Multiplying (38) by y_n, and use conditions (d), (27), we get $y_n y_{n+1} \leq 0$, hence any solutions of (21) are oscillatory. Again, from (38), we have $\frac{y_n}{y_0} = 1 - q_n$. Multiplying from 0 to n-1, we obtain $\frac{y_n}{y_0} = (1 - q_0)(1 - q_1)...(1 - q_{n-1})$, from which, we get (37). ■

Example 4 *Consider the difference equation*

$$\Delta\, y_n + (n + 2) y_n = 0. \tag{E$_5$}$$

From (21) and (E_5), $q_n = n + 2 > 1$, which it is condition (27), hence any solutions of (E_5) are oscillatory. And from (37) and (E_5), one solution of (E_5) is $y_n = (-1)^n n!$; $y_0 = 1$, if we compute the numerical solution we find the following Table 3.

Table 3: Numerical solution of the damped difference equation E_5

N	1	2	3	4	5
y_n	-1.0	2.0	-6.0	24.0	-120.0
N	6	7	8	9	10
y_n	720.0	-5040.0	40320.0	-362880	3628800.0

Remark 2 *Any solution of the difference equation*

$$\Delta y_n - (n+2)y_n = 0. \tag{E_6}$$

are non-oscillatory. One solution of (E_6) *is* $y_n = \frac{1}{2}(n+2)!$. *In all numerical solutions with different initials we find these solutions are non-oscillatory as we see in Fig. 3.*

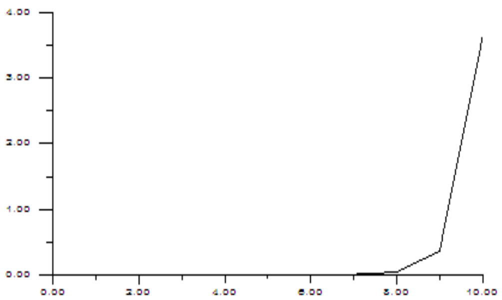

Figure 3: Numerical solution of the damped difference equation E_6

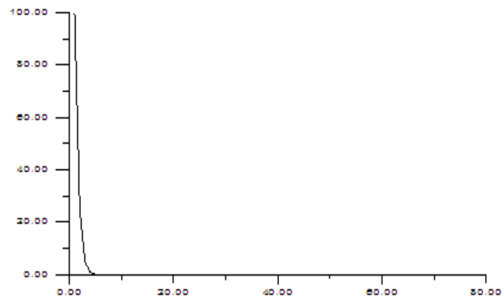

Figure 4: Numerical solution of the damped difference equation E_8

Following the same steps of the proof of Theorem 5, we obtain the following theorem.

Theorem 6 *In addition to (d), assume that (28) is satisfied, then all solutions of (22) are oscillatory, and one solution of (22) is*

$$y_n = y_0 \prod_{s=0}^{n-1} (1+q_s)^{-1}. \tag{39}$$

Example 5 *Consider the damped difference equation*

$$\Delta y_n - (n+2)y_{n+1} = 0. \tag{E_7}$$

From (22) and (E_7), $q_n = -(n+2)$, *from which* $(1+q_n)^{-1} = -(n+1) < 0$, *which it is condition (28), hence any solutions of* (E_7) *are oscillatory. And from (39) and* (E_7), *one solution of* (E_7) *is* $y_n = \frac{(-1)^n}{n!}$. *As it is appearing in the following table 4 the numerical solution is oscillatory.*

Table 4: Numerical solution of the damped difference equation E_7

N	1	2	3	4	5	10
Y_n	1.0	-5.0E-1	1.6E-1	-4.1E-2	8.3E-4	-2.7E-7
N	15	20	25	30	35	38
y_n	7.6E-13	-4.E-19	6.E-26	-3.E-33	9.E-33	-1.E-45

Remark 3 *Any solutions of the damped difference equation*

$$\Delta y_n + (n+2)y_{n+1} = 0. \tag{E_8}$$

are non-oscillatory. The solution is non-oscillatory as in Fig.4

Theorem 7 *In addition to (d) and (e), assume that (29) and (30) are satisfied, then any solution of (23) is oscillatory.*

Proof. From (23), we get

$$y_{n+2} + (p_n - 2)y_{n+1} + (1 - p_n + q_n)y_n = 0. \tag{40}$$

Multiplying (40) by y_{n+1}, we obtain

$$y_{n+2}y_{n+1} = -(p_n - 2)y_{n+1}^2 - (1 - p_n + q_n)y_n y_{n+1}, \tag{41}$$

from which and conditions (d), (e), (29) and (30), we get for $y_n \leq y_{n+1}$,

$$y_{n+1}y_{n+2} \leq -(p_n - 2)y_{n+1}^2 - (1 - p_n + q_n)y_n^2 < 0, \tag{42}$$

and for $y_n \geq y_{n+1}$

$$y_{n+1}y_{n+2} \leq -(p_n - 2)y_{n+1}^2 - (1 - p_n + q_n)y_{n+1}^2 < 0. \tag{43}$$

Hence, $y_{n+1}y_{n+2} \leq 0$, this complete the proof of the theorem. ■

Example 6 *Consider the difference equation*

$$\Delta^2 y_n + (n+2)\Delta y_n + 2ny_n = 0. \tag{E$_9$}$$

From (23) and (E_9), $p_n = n+2 > 2, q_n = 2n$ *, and* $1+q_n-p_n = n-1 \geq 0$*, then any solutions of* (E_9) *are oscillatory. The numerical solution is the same solution as in Fig. 5 and the exact solution is* $y_n = (-1)^n$.

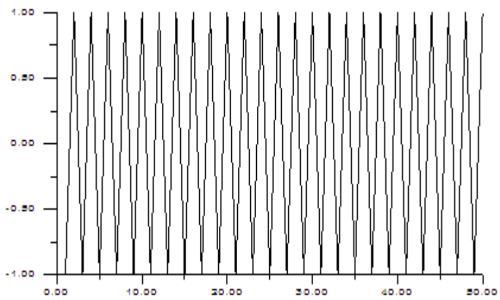

Figure 5: Numerical solution of the damped difference equation E_9

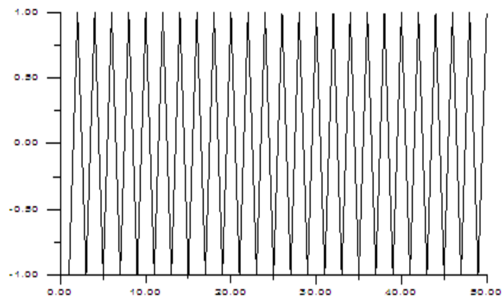

Figure 6: Numerical solution of the damped difference equation E_{12}

Example 7 *Any solution of the difference equation*

$$\Delta^2 y_n + 2(n+1)\Delta y_n + \frac{2(n+1)(2n+1)}{n} y_n = 0, \tag{E$_{10}$}$$

are oscillatory. One solution of (E_{10}) *is* $y_n = (-1)^n n$*. And the numerical solution of the difference equation is oscillatory.*

Following the same steps of Theorem 7, we obtain the following theorems.

Theorem 8 *In addition to (d) and (e), assume that (31) and (32) are satisfied, then any solution of (24) is oscillatory.*

Example 8 *Consider the damped difference equation*

$$\Delta^2 y_n - (n-1)\Delta y_n + \frac{2n^2 + 3n + 3}{n+1} y_{n+1} = 0, \tag{E$_{11}$}$$

From (24) and $(E_{11})p_n < 1, p_n + q_n > 2$*, then any solutions of* (E_{11}) *are oscillatory.*

Our numerical solution is oscillatory.

Example 9 *Any solutions of the damped difference equation*

$$\Delta^2 y_n - (n+1)\Delta y_n + 2(n+3)y_{n+1} = 0, \qquad (E_{12})$$

are oscillatory. We see in the following table that the numerical solution is oscillatory.

Table 5: Numerical solution of the damped difference equation E_{12}

N	5	10	15	20	25
y_n	-2.41E-2	2.43E-2	-2.43E-2	2.43E-2	-2.43E-2
N	30	35	40	45	50
y_n	2.43E-2	-2.43E-2	2.43E-2	-2.43E-2	2.43E-2

In the following Fig. 6 solution of E_{12} with different initials which is oscillatory.

Theorem 9 *In addition to (d), (e) and (f), assume that (33) and (34) are satisfied, then any solutions of (25) are oscillatory.*

Example 10 *Consider the difference equation*

$$\Delta(n\Delta y_n) + 3(n+1)\Delta y_n + (2n+4)y_n = 0. \qquad (E_{13})$$

From (25), (E_{13}), $r_n = n > 0, p_n = 3(n+1)$ and $q_n = 2n+4$, then $r_n + 2r_n < p_n$ and $r_n + q_n - p_n = 1 > 0$, hence any solutions of (E_{13}) are oscillatory. We find the numerical solution is oscillatory as it appear in the Fig. 7.

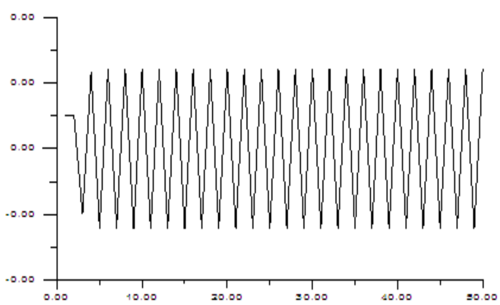

Figure 7: Numerical solution of the damped difference equation E_{13}

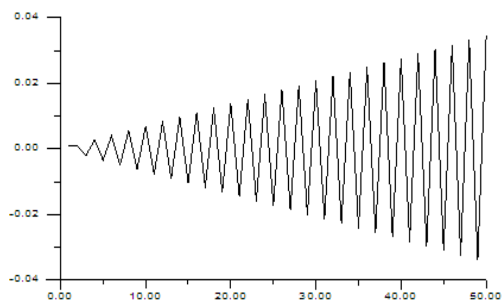

Figure 8: Numerical solution of the damped difference equation E_{14}

Example 11 *Any solution of the difference equation*

$$\Delta((n+1)\Delta y_n) + (3n+7)\Delta y_n + (2n+7)y_n = 0. \qquad (E_{14})$$

are oscillatory. From the following Fig. 8 we see that the solution is oscillatory.

Theorem 10 *In addition to (d), (e) and (f), assume that (35) and (32) are satisfied, then any solutions of (22) are oscillatory.*

Example 12 *Consider the difference equation*

$$\Delta(n\Delta y_n) + (n-1)\Delta y_n + (2n+4)y_{n+1} = 0. \qquad (E_{15})$$

From (22) and (E_{15}), $r_n = n > 0, p_n = n-1$ and $q_n = 2n+4$, then $r_n > p_n$ and $p_n + q_n - r_n - r_{n+1} = n+2 > 0$, hence any solutions of (E_{15}) are oscillatory as we see in Fig. 9.

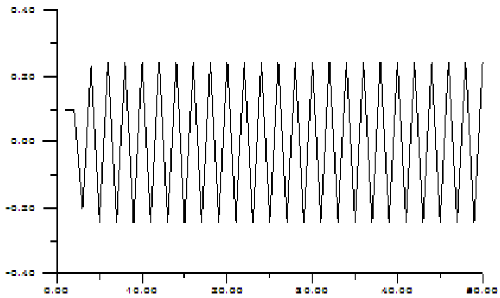

Figure 9: Numerical solution of the damped difference equation E_{15}

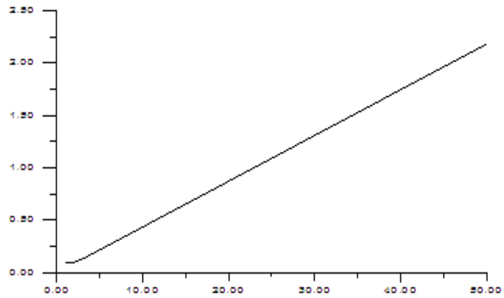

Figure 10: Numerical solution of the damped difference equation E_{16}

Example 13 *Any solutions of the damped difference equation*

$$\Delta((n+1)\Delta y_n) + n\Delta y_n - y_{n+1} = 0. \qquad (E_{16})$$

are non-oscillatory. When we solve this difference equation, we find that all solutions are non-oscillatory as we find in Fig. 10.

3 Conclusions

This study presents the design and implementation of second order damped difference equations. We give new theoretical studies for the sufficient conditions for the oscillation of solution of second order damped difference equations. We give test problems to demonstrate our results with different cases. We believe that the present studies can be useful for the damped difference equations.

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