

On the Rule of Semi-cycle Length for a Class of Fifth-order Nonlinear Difference Equation

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(Received 9 August 2007, accepted 7 March 2008)

Abstract: In this paper we consider the rule of trajectory structure for a kind of fifth-order rational difference equation. With the change of the initial values, we find that the successive lengths of positive and negative semi-cycles for nontrivial solutions of this equation periodically occur with prime period 31. The rule is $4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+, 1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-$ in a period. By the use of the rule, the positive equilibrium point of this equation is proved to be globally asymptotically stable.

Keywords: rational difference equation; trajectory structure rule; semi-cycle length; periodicity; global asymptotic stability

1 Introduction and Preliminaries

Nonlinear difference equations of order greater than one are very importance in application. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics. Some nonlinear difference equations of second, third and even fourth order have been investigated by many authors, for example, see [1-9].

As a kind of typical nonlinear difference equation, rational difference equation is always a subject studied in recent years. Especially, some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. For the systematical investigations of this aspect, refer to the monographs [1, 3, 4] and the papers [8, 9] and the references cited therein.

Motivated by the work [9], we consider in this paper the following fifth-order rational difference equation

$$x_{n+1} = \frac{F(x_n, x_{n-2}, x_{n-3}, x_{n-4})}{G(x_n, x_{n-2}, x_{n-3}, x_{n-4})}, n = 0, 1, \dots, \quad (1)$$

where the functions

$$F(x, y, z, w) = x^u y^v + x^u z^k + x^u w^j + y^v z^k + y^v w^j + z^k w^j + x^u y^v z^k w^j + 1 + a$$

and

$$G(x, y, z, w) = x^u + y^v + z^k + w^j + x^u y^v z^k + x^u y^v w^j + x^u z^k w^j + y^v z^k w^j + a$$

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the parameters $a \in [0, +\infty)$, $u \in (0, 1]$, $v, k, j \in (0, +\infty)$, and the initial values $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, +\infty)$.

Mainly, by analyzing the rule for the length of semi-cycle to occur successively, we describe clearly out the rule for the trajectory structure of its solutions and further derive the global asymptotic stability of positive equilibrium of equation (1). To the best of our knowledge, Eq.(1) has not been investigated so far, therefore, it is theoretically meaningful to study its qualitative properties.

It is easy to see that the positive equilibrium \bar{x} of Eq.(1) satisfies

$$\bar{x} = \frac{1 + a + \bar{x}^{u+v} + \bar{x}^{u+k} + \bar{x}^{u+j} + \bar{x}^{v+k} + \bar{x}^{v+j} + \bar{x}^{k+j} + \bar{x}^{u+v+k+j}}{a + \bar{x}^u + \bar{x}^v + \bar{x}^k + \bar{x}^j + \bar{x}^{u+v+k} + \bar{x}^{u+v+j} + \bar{x}^{u+k+j} + \bar{x}^{v+k+j}}.$$

From this we see that Eq.(1) possesses a unique positive equilibrium $\bar{x} = 1$.

2 Nontrivial Solution

Theorem 2.1 A positive solution $\{x_n\}_{n=-4}^{\infty}$ of Eq.(1) is eventually trivial if and only if

$$(x_{-4} - 1)(x_{-3} - 1)(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) = 0. \quad (2)$$

Proof. Sufficiency. Assume that Eq.(2) holds. Then according to Eq.(1), we know that the following conclusions are true:

- (i) If $x_{-4} = 1$, then $x_n = 1$ for $n \geq 1$.
- (ii) If $x_{-3} = 1$, then $x_n = 1$ for $n \geq 1$.
- (iii) If $x_{-2} = 1$, then $x_n = 1$ for $n \geq 2$.
- (iv) If $x_{-1} = 1$, then $x_n = 1$ for $n \geq 1$.
- (v) If $x_0 = 1$, then $x_n = 1$ for $n \geq 1$.

Necessity. Conversely, assume that

$$(x_{-4} - 1)(x_{-3} - 1)(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) \neq 0. \quad (3)$$

Then we can show $x_n \neq 1$ for any $n \geq 1$. For the sake of contradiction, assume that for some $N \geq 1$,

$$x_N = 1 \text{ and that } x_n \neq 1 \text{ for any } -4 \leq n \leq N-1. \quad (4)$$

Clearly,

$$1 = x_N = \frac{F(x_{N-1}, x_{N-3}, x_{N-4}, x_{N-5})}{G(x_{N-1}, x_{N-3}, x_{N-4}, x_{N-5})}.$$

From this we can know that

$$0 = x_N - 1 = \frac{(x_{N-1}^u - 1)(x_{N-3}^v - 1)(x_{N-4}^k - 1)(x_{N-5}^j - 1)}{G(x_{N-1}, x_{N-3}, x_{N-4}, x_{N-5})},$$

which implies $x_{N-1} = 1$, or $x_{N-3} = 1$, or $x_{N-4} = 1$, or $x_{N-5} = 1$. This contradicts with Eq.(4). ■

Remark 2.2 Theorem 2.1 actually demonstrates that a positive solution $\{x_n\}_{n=-4}^{\infty}$ of Eq.(1) is eventually nontrivial if $(x_{-4} - 1)(x_{-3} - 1)(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) \neq 0$. So, if a solution is a nontrivial one, then $x_n \neq 1$ for any $n \geq -4$.

3 Oscillation and Non-oscillation

Before stating the oscillation and non-oscillation of solutions, we need the following key lemmas. For any integer a , denote $N_a = \{a, a+1, \dots\}$.

Lemma 3.1 If the integer $i \in N_{-4}$, then

$$x_{n+1} - x_i = \frac{K(x_n, x_{n-2}, x_{n-3}, x_{n-4}, x_i)}{G(x_n, x_{n-2}, x_{n-3}, x_{n-4})}, n = 0, 1, \dots, \tag{5}$$

where

$$K(x, y, z, w, p) = (1 - x^u p)(1 + y^v z^k + y^v w^j + z^k w^j) + a(1 - x_i) + (x^u - p)(y^v + z^k + w^j + y^v z^k w^j)$$

and

$$G(x, y, z, w) = x^u + y^v + z^k + w^j + x^u y^v z^k + x^u y^v w^j + x^u z^k w^j + y^v z^k w^j + a$$

Lemma 3.2 If the integers $i \in N_{-4}$ and $t \in N_1$, then

$$1 - x_{n+1} x_i^{\frac{1}{u^t}} = \frac{M(x_n, x_{n-2}, x_{n-3}, x_{n-4}, x_i)}{G(x_n, x_{n-2}, x_{n-3}, x_{n-4})}, n = 0, 1, \dots, \tag{6}$$

where

$$M(x, y, z, w, p) = (x^u - p^{\frac{1}{u^t}})(1 + y^v z^k + y^v w^j + z^k w^j) + a(1 - p^{\frac{1}{u^t}}) + (1 - x^u p^{\frac{1}{u^t}})(y^v + z^k + w^j + y^v z^k w^j).$$

Lemma 3.3 If the integers $i \in N_{-4}$ and $t \in N_1$, then

$$x_{n+1} - x_i^{\frac{1}{u^t}} = \frac{N(x_n, x_{n-2}, x_{n-3}, x_{n-4}, x_i)}{G(x_n, x_{n-2}, x_{n-3}, x_{n-4})}, n = 0, 1, \dots, \tag{7}$$

where

$$N(x, y, z, w, p) = (1 - x^u p^{\frac{1}{u^t}})(1 + y^v z^k + y^v w^j + z^k w^j) + a(1 - p^{\frac{1}{u^t}}) + (x^u - p^{\frac{1}{u^t}})(y^v + z^k + w^j + y^v z^k w^j).$$

The results of the above Lemmas (3.1),(3.2) and (3.3) may be immediately derived from Eq.(1). So, we omit their proofs here.

Lemma 3.4 Let $\{x_n\}_{n=-4}^\infty$ be a positive solution of Eq.(1) which is not eventually equal to 1, then the following conclusions are valid:

- (a) $(x_{n+1} - 1)(x_n - 1)(x_{n-2} - 1)(x_{n-3} - 1)(x_{n-4} - 1) > 0$, for $n \geq 0$;
- (b) $(x_{n+1} - x_n)(x_n - 1) < 0$, for $n \geq 0$;
- (c) $(x_{n+1} - x_{n-1})(x_{n-1} - 1) < 0$, for $n \geq 0$;
- (d) $(x_{n+1} - x_{n-2})(x_{n-2} - 1) < 0$, for $n \geq 0$;
- (e) $(x_{n+1} - x_{n-3})(x_{n-3} - 1) < 0$, for $n \geq 0$;
- (f) $(x_{n+1} - x_{n-4})(x_{n-4} - 1) < 0$, for $n \geq 0$.

Theorem 3.1 Let $\{x_n\}_{n=-4}^\infty$ be a positive solution of Eq.(1) which is not eventually equal to 1, then $(x_{n+1} - x_{n-5})(x_{n-5} - 1) < 0$ for $n \geq 0$.

Theorem 3.2 There exist non-oscillatory solutions of Eq.(1) with $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in (1, +\infty)$, which must be eventually positive. There don't exist eventually negative non-oscillatory solutions of Eq.(1).

Proof. Consider a solution of Eq.(1) with

$$x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in N_1.$$

We then know from Lemma 3.4(a) that $x_n > 1$ for $n \in N_{-4}$. So, this solution is just a non-oscillatory solution and furthermore eventually positive.

Suppose that there exist eventually negative non-oscillatory of Eq.(1). Then, there exists a positive integer N such that $x_n < 1$ for $n \geq N$. Thereout, for $n \geq N + 4$,

$$(x_{n+1} - 1)(x_n - 1)(x_{n-2} - 1)(x_{n-3} - 1)(x_{n-4} - 1) \leq 0.$$

This contradicts Lemma 3.4(a). So, there don't exist eventually negative non-oscillatory of Eq.(1), as desired. ■

4 Rule of Cycle Length

Theorem 4.1 Let $\{x_n\}_{n=-4}^{\infty}$ be a strictly oscillatory of Eq.(1), then the rule for the lengths of positive and negative semi-cycles of this solution to occur successively is $\cdots, 4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+, 1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-, 4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+, 1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-, \cdots$

Proof. By Lemma 3.4 (a), one can see that the length of a negative semi-cycle is at most 5, and a positive semi-cycle is at most 4. On the basis of the strictly oscillatory character of the solution, we see that, for some integer $p \geq 0$, one of the following 2 cases must occur:

case 1: $x_p > 1, x_{p+1} > 1, x_{p+2} > 1, x_{p+3} > 1, \text{and } x_{p+4} > 1$;

case 2: $x_p > 1, x_{p+1} > 1, x_{p+2} > 1, x_{p+3} > 1, \text{and } x_{p+4} < 1$.

Case 1 can't occur. Otherwise, the solution is a non-oscillatory solution of Eq.(1).

If Case 2 occurs, it follows from Lemma 3.4(a) that $x_{p+5} < 1, x_{p+6} < 1, x_{p+7} > 1, x_{p+8} > 1, x_{p+9} < 1, x_{p+10} < 1, x_{p+11} > 1, x_{p+12} < 1, x_{p+13} < 1, x_{p+14} < 1, x_{p+15} < 1, x_{p+16} < 1, x_{p+17} > 1, x_{p+18} < 1, x_{p+19} > 1, x_{p+20} > 1, x_{p+21} > 1, x_{p+22} < 1, x_{p+23} > 1, x_{p+24} > 1, x_{p+25} < 1, x_{p+26} > 1, x_{p+27} < 1, x_{p+28} > 1, x_{p+29} < 1, x_{p+30} < 1, x_{p+31} > 1, x_{p+32} > 1, x_{p+33} > 1, x_{p+34} > 1, x_{p+35} < 1, x_{p+36} < 1, x_{p+37} < 1, x_{p+38} > 1, x_{p+39} > 1, x_{p+40} < 1, x_{p+41} < 1, x_{p+42} > 1, x_{p+43} < 1, x_{p+44} < 1, x_{p+45} < 1, x_{p+46} < 1, x_{p+47} < 1, x_{p+48} > 1, x_{p+49} < 1, x_{p+50} > 1, x_{p+51} > 1, x_{p+52} > 1, x_{p+53} < 1, x_{p+54} > 1, x_{p+55} > 1, x_{p+56} < 1, x_{p+57} > 1, x_{p+58} < 1, x_{p+59} > 1, x_{p+60} < 1, x_{p+61} < 1, \cdots$

This means that rule for the lengths of positive and negative semi-cycles of the solution of Eq.(1) to occur successively is $\cdots, 4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+, 1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-, \cdots$. So, the proof for this theorem is complete. ■

5 Global Asymptotic Stability

First, we consider the local asymptotic stability for unique positive equilibrium point \bar{x} of Eq.(1). We have the following results.

Theorem 5.1 The positive equilibrium point of Eq.(1) is locally asymptotically stable.

Proof. The linearized equation of Eq.(1) about the positive equilibrium point \bar{x} is

$$y_{n+1} = 0 \cdot y_n + 0 \cdot y_{n-2} + 0 \cdot y_{n-3} + 0 \cdot y_{n-4}, n = 0, 1, \cdots,$$

and so it is clear from the paper [2, Remark 1.3.7] that the positive equilibrium point \bar{x} of Eq.(1) is locally asymptotically stable. The proof is complete. ■

We are now in a position to study the global asymptotically stability of positive equilibrium point \bar{x} .

Theorem 5.2 The positive equilibrium point of Eq.(1) is globally asymptotically stable.

Proof. We must prove that the positive equilibrium point \bar{x} of Eq.(1) is both locally asymptotically stable and globally attractive. Theorem (5.1) has shown the local asymptotic stability of \bar{x} . Hence it remains to verify that every positive solution $\{x_n\}_{n=-4}^{\infty}$ of Eq.(1) converges to \bar{x} as $n \rightarrow \infty$. Namely, we want to prove

$$\lim_{n \rightarrow \infty} x_n = \bar{x} = 1. \quad (8)$$

We can divide the solutions of Eq.(1) into two kinds of types.

- i) Trivial solutions;
- ii) Nontrivial solutions.

If the the solution is a trivial one, then it is obvious for(8) to hold because $x_n = 1$ holds eventually.

If the the solution is a nontrivial solutions, then we can further divide the solution into two cases.

- a) Non-oscillatory solution;
- b) Oscillatory solution.

Consider now $\{x_n\}$ to be non-oscillatory about the positive equilibrium point \bar{x} of Eq.(1). By virtue of Lemma 3.4 (b), it follows that the solution is monotonic and bounded. So $\lim_{n \rightarrow \infty} x_n$ exists and is finite. Taking limits on both sides of Eq.(1), one can easily see that (8) holds.

Now let $\{x_n\}$ be strictly oscillatory about the positive equilibrium point of Eq.(1). By virtue of Theorem 4.1, one understands that the rule for the lengths of positive and negative semi-cycles occurring successively is $\dots, 4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+, 1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-, \dots$. For simplicity, for some non-negative integer p , we denote by $\{x_p, x_{p+1}, x_{p+2}, x_{p+3}\}^+$ the terms of a positive semicycle of length four, followed by $\{x_{p+4}, x_{p+5}, x_{p+6}\}^-$, a negative semicycle with semicycle length three, then a positive semicycle of length two and a negative semicycle of length two, and so on. Namely, the rule for the lengths of positive and negative semicycles to occur successively can be periodically expressed as follows:

$$\begin{aligned} & \{x_{p+31n}, x_{p+31n+1}, x_{p+31n+2}, x_{p+31n+3}\}^+, \{x_{p+31n+4}, x_{p+31n+5}, x_{p+31n+6}\}^-, \\ & \{x_{p+31n+7}, x_{p+31n+8}\}^+, \{x_{p+31n+9}, x_{p+31n+10}\}^-, \{x_{p+31n+11}\}^+, \\ & \{x_{p+31n+12}, x_{p+31n+13}, x_{p+31n+14}, x_{p+31n+15}, x_{p+31n+16}\}^-, \{x_{p+31n+17}\}^+, \\ & \{x_{p+31n+18}\}^-, \{x_{p+31n+19}, x_{p+31n+20}, x_{p+31n+21}\}^+, \{x_{p+31n+22}\}^-, \\ & \{x_{p+31n+23}, x_{p+31n+24}\}^+, \{x_{p+31n+25}\}^-, \{x_{p+31n+26}\}^+, \{x_{p+31n+27}\}^-, \\ & \{x_{p+31n+28}\}^+, \{x_{p+31n+29}, x_{p+31n+30}\}^-, n = 0, 1, 2, \dots \end{aligned}$$

Lemma (3.4) (b), (c), (d), (e), (f) and Theorem 3.1 teaches us that the following results are true:

(A)

$$\begin{aligned} x_{p+31n} &> x_{p+31n+1} > x_{p+31n+2} > x_{p+31n+3} > x_{p+31n+7} \\ &> x_{p+31n+8} > x_{p+31n+11} > x_{p+31n+17} > x_{p+31n+19} \\ &> x_{p+31n+20} > x_{p+31n+21} > x_{p+31n+23} > x_{p+31n+24} \\ &> x_{p+31n+26} > x_{p+31n+28} > x_{p+31(n+1)}, n = 0, 1, 2, \dots \end{aligned}$$

(B)

$$\begin{aligned} x_{p+31n+4} &< x_{p+31n+5} < x_{p+31n+6} < x_{p+31n+9} < x_{p+31n+10} \\ &< x_{p+31n+12} < x_{p+31n+13} < x_{p+31n+14} < x_{p+31n+15} \\ &< x_{p+31n+16} < x_{p+31n+18} < x_{p+31n+22} < x_{p+31n+25} \\ &< x_{p+31n+27} < x_{p+31n+29} < x_{p+31n+30} < x_{p+31(n+1)+4}, \\ &n = 0, 1, 2, \dots \end{aligned}$$

So, from (A) one can see that $\{x_{p+31n}\}_{n=0}^\infty$ is decreasing with lower bound 1. So, the limit $S = \lim_{n \rightarrow \infty} x_{p+31n}$ exist and is finite.

Furthermore, From (A) one can further obtain

$$\begin{aligned} S &= \lim_{n \rightarrow \infty} x_{p+31n+1} = \lim_{n \rightarrow \infty} x_{p+31n+2} = \lim_{n \rightarrow \infty} x_{p+31n+3} \\ &= \lim_{n \rightarrow \infty} x_{p+31n+7} = \lim_{n \rightarrow \infty} x_{p+31n+8} = \lim_{n \rightarrow \infty} x_{p+31n+11} \\ &= \lim_{n \rightarrow \infty} x_{p+31n+17} = \lim_{n \rightarrow \infty} x_{p+31n+19} = \lim_{n \rightarrow \infty} x_{p+31n+20} \\ &= \lim_{n \rightarrow \infty} x_{p+31n+21} = \lim_{n \rightarrow \infty} x_{p+31n+23} = \lim_{n \rightarrow \infty} x_{p+31n+24} \\ &= \lim_{n \rightarrow \infty} x_{p+31n+26} = \lim_{n \rightarrow \infty} x_{p+31n+28} \end{aligned}$$

Similarly, by (B) one can see that $\{x_{p+31n+4}\}_{n=0}^\infty$ is increasing with upper bound 1. So, the limit $T = \lim_{n \rightarrow \infty} x_{p+31n+4}$ exist and is finite.

Furthermore, from (B) one can further obtain

$$\begin{aligned} T &= \lim_{n \rightarrow \infty} x_{p+31n+4} = \lim_{n \rightarrow \infty} x_{p+31n+5} = \lim_{n \rightarrow \infty} x_{p+31n+6} \\ &= \lim_{n \rightarrow \infty} x_{p+31n+7} = \lim_{n \rightarrow \infty} x_{p+31n+9} = \lim_{n \rightarrow \infty} x_{p+31n+10} \\ &= \lim_{n \rightarrow \infty} x_{p+31n+12} = \lim_{n \rightarrow \infty} x_{p+31n+13} = \lim_{n \rightarrow \infty} x_{p+31n+14} \\ &= \lim_{n \rightarrow \infty} x_{p+31n+15} = \lim_{n \rightarrow \infty} x_{p+31n+16} = \lim_{n \rightarrow \infty} x_{p+31n+18} \\ &= \lim_{n \rightarrow \infty} x_{p+31n+22} = \lim_{n \rightarrow \infty} x_{p+31n+25} = \lim_{n \rightarrow \infty} x_{p+31n+27} \\ &= \lim_{n \rightarrow \infty} x_{p+31n+29} = \lim_{n \rightarrow \infty} x_{p+31n+30}. \end{aligned}$$

Now, it suffices to prove $S = T = 1$.

Noting that

$$x_{p+31n+25} = \frac{F(x_{p+31n+24}, x_{p+31n+22}, x_{p+31n+21}, x_{p+31n+20})}{G(x_{p+31n+24}, x_{p+31n+22}, x_{p+31n+21}, x_{p+31n+20})}, \quad (9)$$

$$x_{p+31n+24} = \frac{F(x_{p+31n+23}, x_{p+31n+21}, x_{p+31n+20}, x_{p+31n+19})}{G(x_{p+31n+23}, x_{p+31n+21}, x_{p+31n+20}, x_{p+31n+19})}. \quad (10)$$

Taking limits on both sides of the Eq.(9) and Eq.(10) respectively, we get

$$T = \frac{T^v S^u + S^{u+k} + S^{u+j} + T^v S^k + T^v S^j + S^{k+j} + T^v S^{u+k+j} + 1 + a}{S^u + T^v + S^k + S^j + T^v S^{u+k} + T^v S^{u+j} + T^v S^{k+j} + S^{u+k+j} + a}, \quad (11)$$

$$S = \frac{S^{u+v} + S^{u+k} + S^{u+j} + S^{v+k} + S^{v+j} + S^{k+j} + S^{u+v+k+j} + 1 + a}{S^u + S^v + S^k + S^j + S^{u+v+k} + S^{u+v+j} + S^{u+k+j} + S^{v+k+j} + a}, \quad (12)$$

From (12) we can get

$$\begin{aligned} (a + S^{u+v+k+j})(1 - S) + (1 - S^{u+1})(1 + S^{u+v} + S^{u+k} + S^{u+j}) \\ + (S^u - S)(S^u + S^k + S^j) = 0. \end{aligned} \quad (13)$$

From this one can see $S = 1$. Again, by(11), we have $T = 1$, too. These show that Eq.(8) is true. The proof for Theorem 5.2 is complete. ■

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