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The Absolute Continuity of a Family of Self-Similar Measures

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Abstract. For a class of self-similar measures defined by iterated function system on the line with equal equal probability weights, the absolute continuity has been proved by using a technique of polynomial. This extends a result of Garsia on infinite Bernulli convolution. Also, this result provides examples of absolutely continuous self-similar measure without weak separation condition.

Keywords: absolute continuity; self-similar measures; polynomial

1 Introduction

Let $S_j : \mathbb{R}^d \mapsto \mathbb{R}^d$ $(j = 1, 2, \cdots, m)$ be contractive maps on the d-dimensional Euclidian space $\mathbb{R}^d$, i.e.,

$$|S_j(x) - S_j(y)| \le r_j|x - y|, \quad \text{for all } x, y \in \mathbb{R}^d,$$

where $0 < r_j < 1$ and $|\cdot|$ is a metric. We call the family $\{S_j\}_{j=1}^m$ an *iterated function system* (IFS). By making use of the contraction mapping principle, Hutchinson proved the following well known existence and uniqueness theorem [1]: *There exists a unique non-empty compact subset $K \subset \mathbb{R}^d$ such that*

$$K = \bigcup_{j=1}^{m} S_j(K).$$

Furthermore, if we associate the family with a set of probability weight $\{p_j\}_{j=1}^m$ (i.e., $p_j > 0$, $j = 1, 2, \cdots, m$ and $\sum_{j=1}^m p_j = 1$), *then there exists a unique probability measure μ supported on K satisfying*

$$\mu = \sum_{j=1}^{m} p_j \mu \circ S_j^{-1}.$$

We call K the *invariant set* (or *attractor*) of the IFS, μ the *invariant measure* with respect to $\{p_j\}_{j=1}^m$.

The first systematic account of IFS seems to be that of Hutchinson [1]. The name of IFS is from Barnsley and Demko [2].

A classical case of IFS is provided in $\mathbb{R}$ by the maps

$$S_1(x) = \rho x, \quad S_2(x) = \rho(x + 1). \tag{1.1}$$

with $0 < \rho < 1$. The self-similar measure, denoted by μ_ρ, associated with the weight $\frac{1}{2}$ has been studied extensively in the context of Bernoulli convolution ([3], [4], [5] and [6]). It is known that if $0 < \rho < \frac{1}{2}$, then μ_ρ is a Cantor-type measure with Hausdorff dimension

$$\dim_{\mathrm{H}}(\mu_\rho) := \inf\{\dim_{\mathrm{H}}(E) : \ \mu_\rho(E) = 1\} = -\frac{\ln 2}{\ln \rho}.$$

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If $\rho = \frac{1}{2}$, then μ_ρ is the uniform distribution on $[0,\ \frac{1}{1-\rho}]$. For $\frac{1}{2} < \rho < 1$, the situation is completely different: it was a conjecture in the 1930's that such μ_ρ should be absolutely continuous. This was disproved by Erdös in [3]; it is then known that if $\rho = 2^{\frac{1}{k}}$, $k = 2,\ 3,\ \cdots$, or for almost all ρ in $[\frac{1}{2},\ 1)$, μ_ρ is absolutely continuous ([7], [8]). More fascinate results are known if $\beta = \rho^{-1}$ is an algebraic integer: Let $\beta_1,\ \beta_2,\ \cdots,\ \beta_s$ denote the algebraic conjugates of β, then

(a) β is a P.V. number (i.e., $|\beta| > 1$ and $|\beta_j| < 1, j = 1,\ 2,\ \cdots,\ s$) if and only if $\hat{\mu}_\rho(\xi) \not\to 0$ as $\xi \to \infty$, where $\hat{\mu}_\rho$ is the Fourier's transform of μ_ρ. In particular μ_ρ is singular [6].

(b) If $\beta \prod_{|\beta_j|>1} \beta_j = 2$, then μ_ρ is absolutely continuous. Note that in this case $|\beta_j| > 1$, $j = 1,\ 2,\ \cdots,\ s$, necessarily [4].

Lau ([5]) introduced a concept, F-number, which is equivalent to that the IFS (1.1) satisfies the finite type condition ([9], [10]). Assuming $\beta \hat{=} \rho^{-1}$ to be an F-number, Lau studied the α-mean quadratic variation (m.q.v.) of μ_ρ and proved that μ_ρ is singular.

For other related topics, please see [11], [12], [13] and [14].

Our main purpose in this paper is to extend the idea of [4] to the IFS

$$S_j(x) = \rho(x + j), \quad j = 0,\ 1,\ \cdots,\ m-1, \quad x \in \mathbb{R}, \quad 0 < \rho < 1, \tag{1.2}$$

and the self-similar measure defined by

$$\mu(\cdot) = \sum_{j=0}^{m-1} \frac{1}{m} \mu \circ S_j^{-1}(\cdot). \tag{1.3}$$

If we let $\{\varepsilon_n\}_{n>0}$ be a sequence of identically and independently distributed (i.i.d.) random variables with the common distribution $P(\varepsilon_i = k) = \frac{1}{m}$, $k = 0,\ 1,\ \cdots,\ m-1$. Then μ is the distribution of the random variable

$$Y = \sum_{n=1}^{\infty} \varepsilon_n \rho^n,$$

so we will consider μ as a probability distribution of Y. Let

$$Y_k = \sum_{n=1}^{k} \varepsilon_n \rho^n, \quad k = 1,\ 2,\ \cdots,\ N,\ \cdots$$

and let $D_k = \{y_k^1,\ y_k^2,\ \cdots,\ y_k^{N_k}\}$ be the set of all distinct possible values of Y_k.

Our main theorem is

Theorem 1.1 *If $\rho^{-1} \in (1,\ m)$ is an algebraic integer with minimal polynomial $p(x)$ such that $p(0) = m$ and all conjugates of ρ^{-1} are larger than one in modular, then the μ defined in (1.3) is absolutely continuous with a density bounded by*

$$\frac{(m-1)^s}{\prod_{i=1}^{s} (|\beta_i| - 1)}$$

where $\{\beta_1,\ \beta_2,\ \cdots,\ \beta_s\}$ is the set of conjugates of ρ^{-1}.

2 The Proof Of The Theorem

We first give a sufficient condition for μ to be absolutely continuous with a bounded density.

Lemma 2.1 *If there is a constant $c > 0$ and n_0 such that*

$$P(Y_n = y_n^i) \le c \inf_{j:j \ne i} |y_n^i - y_n^j| \tag{2.1}$$

for all $i \in \{1,\ 2,\ \cdots,\ N_n\}$ and $n > n_0$. Then μ is absolutely continuous with a density bounded by c.

Proof. Since $K = [0,\ \frac{\rho(m-1)}{1-\rho}]$ is an interval, it has non singleton. Hence $\lim_{n\to+\infty} \inf_{i\neq j} |y_n^i - y_n^j| = 0$ and so

$$\lim_{n\to+\infty} P(Y_n = y_n^i) = 0, \quad \forall\, i. \tag{2.2}$$

For any given interval $(a,\ b]$, let

$$a < y_n^{i_1} < y_n^{i_2} < \cdots < y_n^{i_p} \leq b.$$

be the set of all the possible values of Y_n belonging to $(a,\ b]$. Then the assumption (2.1) implies

$$\begin{aligned} c(b-a) >& c(y_n^{i_p} - y_n^{i_{p-1}}) + c(y_n^{i_{p-1}} - y_n^{i_{p-2}}) + \cdots + c(y_n^{i_2} - y_n^{i_1}) \\ \geq& P(Y_n = y_n^{i_p}) + P(Y_n = y_n^{i_{p-1}}) + \cdots + P(Y_n = y_n^{i_2}) \\ \geq& P(Y_n \leq b) - P(Y_n \leq a) - P(Y_n = y_n^{i_1}). \end{aligned}$$

Let $n \to +\infty$, then (2.2) and the above inequality imply

$$c(b-a) \geq \mu((a,\ b])) \geq 0,$$

therefore μ is absolutely continuous with a density bounded by c. □

Lemma 2.2 *Let*

$$\mathcal{A}_n = \{(a_1,\ a_2,\ \cdots,\ a_n):\ a_j \in \{0,\ \pm1,\ \cdots,\ \pm(m-1)\}\},$$

$$m_n(\rho) = \min\{|\sum_{j=1}^{n} a_j\rho^j|:\ (a_1,\ a_2,\ \cdots,\ a_n) \in \mathcal{A}_n \setminus \{(0,\ 0,\ \cdots,\ 0)\}\}.$$

If

$$\lim \inf_{n\to+\infty} m_n(\rho) m^n = c > 0,$$

then μ is absolutely continuous with a density bounded by c^{-1}.

Proof. For any constant $\delta \in (0,\ c)$, the assumption implies that there exist n_0 such that

$$m_n(\rho) > m^{-n}\delta, \qquad \forall\, n > n_0.$$

This means that Y_n has m^n distinct possible values and any two distinct possible values y_n and y_n' of the random variable Y_n satisfy

$$|y_n - y_n'| > m^{-n}\delta = \delta P(Y_n = y_n) = \delta P(Y_n = y_n')$$

when $n > n_0$. From Lemma 2.1, we see that μ is absolutely continuous with a density bounded by δ^{-1}.

Therefore, μ is absolutely continuous with a density bounded by c^{-1}. □

Remark. Since all possible values of Y_n are in $[0,\ \frac{\rho(m-1)}{1-\rho})$ and Y_n has at most m^n deferent possible values. Note the definition of $m_n(\rho)$, we see that

$$m^n m_n(\rho) \leq \frac{\rho(m-1)}{1-\rho} < +\infty.$$

In the following, we will prove Theorem 1.1 by several lemmas.

We first prove some results about algebraic integers. For a polynomial $p(x) = \sum_{j=0}^{n} a_j x^j$, we call $\max\{|a_j|\}$ the height of $p(x)$. Let β be an algebraic integer greater than one, $\beta_1,\ \beta_2,\ \cdots,\ \beta_s$ denote the conjugates of β and σ denote the number of i such that $|\beta_i| = 1$. Then we have the following lemma.

Lemma 2.3 *For an algebraic integer $\beta > 1$, if β is not a root of any polynomial with height $l \in \{1, 2, \cdots, m-1\}$, then*

$$m_n(\frac{1}{\beta}) \geq \frac{\prod\limits_{|\beta_i|\neq 1} ||\beta_i| - 1|}{n^\sigma(\beta \prod\limits_{|\beta_i|>1} |\beta_i|)^n (m-1)^s}. \tag{2.3}$$

Proof. First we prove a result about integral polynomials. Let

$$\begin{aligned} l\tau_1 &= x_0 + x_1 + \cdots + x_s \\ \tau_2 &= \sum_{0\leq i<j\leq s} x_i x_j \\ &\cdots \\ \tau_{s+1} &= x_0 x_1 \cdots x_s \end{aligned}$$

Claim. If an integral polynomial $f(x_0, x_1, \cdots, x_s)$ is symmetric, then it is also an integral polynomial of $\tau_1, \tau_2, \cdots, \tau_{s+1}$.

Proof of the claim. For the case $s = 0$, it is obviously true. Assume that it is true when $s < r$. We consider the case $s = r$.

If f has degree zero, it is obviously true. Assume it is true when f has degree less than q. Consider the case that f has degree q. Let $x_s = 0$, then the inductive assumption implies that there exists an integral polynomial g_1 such that

$$f(x_0, x_1, \cdots, x_{s-1}, 0) = g_1(\tau_{1,0}, \tau_{2,0}, \cdots, \tau_{s,0}),$$

where $\tau_{i,0} = \tau_i|_{x_s=0}$. Let

$$f_1(x_0, x_1, x_2, \cdots, x_s) = f(x_0, x_1, \cdots, x_s) - g_1(\tau_1, \tau_2, \cdots, \tau_s),$$

then f_1, which may be zero, is a symmetric polynomial of $x_0, x_1, \cdots, x_s$ with degree at most q and $f_1(x_0, x_1, \cdots, x_{s-1}, 0) = 0$, i.e. $f_1(x_0, x_1, \cdots, x_s)$ has a factor x_s. The fact that $f_1(x_0, x_1, \cdots, x_s)$ is symmetric ensures that $f_1(x_0, x_1, x_2, \cdots, x_s)$ has a factor τ_{s+1}. Let $f_1(x_0, x_1, x_2, \cdots, x_s) = \tau_{s+1} f_2(x_0, x_1, \cdots, x_s)$, then $f_2(x_0, x_1, x_2, \cdots, x_s)$ is also a symmetric integral polynomial of $x_0, x_1, x_2, \cdots, x_s$ with degree less than q. Hence our inductive assumption implies that there exists an integral polynomial g_2 of $\tau_1, \tau_2, \cdots, \tau_{s+1}$ such that

$$f_2(x_0, x_1, \cdots, x_s) = g_2(\tau_1, \tau_2, \cdots, \tau_{s+1}).$$

Therefore

$$\begin{aligned} f(x_0, x_1, \cdots, x_s) =& g_1(\tau_1, \tau_2, \cdots, \tau_{s+1}) + f_1(x_0, x_1, \cdots, x_s) \\ =& g_1(\tau_1, \tau_2, \cdots, \tau_{s+1}) + \tau_{s+1} g_2(\tau_1, \tau_2, \cdots, \tau_{s+1}) \end{aligned}$$

is an integral polynomial of $\tau_1, \tau_2, \cdots, \tau_{n+1}$. The claim is proven.

For any $a_1, a_2, \cdots, a_n \in \{0, \pm 1, \cdots, \pm(m-1)\}$, let $p(x) = \sum_{j=1}^n a_j x^{n-j}$ and $\beta = \beta_0$. Since $\varphi(x_0, x_1, \cdots, x_s) \hat{=} p(x_0)p(x_1)\cdots p(x_s)$ is a symmetric integral polynomial, so the above claim indicates that $\varphi(\beta_1, \beta_2, \cdots, \beta_s)$ is an integral polynomial of $\sum\limits_{i=0}^{s} \beta_i$, $\sum\limits_{0\leq i<j\leq s} \beta_i\beta_j, \cdots, \beta_0\beta_1\cdots\beta_s$.

On the other hand, the minimal polynomial of β is $\prod_{i=0}^s (x - \beta_i)$, its coefficients are $\beta_0 + \beta_1 + \cdots + \beta_s$, $\sum\limits_{0\leq i<j\leq s} \beta_i\beta_j, \cdots, \beta_0\beta_1\cdots\beta_s$, respectively. Hence $\beta_0 + \beta_1 + \cdots + \beta_s$, $\sum\limits_{0\leq i<j\leq s} \beta_i\beta_j, \cdots, \beta_0\beta_1\cdots\beta_s$ are all integers. Therefore

$$\varphi(\beta_0, \beta_1, \cdots, \beta_s) \hat{=} p(\beta_0)p(\beta_1)\cdots p(\beta_s)$$

is an integer. Using the assumption gives that $p(\beta_0) \neq 0$ and $p(\beta_j) \neq 0$, so

$$|p(\beta_0)p(\beta_1)\cdots p(\beta_s)| \geq 1. \tag{2.4}$$

For every β_i, we always have

$$|p(\beta_i)| \leq (m-1)(1+|\beta_i|+\cdots+|\beta_i|^{n-1})$$
$$\leq \begin{cases} ll\dfrac{m-1}{1-|\beta_i|} \text{ if } |\beta_i|<1 \\ n(m-1) \text{ if } |\beta_i|=1 \\ \dfrac{(m-1)\,|\beta_i|^n}{|\beta_i|-1} \text{ if } |\beta_i|>1 \end{cases}$$

Combining the above inequality and (2.4) gives that

$$p(\beta) \geq \frac{\prod\limits_{|\beta_i|\neq 1} ||\beta_i|-1|}{n^\sigma(\prod\limits_{|\beta_i|>1} |\beta_i|)^n(m-1)^s}.$$

Therefore

$$m_n(\frac{1}{\beta}) = \beta^{-n}\min\{p(\beta)\}$$
$$\geq \frac{\prod\limits_{|\beta_i|\neq 1} ||\beta_i|-1|}{n^\sigma(\beta\prod\limits_{|\beta_i|>1} |\beta_i|)^n(m-1)^s},$$

the lemma follows. □

Proof of Theorem 1.1

Let $p(x)$ be the minimal polynomial of β. Since $p(0)=m$, β is not a root of any polynomial with height less than m. Hence Lemma 2.3 implies that the inequality (2.3) holds for all n.

Since all conjugates of ρ^{-1} are larger than one in modular, i.e. $|\beta_i|>1$ for all i, so

$$m_n(\frac{1}{\beta})m^n \geq \frac{\prod\limits_{i=1}^{s}(|\beta_i|-1)}{(m-1)^s}.$$

Therefore, Lemma 2.2 implies that μ is absolutely continuous with a density bounded by $\frac{(m-1)^s}{\prod_{i=1}^{s}(|\beta_i|-1)}$. □

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