

## On the Well-posedness of the Degasperis-Procesi Equation with the Dispersive Term

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**Abstract:** We investigate well-posedness in classes of discontinuous functions for the nonlinear and third order dispersive Degasperis-Procesi equation:

$$\partial_t u - \partial_t^3 u + \partial_x u - \partial_{xxx}^3 u + 4u\partial_x u = 3\partial_x u \partial_{xx}^2 u + u \partial_{xxx}^3 u \quad (DP)$$

This equation can be regarded as a model for shallow water dynamics and its asymptotic accuracy is the same as for the Camassa-Holm equation (one order more accurate than the KdV equation). We prove existence and  $L^1$  stability (uniqueness) results for entropy weak solutions belonging to the class  $L^1 \cap BV$ , while existence of at least one weak solution, satisfying a restricted set of entropy inequalities, is proved in the class  $L^2 \cap L^4$ .

**Key words:** dispersive term; weak solution; entropy condition; hyperbolic equation

### 1 Introduction

Our aim is to investigate well-posedness in classes of discontinuous functions for the Degasperis-Procesi equation[1]

$$\partial_t u - \partial_{txx}^3 u + \partial_x u - \partial_{xxx}^3 u + 4u\partial_x u = 3\partial_x u \partial_{xx}^2 u + u \partial_{xxx}^3 u, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R} \quad (1)$$

with an initial condition  $u_0$ :

$$u(0, x) = u_0(x) \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) \quad (2)$$

### 2 Viscous approximation and a priori estimates

We will prove existence of a solution to the Cauchy problem (1), (2) by analyzing the limiting behavior of a sequence of smooth functions  $\{u_\varepsilon\}_{\varepsilon>0}$ , where each function  $u_\varepsilon$  solves the following viscous problem:

$$\begin{cases} \partial_t u_\varepsilon - \partial_{txx}^3 u_\varepsilon + \partial_x u_\varepsilon + \partial_{xxx}^3 u_\varepsilon + 4u_\varepsilon \partial_x u_\varepsilon \\ \quad = 3\partial_x u_\varepsilon \partial_{xx}^2 u_\varepsilon + u_\varepsilon \partial_{xxx}^3 u_\varepsilon + \varepsilon \partial_{xx}^2 u_\varepsilon - \varepsilon \partial_{xxxx}^4 u_\varepsilon, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u_\varepsilon(0, x) = u_{0,\varepsilon}(x), & x \in \mathbb{R} \end{cases} \quad (3)$$

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This problem can be stated equivalently as a parabolic-elliptic system:

$$\begin{cases} \partial_t u_\varepsilon + \partial_x \left( \frac{u_\varepsilon^2}{2} \right) + \partial_x u_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_{xx}^2 u_\varepsilon, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ -\partial_{xx}^2 P_\varepsilon + P_\varepsilon = \frac{3}{2} u_\varepsilon^2, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u_\varepsilon(0, x) = u_{0,\varepsilon}(x), & x \in \mathbb{R} \end{cases} \quad (4)$$

Observe that we have an explicit expression for  $P_\varepsilon$  in terms of  $u_\varepsilon$ :

$$P_\varepsilon(t, x) = P^{u_\varepsilon}(t, x) = G_1 * \left( \frac{3}{2} u_\varepsilon^2 \right)(t, x) = \frac{3}{4} \int_{\mathbb{R}} e^{-|x-y|} (u_\varepsilon(t, y))^2 dy$$

To begin with, we assume in this section that

$$u_0 \in L^2(\mathbb{R}), \quad (5)$$

and

$$u_{0,\varepsilon} \in H^l(\mathbb{R}), \quad l \geq 2, \quad \|u_{0,\varepsilon}\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}, \quad u_{0,\varepsilon} \rightarrow u_0 \in L^2(\mathbb{R}). \quad (6)$$

We will impose additional condition on the initial data as we make progress.

We begin by stating a lemma as which shows that the viscous problem (3) is well-posed for each fixed  $\varepsilon > 0$ .

**Lemma 2.1** *Assume (5) and (6) hold, and fix any  $\varepsilon > 0$ , then there exists a unique global smooth solution  $u_\varepsilon = u_\varepsilon(t, x)$  to the Cauchy Problem (4) belonging to  $C([0, \infty); H^l(\mathbb{R}))$ .*

**Proof.** We omit the proof since it is similar to the one found in [4, Theorem 2.3] ■

## 2.1 $L^2$ estimates and some consequences

Next we prove a uniform of  $L^2$  bound on the approximate solution  $u_\varepsilon$ , which reinforces the whole analysis in this paper.

**Lemma 2.2** (Energy estimate) *Assume (5) and (6) hold, and fix any  $\varepsilon > 0$ , then the following bounds hold for any  $t \geq 0$ :*

$$\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq 2\sqrt{2} \|u_0\|_{L^2(\mathbb{R})}, \quad \sqrt{\varepsilon} \|\partial_x u_\varepsilon\|_{L^2(\mathbb{R}_+ \times \mathbb{R})} \leq 2 \|u_0\|_{L^2(\mathbb{R})}$$

For the proof of this Lemma we introduce the quantity  $v_\varepsilon = v_\varepsilon(t, x)$ :

$$v_\varepsilon(t, x) = (G_2 * u_\varepsilon) = \int_{\mathbb{R}} e^{-2|x-y|} u_\varepsilon(t, y) dy, \quad t \geq 0, \quad x \in \mathbb{R}.$$

Since  $G_2(x) = e^{-2|x|}$  is Green's function of operator  $4 - \partial_{xx}^2$ , we see that  $v_\varepsilon$  also satisfies the equation

$$-\partial_{xx}^2 v_\varepsilon + 4v_\varepsilon = u_\varepsilon \quad \text{in } \mathbb{R}_+ \times \mathbb{R} \quad (7)$$

The use of the quantity  $v_\varepsilon$  is motivated by the fact that  $\int_{\mathbb{R}} (u - u_{xx}) dx$  is a conserved quantity, where  $4v_\varepsilon - \partial_{xx}^2 v_\varepsilon = u_\varepsilon$  and  $u$  solves (1) (see [3, 6, 8]).

To prove Lemma 2.2 we shall need the following estimates on  $v_\varepsilon$ :

**Lemma 2.3** *Assume (5) and (6) hold, and fix any  $\varepsilon > 0$ , then the following identity holds for any  $t \geq 0$ :*

$$\begin{aligned} & \|\partial_{xx}^2 v_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 5 \|\partial_x v_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 4 \|v_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & + 2\varepsilon \left( \int_0^t \left( \|\partial_{xxx}^3 v_\varepsilon(\tau, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + 5 \|\partial_{xx}^2 v_\varepsilon(\tau, \cdot)\|_{L^2(\mathbb{R})}^2 + 4 \|\partial_x v_\varepsilon\|_{L^2(\mathbb{R})}^2 \right) d\tau \\ & = \|\partial_{xx}^2 v_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R})}^2 + 5 \|\partial_x v_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R})}^2 + 4 \|v_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R})}^2 \end{aligned} \quad (8)$$

**Proof.** Multiplying the first equation of (4) by  $v_\varepsilon - \partial_{xx}^2 v_\varepsilon$  and integrating over  $R$ , we get

$$\begin{aligned} \int_{\mathbb{R}} \partial_t u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx + \int_{\mathbb{R}} \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx - \varepsilon \int_{\mathbb{R}} \partial_{xx}^2 (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ = - \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx - \int_{\mathbb{R}} \partial_x P_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \end{aligned} \quad (9)$$

For the left-hand side of this identity, using (7), we have

$$\begin{aligned} \int_{\mathbb{R}} \partial_t u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx + \int_{\mathbb{R}} \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx - \varepsilon \int_{\mathbb{R}} \partial_{xx}^2 (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ = \int_{\mathbb{R}} (4\partial_t v_\varepsilon - \partial_{txx}^3 v_\varepsilon) (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx + \int_{\mathbb{R}} (4v_\varepsilon - \partial_{xx}^2 v_\varepsilon) (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ - \varepsilon \int_{\mathbb{R}} (4\partial_{xx}^2 v_\varepsilon - \partial_{xxxx}^4 v_\varepsilon) (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ = \frac{1}{2} \frac{d}{dt} \left( 4v_x^2 + 5(\partial_x v_\varepsilon)^2 + (\partial_{xx}^2 v_\varepsilon)^2 \right) dx + \varepsilon \int_{\mathbb{R}} \left( 4(\partial_x v_\varepsilon)^2 + 5(\partial_{xx}^2 v_\varepsilon)^2 + (\partial_{xxx}^3 v_\varepsilon)^2 \right) dx \end{aligned} \quad (10)$$

For the right-hand side of (9), we calculate

$$\begin{aligned} - \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx - \int_{\mathbb{R}} \partial_x P_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ = - \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx - \int_{\mathbb{R}} (P_\varepsilon - \partial_{xx}^2 P_\varepsilon) (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ = - \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (4v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ = - \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon dx = 0 \end{aligned} \quad (11)$$

Substituting (10) and (11) into (9) yields

$$\frac{d}{dt} \left( 4v_x^2 + 5(\partial_x v_\varepsilon)^2 + (\partial_{xx}^2 v_\varepsilon)^2 \right) dx + 2\varepsilon \int_{\mathbb{R}} \left( 4(\partial_x v_\varepsilon)^2 + 5(\partial_{xx}^2 v_\varepsilon)^2 + (\partial_{xxx}^3 v_\varepsilon)^2 \right) dx = 0$$

Integrating this inequality over  $[0, t]$  we obtain (8). ■

We conclude this subsection with some bounds on the nonlocal term  $P_\varepsilon$ , which all are consequences of the  $L^2$  bound in Lemma 2.2.

**Lemma 2.4** Assume (5) and (6) hold, and fix any  $\varepsilon > 0$ , then

$$P_\varepsilon \geq 0, \quad (12)$$

$$\|P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})}, \|\partial_x P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 12 \|u_0\|_{L^2(\mathbb{R})}^2, \quad t \geq 0, \quad (13)$$

$$\|P_\varepsilon\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})}, \|\partial_x P_\varepsilon\|_{L^2(\mathbb{R}_+ \times \mathbb{R})} \leq 6 \|u_0\|_{L^2(\mathbb{R})}^2,$$

$$\|\partial_x^2 P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 24 \|u_0\|_{L^2(\mathbb{R})}^2, \quad t \geq 0.$$

## 2.2 $L^1$ estimate

As a bounded consequence of  $L^2$  in Lemma 2.2, we can bound  $u_\varepsilon$  in  $L^1$ , as long as we assume, in additions to (5) and (6),

$$u_0, u_{0,\varepsilon} \in L^1(\mathbb{R}), \quad \|u_{0,\varepsilon}\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}. \quad (14)$$

**Lemma 2.5** ( $L^1$ -estimate). Assume (5), (6), and (14) hold, and fix any  $\varepsilon > 0$ , then

$$\|u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} + 12 \|u_0\|_{L^2(\mathbb{R})}^2, \quad t \geq 0. \quad (15)$$

**Proof.** Let  $\eta \in C^2(\mathbb{R})$  and  $q : \mathbb{R} \rightarrow \mathbb{R}$  be such that  $q'(u) = uq'(u)$ . By multiplying the first equation in (4) with using the chain rule, we get

$$\partial_t \eta(u_\varepsilon) + \partial_x \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) + \eta'(u_\varepsilon) \partial_x P_\varepsilon = \varepsilon \partial_{xx}^2 \eta(u_\varepsilon) - \eta''(u_\varepsilon) (\partial_x u_\varepsilon)^2 \quad (16)$$

Choosing  $\eta(u) = |u|$  (modulo an approximation argument) and then integrating the resulting equation over  $\mathbb{R}$  yield

$$\frac{d}{dt} \int_{\mathbb{R}} |u_\varepsilon| dx \leq \int_{\mathbb{R}} \text{sign}(u_\varepsilon) \partial_x P_\varepsilon dx$$

By (13),

$$\int_{\mathbb{R}} \text{sign}(u_\varepsilon) \partial_x P_\varepsilon dx \leq \|\partial_x P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 12 \|u_0\|_{L^2(\mathbb{R})}^2,$$

and hence

$$\frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 12 \|u_0\|_{L^2(\mathbb{R})}^2 \quad (17)$$

Integrating (17) over  $[0, t]$  we can obtain (15).

### 2.3 BV and $L^\infty$ estimates

In this subsection we derive supplementary a priori estimate for the viscous approximations, which also are consequences of the  $L^2$  bound in Lemma 2.2. In particular, we prove that the sequence  $\{u_\varepsilon\}_{\varepsilon>0}$  is bounded in BV, which yields strong compactness of this sequence. To this end, we need to assume, in addition to (5) and (6),

$$u_0, u_{0,\varepsilon} \in BV(\mathbb{R}), \quad |u_{0,\varepsilon}|_{BV(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})}. \quad (18)$$

**Lemma 2.6** (BV estimate in space). Assume (5), (6), and (18) hold, and fix any  $\varepsilon > 0$ , then

$$\|\partial_x u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})} + 24t \|u_0\|_{L^2(\mathbb{R})}^2, \quad t \geq 0. \quad (19)$$

**Proof.** Set  $q_\varepsilon := \partial_x u_\varepsilon$ . Then  $q_\varepsilon$  satisfies the equation

$$\partial_t q_\varepsilon + \partial_x q_\varepsilon + u_\varepsilon \partial_x q_\varepsilon + q_\varepsilon^2 + \partial_{xx}^2 P_\varepsilon = \varepsilon \partial_{xx}^2 q_\varepsilon.$$

If  $\eta \in C^2(\mathbb{R})$  and  $q : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $q'(u) = uq'(u)$ , then by the chain rule

$$\begin{aligned} \partial_t \eta(q_\varepsilon) + \partial_x \eta(q_\varepsilon) + \partial_x (u_\varepsilon q_\varepsilon) - q_\varepsilon \eta(q_\varepsilon) + \eta'(q_\varepsilon) q_\varepsilon^2 \\ + \eta'(u_\varepsilon) \partial_{xx}^2 P_\varepsilon = \varepsilon \partial_{xx}^2 \eta(q_\varepsilon) - \eta''(q_\varepsilon) (\partial_x q_\varepsilon)^2. \end{aligned}$$

Choosing  $\eta(u) = |u|$  (modulo an approximation argument) and then integrating the resulting equation over  $\mathbb{R}$  yield

$$\int_{\mathbb{R}} \text{sign}(q_\varepsilon) \partial_{xx}^2 P_\varepsilon dx \leq \|\partial_{xx}^2 P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 24 \|u_0\|_{L^2(\mathbb{R})}^2,$$

and hence

$$\frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 24 \|u_0\|_{L^2(\mathbb{R})}^2. \quad (20)$$

Integrating (20) over  $[0, t]$  we get (19). ■

**Lemma 2.7** ( $L^\infty$ -estimate[7]). Assume (5), (6), (18) hold, and fix any  $\varepsilon > 0$ , then

$$\|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})} + 24 \|u_0\|_{L^2(\mathbb{R})}^2, \quad t \geq 0. \quad (21)$$

**Lemma 2.8** (BV estimate in time). Assume (5), (6), and (18) hold, and fix any  $\varepsilon > 0$ , then

$$\|\partial_t u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq C_t, \quad t \geq 0.$$

where the constant  $C_t := \left(|u_0|_{BV(\mathbb{R})} + 24t \|u_0\|_{L^2(\mathbb{R})}^2\right)^2 + \left(|u_0|_{BV(\mathbb{R})} + 24t \|u_0\|_{L^2(\mathbb{R})}^2\right) + 12 \|u_0\|_{L^2(\mathbb{R})}^2$  is independent of  $\varepsilon$  but dependent on  $t$ .

**Proof.** We have, by (21), (19), and (12),

$$\begin{aligned} \|\partial_t u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} &\leq \int_{\mathbb{R}} |\partial_x u_\varepsilon| dx + \int_{\mathbb{R}} |u_\varepsilon \partial_x u_\varepsilon| dx + \int_{\mathbb{R}} \partial_x P_\varepsilon dx \\ &\leq |u_\varepsilon(t, \cdot)|_{BV} + \|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} |u_\varepsilon(t, \cdot)|_{BV} + \|\partial_x P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq C_t. \end{aligned}$$

■

**Lemma 2.9** . Assume (5), (6), and (18) hold, and fix any  $\varepsilon > 0$ . Then

$$\|\partial_{xx}^2 P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq 6 \|u_0\|_{L^2(\mathbb{R})}^2 + \frac{3}{2} \left( |u_0|_{BV(\mathbb{R})} + 24t \|u_0\|_{L^2(\mathbb{R})}^2 \right)^2,$$

for any  $t \geq 0$ .

## 2.4 $L^4$ -estimate and Oleinik type estimate

Next we prove that the viscous approximation are uniformly bounded in  $L^4$ , a fact that we use later to prove the existence of at least one weak solution to (1), (2) under the mere assumption that  $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$  holds. For this purpose, we need to assume, in addition to (5) and (6),

$$u_0, u_{0,\varepsilon} \in L^4(\mathbb{R}), \quad \|u_{0,\varepsilon}\|_{L^4(\mathbb{R})} \leq \|u_0\|_{L^4(\mathbb{R})}. \quad (22)$$

**Lemma 2.10** ( $L^4$  - estimate). Assume (5), (6), and (22) hold, and fix any  $\varepsilon > 0$ , then

$$\|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 \leq e^{12\|u_0\|_{L^2(\mathbb{R})}^2} \|u_0\|_{L^4(\mathbb{R})}^4 + 8 \|u_0\|_{L^2(\mathbb{R})}^2 \left( e^{12t\|u_0\|_{L^2(\mathbb{R})}^2} - 1 \right),$$

for any  $t \geq 0$ .

**Lemma 2.11** (Oleinik type estimate ). Assume (5), (6), and (18) hold, and fix any  $\varepsilon > 0$ , then for each  $t \in (0, T]$ , with  $T > 0$  being fixed,

$$\partial_x u_\varepsilon(t, x) \leq \frac{1}{t} + K_T, \quad x \in \mathbb{R},$$

where  $K_T := \left[ 6 \|u_0\|_{L^2(\mathbb{R})}^2 + \frac{3}{2} \left( |u_0|_{BV(\mathbb{R})} + 24T \|u_0\|_{L^2(\mathbb{R})}^2 \right)^2 \right]^{1/2}$

## 3 Well-posedness in $L^1 \cap BV$

Relying on the a priori estimate derive in Section 2, we prove existence, uniqueness, and  $L^1$  stability of entropy weak solutions to (1), (2), under the  $L^1 \cap BV$  assumption  $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$ .

**Theorem 3.1** (well-posedness ). Suppose  $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$ , then there exists an entropy weak solution to the Cauchy problem (1), (2). Fix any  $T > 0$ , and let  $u, v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  be two entropy weak solutions to (1), (2) with initial data  $u_0, v_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$ , respectively.

Then for any  $t \in (0, T)$

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^1(\mathbb{R})} \leq e^{M_T t} \|u_0 - v_0\|_{L^1(\mathbb{R})},$$

where

$$M_T := \frac{3}{2} \left( \|u\|_{L^\infty((0,T) \times \mathbb{R})} \right) + \|v\|_{L^\infty((0,T) \times \mathbb{R})} < \infty.$$

Consequently, there exists at most one entropy weak solution to (1), (2).

The entropy weak solution  $u$  satisfies the following estimates for any  $t \in (0, T)$  :

$$\|u(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} + 12t \|u_0\|_{L^2(\mathbb{R})}^2, \quad (23)$$

$$\|u(t, \cdot)\|_{BV(\mathbb{R})} \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{BV(\mathbb{R})} + 24t \|u_0\|_{L^2(\mathbb{R})}^2, \quad (24)$$

$$\|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 \leq e^{12\|u_0\|_{L^2(\mathbb{R})}^2 t} \|u_0\|_{L^4(\mathbb{R})}^4 + 8 \|u_0\|_{L^2(\mathbb{R})}^2 \left( e^{12\|u_0\|_{L^2(\mathbb{R})}^2 t} - 1 \right). \quad (25)$$

Furthermore,

$$\|u(t_2, \cdot) - u(t_1, \cdot)\|_{L^1(\mathbb{R})} \leq C_T |t_2 - t_1|, \quad \forall t_1, t_2 \in [0, t], \quad (26)$$

where  $C_T := \left( \|u_0\|_{L^1(\mathbb{R})} + 12T \|u_0\|_{L^2(\mathbb{R})}^2 \right)^2 + \left( \|u_0\|_{L^1(\mathbb{R})} + 12T \|u_0\|_{L^2(\mathbb{R})}^2 \right) + 12 \|u_0\|_{L^2(\mathbb{R})}^2$ .

Finally, the following Oleinik type estimate holds for a.e.  $(t, x) \in (0, T] \times \mathbb{R}$ ,

$$\partial_x u(t, x) \leq \frac{1}{t} + K_T, \quad (27)$$

where  $K_T := \left[ 6 \|u_0\|_{L^2(\mathbb{R})}^2 + \frac{3}{2} \left( \|u_0\|_{BV(\mathbb{R})} + 24T \|u_0\|_{L^2(\mathbb{R})}^2 \right)^2 \right]^{1/2}$ .

This theorem is an immediate consequence of Theorems 3.2, 3.3 and Corollary 3.1.

**Theorem 3.2** (Existence of entropy weak solutions) Suppose (2) holds, then there exists at least one entropy weak solution to (1), (2).

**Proof.** We assume that the approximating sequence  $\{u_{0,\varepsilon}\}_{\varepsilon>0}$  is chosen such that (5), (6), and (14) hold. Then, in view of the a priori estimates obtained in Section 2, it takes a standard argument to see that there exists a sequence of strictly positive numbers  $\{\varepsilon_k\}_{k=1}^\infty$  tending to zero such that  $u_{\varepsilon_k} \rightarrow u$  a.e. while  $k \rightarrow \infty$  in  $\mathbb{R}_+ \times \mathbb{R}$ , and hence  $u_{\varepsilon_k} \rightarrow u$  in  $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R})$  for all  $p \in [1, \infty)$ . Thanks to (15) and (21), there also holds  $u_{\varepsilon_k} \rightarrow u$  in  $L^p_{loc}((0, T) \times \mathbb{R}) \quad \forall T > 0, \quad \forall p \in [1, \infty)$ .

The a priori estimates in Section 2 imply immediately that the limit function  $u$  satisfied (23)-(27).

Let us now prove that as  $k \rightarrow \infty$ ,  $p_{\varepsilon_k} \rightarrow p^u$ ,  $\partial_x p_{\varepsilon_k} \rightarrow \partial_x p^u$  in  $L^p((0, T) \times \mathbb{R})$ ,  $\forall T > 0, \quad \forall p \in [1, \infty)$ ,

Which following from the following calculation:

$$\begin{aligned} & \|P_{\varepsilon_k} - P^u\|_{L^p((0,T)\times\mathbb{R})}^p, \quad \|\partial_x P_{\varepsilon_k} - \partial_x P^u\|_{L^p((0,T)\times\mathbb{R})}^p \\ & \leq \left(\frac{3}{4}\right)^p \int \int_{\Pi_T} \left( \int_{\mathbb{R}} e^{-|x-y|} \left| (u_{\varepsilon_k}(t, y))^2 - (u(t, y))^2 \right| dy \right)^p dx dt \\ & \leq \left(\frac{3}{4}\right)^p \int \int_{\Pi_T} \left( \int_{\mathbb{R}} e^{-|x-y|} e^{-|x-y|/p} \left| (u_{\varepsilon_k}(t, y))^2 - (u(t, y))^2 \right| dy \right)^p dx dt \\ & \leq \left(\frac{3}{4}\right)^p \int \int_{\Pi_T} \left( \int_{\mathbb{R}} e^{-|x-y|} dy \right)^{p-1} \times \left( \int_{\mathbb{R}} e^{-|x-y|} \left| (u_{\varepsilon_k}(t, y))^2 - (u(t, y))^2 \right|^p dy \right) dx dt \\ & \leq \left(\frac{3}{2}\right)^p \int \int_{\Pi_T} \left| (u_{\varepsilon_k}(t, y))^2 - (u(t, y))^2 \right|^p dy dt \rightarrow 0 \\ & \leq C_T \int \int_{\Pi_T} |u_{\varepsilon_k}(t, y) - u(t, y)|^p dy dt \rightarrow 0 \quad \text{as } k \rightarrow \infty \end{aligned}$$

where  $\Pi_T := (0, T) \times \mathbb{R}$ . ■

**Theorem 3.3** ( $L^1$  stability). Let  $u$  and  $v$  be two entropy weak solution of (1) with initial data  $u(0, \cdot) = u_0$  and  $v(0, \cdot) = v_0$  satisfying (2). Fix any  $T > 0$ , then

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^1(\mathbb{R})} \leq e^{M_T t} \|u_0 - v_0\|_{L^1(\mathbb{R})}, \quad t \in (0, T),$$

We omit the proof since it is similar to the one of found in [2, Theorem3.3].

**Corollary 3.1** (Uniqueness) Suppose  $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$ , then the Cauchy problem (1), (2) admits at most one entropy weak solution.

This is an immediate consequence of Theorem 3.3.

## 4 Existence in $L^2 \cap L^4$

In this section we prove that there exists at least one weak solution to (1), (2) under assumption  $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ , in which case we are outside the  $BV/L^\infty$  framework considered in Section 3. Since no  $L^\infty$  bound is available we can only prove that this weak solution satisfies the entropy inequality for convex  $C^2$  entropies possessing a bounded second order derivative. Be that as it may, we are not able to prove  $L^1$  stability/uniqueness based on this restricted class of entropies.

Our main result is the following theorem:

**Theorem 4.1** (Existence) Suppose  $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$  holds, then there exists a function  $u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R})) \cap L^\infty(0, T; L^4(\mathbb{R}))$  for any  $T > 0$ , which solves the Cauchy problem (1), (2) in  $D'([0, T) \times \mathbb{R})$ .

To avoid strict convexity of the flux function, we will use a refinement of Coclite's method found in [3].

**Lemma 4.1** Let  $\Omega$  be a bounded open subset of  $\mathbb{R}_+ \times \mathbb{R}$ . Let  $f \in C^2(\mathbb{R})$  satisfy  $|f(u)| \leq C|u|^{s+1}$  for  $u \in \mathbb{R}$ ,  $|f'(u)| \leq C|u|^s$  for  $u \in \mathbb{R}$ . for some  $s \geq 0$ , and

$$\text{meas} \{u \in \mathbb{R} : f''(u) = 0\} = 0.$$

Define function  $I_l, f_l, F_l : \mathbb{R} \rightarrow \mathbb{R}$  as follow:

$$\begin{cases} I_l \in C^2(\mathbb{R}), & |I_l(u)| \leq |u| \quad \text{for } u \in \mathbb{R}, & |I_l'(u)| \leq 2 \quad \text{for } u \in \mathbb{R}, \\ & |I_l(u)| \leq |u| \quad \text{for } |u| \leq l, & I_l(u) = 0 \quad \text{for } |u| \geq 2l, \end{cases}$$

$$\text{and } f_l(u) = \int_0^u I_l'(\zeta) f'(\zeta) d\zeta, \quad F_l(u) = \int_0^u f_l'(\zeta) f'(\zeta) d\zeta.$$

Suppose  $\{u_n\}_{n=1}^\infty \subset L^{2(s+1)}(\Omega)$  is such that the two sequences

$$\{\partial_t I_l(u_n) + \partial_x f_l(u_n)_x\}_{n=1}^\infty, \quad \{\partial_t f_l(u_n) + \partial_x F_l(u_n)\}_{n=1}^\infty$$

of distributions belong to a compact subset of  $H_{loc}^{-1}(\Omega)$ , for each fixed  $l > 0$ .

Then there exists a subsequence of  $\{u_n\}_{n=1}^\infty$  that converges to a limit function  $u \in L^{2(s+1)}(\Omega)$  strongly in  $L^r(\Omega)$  for any  $1 \leq r < 2(s+1)$ .

**Lemma 4.2** Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^N$ ,  $N \geq 2$ . Suppose the sequence  $\{L_n\}_{n=1}^\infty$  of distributions is bounded in  $W^{-1, \infty}(\Omega)$ . Suppose also that

$$L_n = L_n^1 + L_n^2,$$

where  $\{L_n^1\}_{n=1}^\infty$  lies in a compact subset of  $H_{loc}^{-1}(\Omega)$  and  $\{L_n^2\}_{n=1}^\infty$  lies in a bounded subset of  $M_{loc}(\Omega)$ . Then  $\{L_n^1\}_{n=1}^\infty$  lies in compact subset of  $H_{loc}^{-1}(\Omega)$  (see [5,9]).

**Lemma 4.3** Suppose  $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$  holds. Then there exists a subsequence  $\{u_{\varepsilon_k}\}_{k=1}^\infty$  of  $\{u_\varepsilon\}_{\varepsilon > 0}$  and a limit function

$$u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R})) \cap L^\infty(0, T; L^4(\mathbb{R})) \quad \forall T > 0$$

such that

$$u_{\varepsilon_k} \rightarrow u \in L^p((0, T) \times \mathbb{R}) \quad \forall T > 0, \quad \forall p \in [2, 4]. \quad (28)$$

If, in addition,  $u_0 \in L^1(\mathbb{R})$ , then

$$u_{\varepsilon_k} \rightarrow u \quad \text{in } L^p((0, T) \times \mathbb{R}) \quad \forall T > 0, \quad \forall p \in [1, 4)$$

**Lemma 4.4** Suppose  $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$  holds, then

$$P_{\varepsilon_k} \rightarrow P^u \quad \text{in } L^p(0, T; W^{1,p}(\mathbb{R})) \quad \forall T > 0, \quad \forall p \in [1, 2), \quad (29)$$

where the sequence  $\{\varepsilon_k\}_{k=1}^\infty$  and the function  $u$  are constructed in Lemma 4.3.

**Proof.** Observe that Lemma 4.3 implies  $u_{\varepsilon_k}^2 \rightarrow u^2$  in  $L^p((0, T) \times \mathbb{R})$  for all  $T > 0$  and for all  $p \in [1, 2)$ . Using this fact and arguing as in the proof of theorem 3.2, we find that

$$\begin{aligned} & \|P_{\varepsilon_k} - P^u\|_{L^p((0,T)\times\mathbb{R})}^p, \|\partial_x P_{\varepsilon_k} - \partial_x P^u\|_{L^p((0,T)\times\mathbb{R})}^p \\ & \leq \left(\frac{3}{2}\right)^p \int_0^T \int_{\mathbb{R}} \left| (u_{\varepsilon_k}(t, y))^2 - (u(t, y))^2 \right|^p dy dt \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

■

**Lemma 4.5** Suppose  $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$  holds, then the limit  $u$  from Lemma 4.3 is a weak solution of (1), (2). Moreover,  $u \in L^\infty(0, T; L^4(\mathbb{R}))$  for each  $T > 0$ . Finally, if  $u_0$  also belongs to  $L^1(\mathbb{R})$ , then  $u \in L^\infty(0, T; L^1(\mathbb{R}))$  for each  $T > 0$ .

**Lemma 4.6** Suppose  $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$  holds, then the weak solution  $u$  from Lemma 4.5 satisfies the entropy inequality for any convex  $C^2$  entropy  $\eta : \mathbb{R} \rightarrow \mathbb{R}$  with  $\eta''$  bounded and corresponding entropy flux  $q : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $q'(u) = \eta'(u)u$ .

**Proof.** Let  $(\eta, q)$  be as in the lemma. By (16),

$$\partial_t \eta(u_{\varepsilon_k}) + \partial_x \eta(u_{\varepsilon_k}) + \partial_x q(u_{\varepsilon_k}) + \eta'(u_{\varepsilon_k}) \partial_x P_{\varepsilon_k} \leq \varepsilon_k \partial_{xx}^2 \eta(u_{\varepsilon_k}) \quad \text{in } D'([0, \infty) \times \mathbb{R}). \quad (30)$$

Observing that

$$|\eta(u)| = o(1 + u^2), \quad |\eta'(u)| = o(1 + u), \quad |q(u)| = o(1 + u^3),$$

■

We can use (28) and (29) when sending  $k \rightarrow \infty$  in (30). Then result is

$$\partial_t \eta(u) + \partial_x \eta(u) + \partial_x q(u) + \eta'(u) \partial_x P^u \leq 0 \quad \text{in } D'([0, \infty) \times \mathbb{R}),$$

**Proof of Theorem 4.1.** This follows from Lemma 4.5 and 4.6.

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