

On the Well-posedness of the Degasperis-Procesi Equation with the Dispersive Term

Shu Wen¹ *, Wenbin Zhang², Caiyin Niu¹

¹Department of Computing Science, Huaiyin Institute of Technology
 Huaian, Jiangsu, 223002, P.R.China

²Taizhou Institute of Science and Technology.NJUST
 Taizhou, Jiangsu, 225300, P.R.China

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Abstract: We investigate well-posedness in classes of discontinuous functions for the nonlinear and third order dispersive Degasperis-Procesi equation:

$$\partial_t u - \partial_t^3 u + \partial_x u - \partial_{xxx}^3 u + 4u\partial_x u = 3\partial_x u \partial_{xx}^2 u + u \partial_{xxx}^3 u \quad (DP)$$

This equation can be regarded as a model for shallow water dynamics and its asymptotic accuracy is the same as for the Camassa-Holm equation (one order more accurate than the KdV equation). We prove existence and L^1 stability (uniqueness) results for entropy weak solutions belonging to the class $L^1 \cap BV$, while existence of at least one weak solution, satisfying a restricted set of entropy inequalities, is proved in the class $L^2 \cap L^4$.

Key words: dispersive term; weak solution; entropy condition; hyperbolic equation

1 Introduction

Our aim is to investigate well-posedness in classes of discontinuous functions for the Degasperis-Procesi equation[1]

$$\partial_t u - \partial_{txx}^3 u + \partial_x u - \partial_{xxx}^3 u + 4u\partial_x u = 3\partial_x u \partial_{xx}^2 u + u \partial_{xxx}^3 u, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R} \quad (1)$$

with an initial condition u_0 :

$$u(0, x) = u_0(x) \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) \quad (2)$$

2 Viscous approximation and a priori estimates

We will prove existence of a solution to the Cauchy problem (1), (2) by analyzing the limiting behavior of a sequence of smooth functions $\{u_\varepsilon\}_{\varepsilon>0}$, where each function u_ε solves the following viscous problem:

$$\begin{cases} \partial_t u_\varepsilon - \partial_{txx}^3 u_\varepsilon + \partial_x u_\varepsilon + \partial_{xxx}^3 u_\varepsilon + 4u_\varepsilon \partial_x u_\varepsilon \\ \quad = 3\partial_x u_\varepsilon \partial_{xx}^2 u_\varepsilon + u_\varepsilon \partial_{xxx}^3 u_\varepsilon + \varepsilon \partial_{xx}^2 u_\varepsilon - \varepsilon \partial_{xxxx}^4 u_\varepsilon, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u_\varepsilon(0, x) = u_{0,\varepsilon}(x), & x \in \mathbb{R} \end{cases} \quad (3)$$

*Corresponding author. E-mail address: wenshu119@163.com

This problem can be stated equivalently as a parabolic-elliptic system:

$$\begin{cases} \partial_t u_\varepsilon + \partial_x \left(\frac{u_\varepsilon^2}{2} \right) + \partial_x u_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_{xx}^2 u_\varepsilon, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ -\partial_{xx}^2 P_\varepsilon + P_\varepsilon = \frac{3}{2} u_\varepsilon^2, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u_\varepsilon(0, x) = u_{0,\varepsilon}(x), & x \in \mathbb{R} \end{cases} \quad (4)$$

Observe that we have an explicit expression for P_ε in terms of u_ε :

$$P_\varepsilon(t, x) = P^{u_\varepsilon}(t, x) = G_1 * \left(\frac{3}{2} u_\varepsilon^2 \right)(t, x) = \frac{3}{4} \int_{\mathbb{R}} e^{-|x-y|} (u_\varepsilon(t, y))^2 dy$$

To begin with, we assume in this section that

$$u_0 \in L^2(\mathbb{R}), \quad (5)$$

and

$$u_{0,\varepsilon} \in H^l(\mathbb{R}), \quad l \geq 2, \quad \|u_{0,\varepsilon}\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}, \quad u_{0,\varepsilon} \rightarrow u_0 \in L^2(\mathbb{R}). \quad (6)$$

We will impose additional condition on the initial data as we make progress.

We begin by stating a lemma as which shows that the viscous problem (3) is well-posed for each fixed $\varepsilon > 0$.

Lemma 2.1 *Assume (5) and (6) hold, and fix any $\varepsilon > 0$, then there exists a unique global smooth solution $u_\varepsilon = u_\varepsilon(t, x)$ to the Cauchy Problem (4) belonging to $C([0, \infty); H^l(\mathbb{R}))$.*

Proof. We omit the proof since it is similar to the one found in [4, Theorem 2.3] ■

2.1 L^2 estimates and some consequences

Next we prove a uniform of L^2 bound on the approximate solution u_ε , which reinforces the whole analysis in this paper.

Lemma 2.2 (Energy estimate) *Assume (5) and (6) hold, and fix any $\varepsilon > 0$, then the following bounds hold for any $t \geq 0$:*

$$\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq 2\sqrt{2} \|u_0\|_{L^2(\mathbb{R})}, \quad \sqrt{\varepsilon} \|\partial_x u_\varepsilon\|_{L^2(\mathbb{R}_+ \times \mathbb{R})} \leq 2 \|u_0\|_{L^2(\mathbb{R})}$$

For the proof of this Lemma we introduce the quantity $v_\varepsilon = v_\varepsilon(t, x)$:

$$v_\varepsilon(t, x) = (G_2 * u_\varepsilon) = \int_{\mathbb{R}} e^{-2|x-y|} u_\varepsilon(t, y) dy, \quad t \geq 0, \quad x \in \mathbb{R}.$$

Since $G_2(x) = e^{-2|x|}$ is Green's function of operator $4 - \partial_{xx}^2$, we see that v_ε also satisfies the equation

$$-\partial_{xx}^2 v_\varepsilon + 4v_\varepsilon = u_\varepsilon \quad \text{in } \mathbb{R}_+ \times \mathbb{R} \quad (7)$$

The use of the quantity v_ε is motivated by the fact that $\int_{\mathbb{R}} (u - u_{xx}) dx$ is a conserved quantity, where $4v_\varepsilon - \partial_{xx}^2 v_\varepsilon = u_\varepsilon$ and u solves (1) (see [3, 6, 8]).

To prove Lemma 2.2 we shall need the following estimates on v_ε :

Lemma 2.3 *Assume (5) and (6) hold, and fix any $\varepsilon > 0$, then the following identity holds for any $t \geq 0$:*

$$\begin{aligned} & \|\partial_{xx}^2 v_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 5 \|\partial_x v_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 4 \|v_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & + 2\varepsilon \left(\int_0^t \left(\|\partial_{xxx}^3 v_\varepsilon(\tau, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + 5 \|\partial_{xx}^2 v_\varepsilon(\tau, \cdot)\|_{L^2(\mathbb{R})}^2 + 4 \|\partial_x v_\varepsilon\|_{L^2(\mathbb{R})}^2 \right) d\tau \\ & = \|\partial_{xx}^2 v_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R})}^2 + 5 \|\partial_x v_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R})}^2 + 4 \|v_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R})}^2 \end{aligned} \quad (8)$$

Proof. Multiplying the first equation of (4) by $v_\varepsilon - \partial_{xx}^2 v_\varepsilon$ and integrating over R , we get

$$\begin{aligned} \int_{\mathbb{R}} \partial_t u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx + \int_{\mathbb{R}} \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx - \varepsilon \int_{\mathbb{R}} \partial_{xx}^2 (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ = - \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx - \int_{\mathbb{R}} \partial_x P_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \end{aligned} \quad (9)$$

For the left-hand side of this identity, using (7), we have

$$\begin{aligned} \int_{\mathbb{R}} \partial_t u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx + \int_{\mathbb{R}} \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx - \varepsilon \int_{\mathbb{R}} \partial_{xx}^2 (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ = \int_{\mathbb{R}} (4\partial_t v_\varepsilon - \partial_{txx}^3 v_\varepsilon) (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx + \int_{\mathbb{R}} (4v_\varepsilon - \partial_{xx}^2 v_\varepsilon) (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ - \varepsilon \int_{\mathbb{R}} (4\partial_{xx}^2 v_\varepsilon - \partial_{xxxx}^4 v_\varepsilon) (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ = \frac{1}{2} \frac{d}{dt} \left(4v_x^2 + 5(\partial_x v_\varepsilon)^2 + (\partial_{xx}^2 v_\varepsilon)^2 \right) dx + \varepsilon \int_{\mathbb{R}} \left(4(\partial_x v_\varepsilon)^2 + 5(\partial_{xx}^2 v_\varepsilon)^2 + (\partial_{xxx}^3 v_\varepsilon)^2 \right) dx \end{aligned} \quad (10)$$

For the right-hand side of (9), we calculate

$$\begin{aligned} - \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx - \int_{\mathbb{R}} \partial_x P_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ = - \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx - \int_{\mathbb{R}} (P_\varepsilon - \partial_{xx}^2 P_\varepsilon) (v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ = - \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (4v_\varepsilon - \partial_{xx}^2 v_\varepsilon) dx \\ = - \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon dx = 0 \end{aligned} \quad (11)$$

Substituting (10) and (11) into (9) yields

$$\frac{d}{dt} \left(4v_x^2 + 5(\partial_x v_\varepsilon)^2 + (\partial_{xx}^2 v_\varepsilon)^2 \right) dx + 2\varepsilon \int_{\mathbb{R}} \left(4(\partial_x v_\varepsilon)^2 + 5(\partial_{xx}^2 v_\varepsilon)^2 + (\partial_{xxx}^3 v_\varepsilon)^2 \right) dx = 0$$

Integrating this inequality over $[0, t]$ we obtain (8). ■

We conclude this subsection with some bounds on the nonlocal term P_ε , which all are consequences of the L^2 bound in Lemma 2.2.

Lemma 2.4 Assume (5) and (6) hold, and fix any $\varepsilon > 0$, then

$$P_\varepsilon \geq 0, \quad (12)$$

$$\|P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})}, \|\partial_x P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 12 \|u_0\|_{L^2(\mathbb{R})}^2, \quad t \geq 0, \quad (13)$$

$$\|P_\varepsilon\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})}, \|\partial_x P_\varepsilon\|_{L^2(\mathbb{R}_+ \times \mathbb{R})} \leq 6 \|u_0\|_{L^2(\mathbb{R})}^2,$$

$$\|\partial_x^2 P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 24 \|u_0\|_{L^2(\mathbb{R})}^2, \quad t \geq 0.$$

2.2 L^1 estimate

As a bounded consequence of L^2 in Lemma 2.2, we can bound u_ε in L^1 , as long as we assume, in additions to (5) and (6),

$$u_0, u_{0,\varepsilon} \in L^1(\mathbb{R}), \quad \|u_{0,\varepsilon}\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}. \quad (14)$$

Lemma 2.5 (L^1 -estimate). Assume (5), (6), and (14) hold, and fix any $\varepsilon > 0$, then

$$\|u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} + 12 \|u_0\|_{L^2(\mathbb{R})}^2, \quad t \geq 0. \quad (15)$$

Proof. Let $\eta \in C^2(\mathbb{R})$ and $q : \mathbb{R} \rightarrow \mathbb{R}$ be such that $q'(u) = uq'(u)$. By multiplying the first equation in (4) with using the chain rule, we get

$$\partial_t \eta(u_\varepsilon) + \partial_x \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) + \eta'(u_\varepsilon) \partial_x P_\varepsilon = \varepsilon \partial_{xx}^2 \eta(u_\varepsilon) - \eta''(u_\varepsilon) (\partial_x u_\varepsilon)^2 \quad (16)$$

Choosing $\eta(u) = |u|$ (modulo an approximation argument) and then integrating the resulting equation over \mathbb{R} yield

$$\frac{d}{dt} \int_{\mathbb{R}} |u_\varepsilon| dx \leq \int_{\mathbb{R}} \text{sign}(u_\varepsilon) \partial_x P_\varepsilon dx$$

By (13),

$$\int_{\mathbb{R}} \text{sign}(u_\varepsilon) \partial_x P_\varepsilon dx \leq \|\partial_x P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 12 \|u_0\|_{L^2(\mathbb{R})}^2,$$

and hence

$$\frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 12 \|u_0\|_{L^2(\mathbb{R})}^2 \quad (17)$$

Integrating (17) over $[0, t]$ we can obtain (15).

2.3 BV and L^∞ estimates

In this subsection we derive supplementary a priori estimate for the viscous approximations, which also are consequences of the L^2 bound in Lemma 2.2. In particular, we prove that the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in BV, which yields strong compactness of this sequence. To this end, we need to assume, in addition to (5) and (6),

$$u_0, u_{0,\varepsilon} \in BV(\mathbb{R}), \quad |u_{0,\varepsilon}|_{BV(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})}. \quad (18)$$

Lemma 2.6 (BV estimate in space). Assume (5), (6), and (18) hold, and fix any $\varepsilon > 0$, then

$$\|\partial_x u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})} + 24t \|u_0\|_{L^2(\mathbb{R})}^2, \quad t \geq 0. \quad (19)$$

Proof. Set $q_\varepsilon := \partial_x u_\varepsilon$. Then q_ε satisfies the equation

$$\partial_t q_\varepsilon + \partial_x q_\varepsilon + u_\varepsilon \partial_x q_\varepsilon + q_\varepsilon^2 + \partial_{xx}^2 P_\varepsilon = \varepsilon \partial_{xx}^2 q_\varepsilon.$$

If $\eta \in C^2(\mathbb{R})$ and $q : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $q'(u) = uq'(u)$, then by the chain rule

$$\begin{aligned} \partial_t \eta(q_\varepsilon) + \partial_x \eta(q_\varepsilon) + \partial_x (u_\varepsilon q_\varepsilon) - q_\varepsilon \eta(q_\varepsilon) + \eta'(q_\varepsilon) q_\varepsilon^2 \\ + \eta'(u_\varepsilon) \partial_{xx}^2 P_\varepsilon = \varepsilon \partial_{xx}^2 \eta(q_\varepsilon) - \eta''(q_\varepsilon) (\partial_x q_\varepsilon)^2. \end{aligned}$$

Choosing $\eta(u) = |u|$ (modulo an approximation argument) and then integrating the resulting equation over \mathbb{R} yield

$$\int_{\mathbb{R}} \text{sign}(q_\varepsilon) \partial_{xx}^2 P_\varepsilon dx \leq \|\partial_{xx}^2 P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 24 \|u_0\|_{L^2(\mathbb{R})}^2,$$

and hence

$$\frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 24 \|u_0\|_{L^2(\mathbb{R})}^2. \quad (20)$$

Integrating (20) over $[0, t]$ we get (19). ■

Lemma 2.7 (L^∞ -estimate[7]). Assume (5), (6), (18) hold, and fix any $\varepsilon > 0$, then

$$\|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})} + 24 \|u_0\|_{L^2(\mathbb{R})}^2, \quad t \geq 0. \quad (21)$$

Lemma 2.8 (BV estimate in time). Assume (5), (6), and (18) hold, and fix any $\varepsilon > 0$, then

$$\|\partial_t u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq C_t, \quad t \geq 0.$$

where the constant $C_t := \left(|u_0|_{BV(\mathbb{R})} + 24t \|u_0\|_{L^2(\mathbb{R})}^2\right)^2 + \left(|u_0|_{BV(\mathbb{R})} + 24t \|u_0\|_{L^2(\mathbb{R})}^2\right) + 12 \|u_0\|_{L^2(\mathbb{R})}^2$ is independent of ε but dependent on t .

Proof. We have, by (21), (19), and (12),

$$\begin{aligned} \|\partial_t u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} &\leq \int_{\mathbb{R}} |\partial_x u_\varepsilon| dx + \int_{\mathbb{R}} |u_\varepsilon \partial_x u_\varepsilon| dx + \int_{\mathbb{R}} \partial_x P_\varepsilon dx \\ &\leq |u_\varepsilon(t, \cdot)|_{BV} + \|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} |u_\varepsilon(t, \cdot)|_{BV} + \|\partial_x P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq C_t. \end{aligned}$$

■

Lemma 2.9 . Assume (5), (6), and (18) hold, and fix any $\varepsilon > 0$. Then

$$\|\partial_{xx}^2 P_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq 6 \|u_0\|_{L^2(\mathbb{R})}^2 + \frac{3}{2} \left(|u_0|_{BV(\mathbb{R})} + 24t \|u_0\|_{L^2(\mathbb{R})}^2 \right)^2,$$

for any $t \geq 0$.

2.4 L^4 -estimate and Oleinik type estimate

Next we prove that the viscous approximation are uniformly bounded in L^4 , a fact that we use later to prove the existence of at least one weak solution to (1), (2) under the mere assumption that $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ holds. For this purpose, we need to assume, in addition to (5) and (6),

$$u_0, u_{0,\varepsilon} \in L^4(\mathbb{R}), \quad \|u_{0,\varepsilon}\|_{L^4(\mathbb{R})} \leq \|u_0\|_{L^4(\mathbb{R})}. \quad (22)$$

Lemma 2.10 (L^4 - estimate). Assume (5), (6), and (22) hold, and fix any $\varepsilon > 0$, then

$$\|u_\varepsilon(t, \cdot)\|_{L^4(\mathbb{R})}^4 \leq e^{12\|u_0\|_{L^2(\mathbb{R})}^2} \|u_0\|_{L^4(\mathbb{R})}^4 + 8 \|u_0\|_{L^2(\mathbb{R})}^2 \left(e^{12t\|u_0\|_{L^2(\mathbb{R})}^2} - 1 \right),$$

for any $t \geq 0$.

Lemma 2.11 (Oleinik type estimate). Assume (5), (6), and (18) hold, and fix any $\varepsilon > 0$, then for each $t \in (0, T]$, with $T > 0$ being fixed,

$$\partial_x u_\varepsilon(t, x) \leq \frac{1}{t} + K_T, \quad x \in \mathbb{R},$$

where $K_T := \left[6 \|u_0\|_{L^2(\mathbb{R})}^2 + \frac{3}{2} \left(|u_0|_{BV(\mathbb{R})} + 24T \|u_0\|_{L^2(\mathbb{R})}^2 \right)^2 \right]^{1/2}$

3 Well-posedness in $L^1 \cap BV$

Relying on the a priori estimate derive in Section 2, we prove existence, uniqueness, and L^1 stability of entropy weak solutions to (1), (2), under the $L^1 \cap BV$ assumption $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$.

Theorem 3.1 (well-posedness). Suppose $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then there exists an entropy weak solution to the Cauchy problem (1), (2). Fix any $T > 0$, and let $u, v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be two entropy weak solutions to (1), (2) with initial data $u_0, v_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, respectively.

Then for any $t \in (0, T)$

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^1(\mathbb{R})} \leq e^{M_T t} \|u_0 - v_0\|_{L^1(\mathbb{R})},$$

where

$$M_T := \frac{3}{2} \left(\|u\|_{L^\infty((0,T) \times \mathbb{R})} \right) + \|v\|_{L^\infty((0,T) \times \mathbb{R})} < \infty.$$

Consequently, there exists at most one entropy weak solution to (1), (2).

The entropy weak solution u satisfies the following estimates for any $t \in (0, T)$:

$$\|u(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} + 12t \|u_0\|_{L^2(\mathbb{R})}^2, \quad (23)$$

$$\|u(t, \cdot)\|_{BV(\mathbb{R})} \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{BV(\mathbb{R})} + 24t \|u_0\|_{L^2(\mathbb{R})}^2, \quad (24)$$

$$\|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 \leq e^{12\|u_0\|_{L^2(\mathbb{R})}^2 t} \|u_0\|_{L^4(\mathbb{R})}^4 + 8 \|u_0\|_{L^2(\mathbb{R})}^2 \left(e^{12\|u_0\|_{L^2(\mathbb{R})}^2 t} - 1 \right). \quad (25)$$

Furthermore,

$$\|u(t_2, \cdot) - u(t_1, \cdot)\|_{L^1(\mathbb{R})} \leq C_T |t_2 - t_1|, \quad \forall t_1, t_2 \in [0, t], \quad (26)$$

where $C_T := \left(\|u_0\|_{L^1(\mathbb{R})} + 12T \|u_0\|_{L^2(\mathbb{R})}^2 \right)^2 + \left(\|u_0\|_{L^1(\mathbb{R})} + 12T \|u_0\|_{L^2(\mathbb{R})}^2 \right) + 12 \|u_0\|_{L^2(\mathbb{R})}^2$.

Finally, the following Oleinik type estimate holds for a.e. $(t, x) \in (0, T] \times \mathbb{R}$,

$$\partial_x u(t, x) \leq \frac{1}{t} + K_T, \quad (27)$$

where $K_T := \left[6 \|u_0\|_{L^2(\mathbb{R})}^2 + \frac{3}{2} \left(\|u_0\|_{BV(\mathbb{R})} + 24T \|u_0\|_{L^2(\mathbb{R})}^2 \right)^2 \right]^{1/2}$.

This theorem is an immediate consequence of Theorems 3.2, 3.3 and Corollary 3.1.

Theorem 3.2 (Existence of entropy weak solutions) Suppose (2) holds, then there exists at least one entropy weak solution to (1), (2).

Proof. We assume that the approximating sequence $\{u_{0,\varepsilon}\}_{\varepsilon>0}$ is chosen such that (5), (6), and (14) hold. Then, in view of the a priori estimates obtained in Section 2, it takes a standard argument to see that there exists a sequence of strictly positive numbers $\{\varepsilon_k\}_{k=1}^\infty$ tending to zero such that $u_{\varepsilon_k} \rightarrow u$ a.e. while $k \rightarrow \infty$ in $\mathbb{R}_+ \times \mathbb{R}$, and hence $u_{\varepsilon_k} \rightarrow u$ in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R})$ for all $p \in [1, \infty)$. Thanks to (15) and (21), there also holds $u_{\varepsilon_k} \rightarrow u$ in $L^p_{loc}((0, T) \times \mathbb{R}) \quad \forall T > 0, \quad \forall p \in [1, \infty)$.

The a priori estimates in Section 2 imply immediately that the limit function u satisfied (23)-(27).

Let us now prove that as $k \rightarrow \infty$, $p_{\varepsilon_k} \rightarrow p^u$, $\partial_x p_{\varepsilon_k} \rightarrow \partial_x p^u$ in $L^p((0, T) \times \mathbb{R})$, $\forall T > 0, \quad \forall p \in [1, \infty)$,

Which following from the following calculation:

$$\begin{aligned} & \|P_{\varepsilon_k} - P^u\|_{L^p((0,T) \times \mathbb{R})}^p, \quad \|\partial_x P_{\varepsilon_k} - \partial_x P^u\|_{L^p((0,T) \times \mathbb{R})}^p \\ & \leq \left(\frac{3}{4}\right)^p \int \int_{\Pi_T} \left(\int_{\mathbb{R}} e^{-|x-y|} \left| (u_{\varepsilon_k}(t, y))^2 - (u(t, y))^2 \right| dy \right)^p dx dt \\ & \leq \left(\frac{3}{4}\right)^p \int \int_{\Pi_T} \left(\int_{\mathbb{R}} e^{-|x-y|} e^{-|x-y|/p} \left| (u_{\varepsilon_k}(t, y))^2 - (u(t, y))^2 \right| dy \right)^p dx dt \\ & \leq \left(\frac{3}{4}\right)^p \int \int_{\Pi_T} \left(\int_{\mathbb{R}} e^{-|x-y|} dy \right)^{p-1} \times \left(\int_{\mathbb{R}} e^{-|x-y|} \left| (u_{\varepsilon_k}(t, y))^2 - (u(t, y))^2 \right|^p dy \right) dx dt \\ & \leq \left(\frac{3}{2}\right)^p \int \int_{\Pi_T} \left| (u_{\varepsilon_k}(t, y))^2 - (u(t, y))^2 \right|^p dy dt \rightarrow 0 \\ & \leq C_T \int \int_{\Pi_T} |u_{\varepsilon_k}(t, y) - u(t, y)|^p dy dt \rightarrow 0 \quad \text{as } k \rightarrow \infty \end{aligned}$$

where $\Pi_T := (0, T) \times \mathbb{R}$. ■

Theorem 3.3 (L^1 stability). Let u and v be two entropy weak solution of (1) with initial data $u(0, \cdot) = u_0$ and $v(0, \cdot) = v_0$ satisfying (2). Fix any $T > 0$, then

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^1(\mathbb{R})} \leq e^{M_T t} \|u_0 - v_0\|_{L^1(\mathbb{R})}, \quad t \in (0, T),$$

We omit the proof since it is similar to the one of found in [2, Theorem3.3].

Corollary 3.1 (Uniqueness) Suppose $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then the Cauchy problem (1), (2) admits at most one entropy weak solution.

This is an immediate consequence of Theorem 3.3.

4 Existence in $L^2 \cap L^4$

In this section we prove that there exists at least one weak solution to (1), (2) under assumption $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, in which case we are outside the BV/L^∞ framework considered in Section 3. Since no L^∞ bound is available we can only prove that this weak solution satisfies the entropy inequality for convex C^2 entropies possessing a bounded second order derivative. Be that as it may, we are not able to prove L^1 stability/uniqueness based on this restricted class of entropies.

Our main result is the following theorem:

Theorem 4.1 (Existence) Suppose $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ holds, then there exists a function $u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R})) \cap L^\infty(0, T; L^4(\mathbb{R}))$ for any $T > 0$, which solves the Cauchy problem (1), (2) in $D'([0, T) \times \mathbb{R})$.

To avoid strict convexity of the flux function, we will use a refinement of Coclite's method found in [3].

Lemma 4.1 Let Ω be a bounded open subset of $\mathbb{R}_+ \times \mathbb{R}$. Let $f \in C^2(\mathbb{R})$ satisfy $|f(u)| \leq C|u|^{s+1}$ for $u \in \mathbb{R}$, $|f'(u)| \leq C|u|^s$ for $u \in \mathbb{R}$. for some $s \geq 0$, and

$$\text{meas} \{u \in \mathbb{R} : f''(u) = 0\} = 0.$$

Define function $I_l, f_l, F_l : \mathbb{R} \rightarrow \mathbb{R}$ as follow:

$$\begin{cases} I_l \in C^2(\mathbb{R}), & |I_l(u)| \leq |u| \quad \text{for } u \in \mathbb{R}, & |I'_l(u)| \leq 2 \quad \text{for } u \in \mathbb{R}, \\ & |I_l(u)| \leq |u| \quad \text{for } |u| \leq l, & I_l(u) = 0 \quad \text{for } |u| \geq 2l, \end{cases}$$

$$\text{and } f_l(u) = \int_0^u I'_l(\zeta) f'(\zeta) d\zeta, \quad F_l(u) = \int_0^u f'_l(\zeta) f'(\zeta) d\zeta.$$

Suppose $\{u_n\}_{n=1}^\infty \subset L^{2(s+1)}(\Omega)$ is such that the two sequences

$$\{\partial_t I_l(u_n) + \partial_x f_l(u_n)_x\}_{n=1}^\infty, \quad \{\partial_t f_l(u_n) + \partial_x F_l(u_n)\}_{n=1}^\infty$$

of distributions belong to a compact subset of $H_{loc}^{-1}(\Omega)$, for each fixed $l > 0$.

Then there exists a subsequence of $\{u_n\}_{n=1}^\infty$ that converges to a limit function $u \in L^{2(s+1)}(\Omega)$ strongly in $L^r(\Omega)$ for any $1 \leq r < 2(s+1)$.

Lemma 4.2 Let Ω be a bounded open subset of \mathbb{R}^N , $N \geq 2$. Suppose the sequence $\{L_n\}_{n=1}^\infty$ of distributions is bounded in $W^{-1, \infty}(\Omega)$. Suppose also that

$$L_n = L_n^1 + L_n^2,$$

where $\{L_n^1\}_{n=1}^\infty$ lies in a compact subset of $H_{loc}^{-1}(\Omega)$ and $\{L_n^2\}_{n=1}^\infty$ lies in a bounded subset of $M_{loc}(\Omega)$. Then $\{L_n^1\}_{n=1}^\infty$ lies in compact subset of $H_{loc}^{-1}(\Omega)$ (see [5,9]).

Lemma 4.3 Suppose $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ holds. Then there exists a subsequence $\{u_{\varepsilon_k}\}_{k=1}^\infty$ of $\{u_\varepsilon\}_{\varepsilon > 0}$ and a limit function

$$u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R})) \cap L^\infty(0, T; L^4(\mathbb{R})) \quad \forall T > 0$$

such that

$$u_{\varepsilon_k} \rightarrow u \in L^p((0, T) \times \mathbb{R}) \quad \forall T > 0, \quad \forall p \in [2, 4]. \quad (28)$$

If, in addition, $u_0 \in L^1(\mathbb{R})$, then

$$u_{\varepsilon_k} \rightarrow u \quad \text{in } L^p((0, T) \times \mathbb{R}) \quad \forall T > 0, \quad \forall p \in [1, 4)$$

Lemma 4.4 Suppose $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ holds, then

$$P_{\varepsilon_k} \rightarrow P^u \quad \text{in } L^p(0, T; W^{1,p}(\mathbb{R})) \quad \forall T > 0, \quad \forall p \in [1, 2), \quad (29)$$

where the sequence $\{\varepsilon_k\}_{k=1}^\infty$ and the function u are constructed in Lemma 4.3.

Proof. Observe that Lemma 4.3 implies $u_{\varepsilon k}^2 \rightarrow u^2$ in $L^p((0, T) \times \mathbb{R})$ for all $T > 0$ and for all $p \in [1, 2)$. Using this fact and arguing as in the proof of theorem 3.2, we find that

$$\begin{aligned} & \|P_{\varepsilon k} - P^u\|_{L^p((0, T) \times \mathbb{R})}^p, \|\partial_x P_{\varepsilon k} - \partial_x P^u\|_{L^p((0, T) \times \mathbb{R})}^p \\ & \leq \left(\frac{3}{2}\right)^p \int_0^T \int_{\mathbb{R}} \left| (u_{\varepsilon k}(t, y))^2 - (u(t, y))^2 \right|^p dy dt \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

■

Lemma 4.5 Suppose $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ holds, then the limit u from Lemma 4.3 is a weak solution of (1), (2). Moreover, $u \in L^\infty(0, T; L^4(\mathbb{R}))$ for each $T > 0$. Finally, if u_0 also belongs to $L^1(\mathbb{R})$, then $u \in L^\infty(0, T; L^1(\mathbb{R}))$ for each $T > 0$.

Lemma 4.6 Suppose $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ holds, then the weak solution u from Lemma 4.5 satisfies the entropy inequality for any convex C^2 entropy $\eta : \mathbb{R} \rightarrow \mathbb{R}$ with η'' bounded and corresponding entropy flux $q : \mathbb{R} \rightarrow \mathbb{R}$ defined by $q'(u) = \eta'(u)u$.

Proof. Let (η, q) be as in the lemma. By (16),

$$\partial_t \eta(u_{\varepsilon k}) + \partial_x \eta(u_{\varepsilon k}) + \partial_x q(u_{\varepsilon k}) + \eta'(u_{\varepsilon k}) \partial_x P_{\varepsilon k} \leq \varepsilon_k \partial_{xx}^2 \eta(u_{\varepsilon k}) \quad \text{in } D'([0, \infty) \times \mathbb{R}). \quad (30)$$

Observing that

$$|\eta(u)| = o(1 + u^2), \quad |\eta'(u)| = o(1 + u), \quad |q(u)| = o(1 + u^3),$$

■

We can use (28) and (29) when sending $k \rightarrow \infty$ in (30). Then result is

$$\partial_t \eta(u) + \partial_x \eta(u) + \partial_x q(u) + \eta'(u) \partial_x P^u \leq 0 \quad \text{in } D'([0, \infty) \times \mathbb{R}),$$

Proof of Theorem 4.1. This follows from Lemma 4.5 and 4.6.

References

- [1] Octavian G Mustafa: A note on the Degasperis-Procesi equation. *Journal of Nonlinear Mathematical physics*.12(1):10-14(2005)
- [2] Giuseppe M. Coclite, Kenneth H: Karlsen. On the well-posedness of the Degasperis-Procesi equation. *Journal of functional analysis*.233:60-91(2006)
- [3] A. Degasperis, D. D. Holm, A. N. W. Home: Integrable and non-integrable equations with peakons. *In Nonlinear physics: theory and experiment*(Gallipoli):37-43(2002).World Sci. Publishing, River Edge, NJ.(2003)
- [4] G.M. Coclite, K.H. Karlsen, H. Holden: Wellposedness for a parabolic-elliptic system. *Discrete Contin. Dyn. Systems*. 13(3):659-682(2005)
- [5] Tian Li-xin: Attactor of dissipative soliton equation. *Applied Mathematics and Mechanics*.15(6):571-578(1994)
- [6] Lixin Tian, Gochang Fang, Guilong Gui: Well-posedness and blow up for an integrable shallow water equation with strong dispersive term. *International Journal of Nonlinear Sciece*. 1:3-13(2006)
- [7] Danping Ding, Lixin Tian: The study of solution of dissipative Camassa-Holm equation on total space.*International Journal of Nonlinear Sciece*.1:37-42(2006)
- [8] Mei Sun, Lixin Tian, Jian Yin: Hopf bifurcation analysis of energy resource chaotic system. *International Journal of Nonlinear Sciece*.1(1):49-53(2006)
- [9] Lixin Tian, Lihong Ren, Shaoguang Shi: N-multiple nonwandering unilateral weighted backward shift operators and the property of direct sum operators in Banach space. *International Journal of Nonlinear Science*.2(2):104-110(2006)