ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.5(2008) No.2, pp. 164-169

A Hierarchy of Discrete Integrable Equation and its Hamiltonian Structure

Weimin Zhang[1,2] *
[1]Nonlinear Scientific Research Center, Jiangsu University
Zhenjiang, Jiangsu,212013,P.R.China
[2]Department of Mathematics, Jiaying University, Meizhou, Guangdong 514015, P. R. China
(Received 8 February 2007, accepted 18 March 2007)

Abstract: In this paper a new discrete isospectral problem is constructed, from which a new Lax integrable discrete equation is generated and obtain its hierarchy structure. The Hamiltonian structures for the hierarchy of the Lax integrable discrete equation is established by using the trace identity which was proposed by Tu.

Key words:discrete Lax pair ; discrete integrable system ; discrete Hamiltonian System

1 Introduction

As is well known, the solitary theory is a powerful tool used for explaining and describing the nonlinear phenomena such as nonlinear optics, plasma physics, cellular automata, fluid mechanics, solid state physics and so on. In the solitary theory, two important subjects are to search for the new Lax integrable lattice equations and to study their Hamiltonian structure such as KdV equation, Toda equation, Camassa-Holm equation, etc [1-6]. The theory of continuous integrable systems has extensively and actively been developed in the past twenty years, and Tu scheme is one of the effective methods in studying continuous integrable systems[7-9]. In recent years, the nonlinear integrable discrete equations have become the focus of common concern in the theory of integrable systems. The mathematical structures and physical applications of the discrete lattice systems, such as the bi-Hamiltonian structures , integrable symplectic maps, Backlund transformations, nonlinear superposition formulae, soliton solutions, mater symmetries and so on, were investigated in [10–13]. The method of Lax pairs is an important way to generate the new Lax integrable discrete lattice equations. The Hamiltonian structure for a hierarchy of Lax integrable discrete lattice equation may be established by the trace identity which was proposed by Tu [14]. In this paper we introduce a new isospecial problem and obtain a hierarchy of lattice equation , give its Hamiltonian structure by using Tu scheme.

This paper is organized as follows: In Section 2, we recall briefly some notation from the discrete integrable systems. In Section 3, a new discrete Lax integrable equation is constructed. In Section 4, we give the Hamiltonian structure of this new discrete Lax integrable equation by using Tu scheme.

2 The Lax pair of discrete lattice equation

2.1 Concepts and notations

The shift operator E and the difference operator D are defined as follows :

$$(Ef)(n) = f(n+1) \equiv f^1(n), E^k f = f^{(k)}$$

$$(Df)(n) = f(n+1) - f(n) = (E-1)f(n)$$

* *E-mail address*: zhang-weimin215@163.com

IJNS.2008.04.15/140

For a given function $f = f(n)$, if there exists a function such as $f = Dh$, then we call $f \sim 0(mod)$, i.e. $f \sim 0(modD) \Leftrightarrow$ exist h, such as $f = Dh$.

For the vector function $f(u)$, its gradient $(\nabla f)(u)$ is defined as

$$(\nabla f, v) = (d/d\epsilon)f(u + \epsilon v)|_{\epsilon=0},$$

where

$$u = (u_1, \dots, u_p),$$
$$v = (v_1, \dots, v_p)^T,$$
$$\nabla f = ((\nabla f)_1, \dots, (\nabla f)_p)$$

and

$$(\nabla f)_i = \frac{\delta f}{\delta u_i} = \sum_{n\epsilon\mathbb{Z}} E^{-n}(\frac{\partial f}{\partial u_i^{(n)}})$$

The Poisson bracket for arbitrary funtions f and g are defined as

$$\{f, g\} = (J\nabla f, \nabla g) = \sum_{m\epsilon\mathbb{Z}} \sum_{i=1}^{p} (J\frac{\delta f}{\delta u_i})(\frac{\delta g}{\delta u_i})$$

If

$$\{f, g\} = -\{g, f\},$$
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

then J is called the Hamiltonian operator.

A discrete integrable system $u_{t_n} = J\frac{\delta H_n}{\delta u}$ is called Liouville integrable, if there are infinitely conservation law $\{H_n\}_{n=1}^{\infty}$,which satisfy $\{H_n, H_m\} = 0, m, n \geq 0$

2.2 Discrete Lax pair

Given a properly discrete isospectral problem and its auxiliary problem

$$E\Psi_n = U_n(u, \lambda)\Psi_n \tag{2.1}$$

$$\partial\Psi_n/\partial t = V_n(u, \lambda)\Psi_n \tag{2.2}$$

with λ being the spectral parameter for $\lambda_t = 0$, where U_n and V_n are $N \times N$ matrixs. The compatibility condition of system (2.1) and (2.2) is

$$\frac{\partial}{\partial t}E\Psi_n = E\frac{\partial}{t}\Psi_n \qquad (2.3).$$

From (2.1) and (2.2) we have

$$\frac{\partial}{\partial t}E\Psi_n = \frac{\partial}{\partial t}(U_n\Psi_n) = \frac{\partial U_n}{\partial t}\Psi_n + U_n\frac{\partial\Psi_n}{\partial t} = \frac{\partial U_n}{\partial t}\Psi_n + U_nV_n\Psi_n \tag{2.4}$$

and

$$E\frac{\partial\Psi_n}{\partial t} = E(V_n\Psi_n = E(V_n)E(\Psi_n) = V_{n+1}U_n\Psi_n \tag{2.5}$$

Thus, the compatibility condition (2.3) leads to

$$\frac{\partial U_n}{\partial t} = V_{n+1}U_n - U_nV_n \tag{2.6}$$

Equation (2.6) is called discrete Lax pair or discrete zero curvature equation.

3 A new discrete Lax integrable equation

Consider the following isospectral problem

$$E\phi = U\phi, U = \begin{pmatrix} 0 & \frac{1}{Ar^2+Br} \\ r & \lambda + s \end{pmatrix} \tag{3.1}$$

where $r = r(n,t), s = s(n,t)$ are potential functions and $\lambda_t = 0$.

To deduce a discrete Lax integrable hierarchy of equation (3.1), letting $\Gamma = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, solving the discrete stationary zero curvature equation

$$(E\Gamma) - U\Gamma = 0 \qquad (3.2).$$

yield that

$$\begin{cases} rb^{(1)} - \frac{c}{Ar^2+Br} = 0 \\ \frac{a^{(1)}+a}{Ar^2+Br} + sb^{(1)} + \lambda b^{(1)} = 0 \\ r(a^{(1)} + a) + sc + \lambda c = 0 \\ \frac{c^{(1)}}{Ar^2+Br} - rb - s(a^{(1)} - a) - \lambda(a^{(1)} - a) = 0. \end{cases} \tag{3.3}$$

Substituting

$$a = \sum_{m\geq 0} a_m\lambda^{-m}, b = \sum_{m\geq 0} b_m\lambda^{-m}, c = \sum_{m\geq 0} c_m\lambda^{-m} \tag{3.4}$$

into (3.3) and giving rise to

$$\begin{cases} r\sum_{m\geq 0} b_m^{(1)} - \frac{\sum_{m\geq 0} c_m\lambda^{-m}}{} = 0 \\ \frac{\sum_{m\geq 0}(a_m^{(1)}+a_m)\lambda^{-m}}{Ar^2+Br} + s\sum_{m\geq 0} b_m^{(1)}\lambda^{-m} + \lambda\sum_{m\geq 0} b_m^{(1)}\lambda^{-m} = 0 \\ r\sum_{m\geq 0}(a_m^{(1)} + a_m\lambda^{-m} + s\sum_{m\geq 0} c_m\lambda^{-m} + \lambda\sum_{m\geq 0} c_m\lambda^{-m}) = 0 \\ \frac{\sum_{m\geq 0} c_m^{(1)}\lambda^{-m}}{Ar^2+Br} - r\sum_{m\geq 0} b_m\lambda^{-m} - s\sum_{m\geq 0}(a_m^{(1)} - a_m)\lambda^{-m} - \lambda\sum_{m\geq 0}(a_m^{(1)} - a_m)\lambda^{-m} = 0. \end{cases} \tag{3.5}$$

Then we have the following identities

$$\begin{cases} \frac{a_m^{(1)}+a_m}{Ar^2+Br} + sb_m^{(1)} + b_{m+1}^{(1)} = 0 \\ rb_m^{(1)} - \frac{c_m}{Ar^2+Br} = 0 \\ r(a_m^{(1)} + a_m) + sc_m + c_{m+1} = 0 \\ \frac{c_m^{(1)}}{Ar^2+Br} - s(a_m^{(1)} - a_m) - (a_{m+1}^{(1)} - a_{m+1}) = 0, (m \geq 1) \end{cases} \tag{3.6}$$

From (3.6) we have $b_0 = c_0 = 0$. If we take $a_0 = -\frac{1}{2}$ as the initial values, and assume $a_m|_{u=0} = 0 (m \geq 1), b_m|_{u=0} = 0 (m \geq 0)$ and $c_m|_{u=0} = 0 (m \geq 0)$, then a_i, b_i and c_i can be uniquely determined by (3.6). For example, when $i = 0, 1, 2$, we have

$$a_0 = \frac{1}{2}, b_0 = 0, c_0 = 0;$$

$$a_1 = 0, b_1 = \frac{1}{A(r^{(-1)})^2 + Br^{(-1)}}, c_1 = r;$$

and

$$a_2 = \frac{r}{E^{-1}(Ar^2 + Br)} = \frac{r}{A(r^{(-1)})^2 + Br^{(-1)}}, b_2 = -\frac{s^{(-1)}}{A(r^{(-1)})^2 + Br^{(-1)}}, c_2 = -sr$$

Letting $h = \sum_{m\in\mathbb{Z}} h_m\lambda^m$, and noting $h_+ = \sum_{m\geq 0} h_m\lambda^m$, we have

$$(\Gamma\lambda^m)_+ = \begin{pmatrix} \sum_{i=0}^m a_i\lambda^{m-i} & \sum_{i=0}^m b_i\lambda^{m-i} \\ \sum_{i=0}^m c_i\lambda^{m-i} & -\sum_{i=0}^m a_i\lambda^{m-i} \end{pmatrix}$$

It is easy to check

$$(E(\Gamma\lambda^m)_+)U - U(\Gamma\lambda^m)_+ = \begin{pmatrix} 0 & b_{m+1}^{(1)} \\ c_{m+1} & a_{m+1}^{(1)} - a_{m+1} \end{pmatrix} \tag{3.7}$$

Taking $V^m = (\Gamma\lambda^m)_m$, gives rise to zero-curvature equation

$$U_{t_m} = (EV^{(m)})U - UV^{(m)}. \tag{3.8}$$

Then we obtain discrete nonlinear systems of Lax integrable

$$\begin{cases} r_{t_m} = c_{m+1} \\ s_{t_m} = a_{m+1}^{(1)} - a_{m+1} \end{cases} \tag{3.9}$$

where $m \geq 0$. Especially, when $m = 0, 1$, we have

$$\begin{cases} r_{t_0} = c_1 = r, s_{t_0} = a_1^{(1)} - a_1 = 0 \\ r_{t_1} = c_2 = -sr \\ s_{t_1} = \frac{r^{(1)}}{Ar^2+Br} - \frac{r}{A(r^{(-1)})^2+Br^{(-1)}} \end{cases}$$

4 The Hamiltonian Structure of Discrete Equations

In this section, we will discuss Hamiltonian structure of nonlinear discrete equations (3.9). Take

$$V = \Gamma U^{-1} = \begin{pmatrix} (Ar+B)[-a(\lambda+s)+br] & \frac{a}{r} \\ (Ar+B)[-c(\lambda+s)-ar] & \frac{c}{r} \end{pmatrix}.$$

If C and D are the same order matrix, we define

$$\langle A, B\rangle = Tr(AB)$$

Because

$$U_\lambda = \frac{\partial U}{\partial \lambda} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, U_s = \frac{\partial u}{\partial s} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$U_r = \frac{\partial U}{\partial r} = \begin{pmatrix} 0 & -\frac{2Ar+B}{(Ar^2+Br)^2} \\ 1 & 0 \end{pmatrix}$$

then we have

$$\langle V, U_\lambda\rangle = \frac{c}{r}, \langle V, U_s\rangle = \frac{c}{r},$$

and

$$\langle V, U_r\rangle = \frac{a}{r}(1 + \frac{2Ar+B}{Ar+B}) + \frac{(2Ar+B)(\lambda+s)c}{r^2(Ar+B)}.$$

Using the trace identity

$$\frac{\delta}{\delta u}\langle V, U_\lambda\rangle = (\lambda^{-\tau}\frac{\partial}{\partial\lambda}\lambda^\tau)[< V, U_r >, < V, U_s >]^T$$

we have

$$\frac{\delta}{\delta u}\frac{c}{r} = (\lambda^{-\tau}\frac{\partial}{\partial\lambda}\lambda^\tau) \begin{pmatrix} \frac{a}{r}(1 + \frac{2Ar+B}{Ar+B}) + \frac{(2Ar+B)(\lambda+s)c}{r^2(Ar+B)} \\ \frac{c}{r} \end{pmatrix}. \tag{4.1}$$

Comparing the coefficients of λ^{-m-1} on both side of (4.1) , lead to

$$\frac{\delta}{\delta u}\frac{c_{m+1}}{r} = (\tau - m) \begin{pmatrix} \frac{a_m - a_m^{(1)}}{r} - \frac{Aa_m^{(1)}}{Ar+B} \\ \frac{c_m}{r} \end{pmatrix} \tag{4.2}$$

To determine constant τ, we take $m = 0$ in (4.2) and obtain $\tau = 0$. Then we have

$$\frac{\delta}{\delta u}H_m = \frac{\delta}{\delta u}\frac{c_{m+1}}{(-mr)} = \begin{pmatrix} \frac{a_m - a_m^{(1)}}{r} - \frac{Aa_m^{(1)}}{Ar+B} \\ \frac{c_m}{r} \end{pmatrix} \tag{4.3}$$

where $H_m = \frac{c_{m+1}}{-mr}$.
To discuss Hamiltonian structure of (4.3), we observe the following

$$\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}\begin{pmatrix} c_m \\ a_m^{(1)} - a_m \end{pmatrix} = \begin{pmatrix} \frac{a_m - a_m^{(1)}}{r} - \frac{Aa_m^{(1)}}{Ar+B} \\ \frac{c_m}{r} \end{pmatrix} \tag{4.4}$$

Clearly, (4.4) holds if and only if $A = 0$ and $B \neq 0$. If $A = 0$ and $B \neq 0$, we have the following by (4.4)

$$\frac{\delta}{\delta u}H_m = \begin{pmatrix} \frac{a_m - a_m^{(1)}}{r} \\ \frac{c_m}{r} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{r} \\ \frac{1}{r} & 0 \end{pmatrix}\begin{pmatrix} c_m \\ a_m^{(1)} - a_m \end{pmatrix} \tag{4.5}$$

So when $A = 0$, Hamiltonian structure of discrete nonlinear equations (3.9) is

$$u_{t_m} = J\frac{\delta}{\delta u}H_m = J\begin{pmatrix} \frac{a_{m+1} - a_{m+1}^{(1)}}{r} \\ \frac{c_{m+1}}{r} \end{pmatrix} = JL^m\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.6}$$

where $J = \begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix}$ is the Hamiltonian operator, and

$$L = \begin{pmatrix} -s & -\frac{1}{r^2}Er + E^{(-1)}\frac{1}{r} \\ -(E+1)(1-E)^{-1}s & -s \end{pmatrix}.$$

Acknowledgements

The author would thank Professor Lixin Tian for his valuable discussion and suggestions.

References

[1] Lixin Tian, Guilong Gui, Yue Liu: On the well-posedness problem and the scattering problem for DGH equation. *Comm. Math. Phys*. 257(3):667 - 701(2005)

[2] Lixin Tian, Guochang Fang, Guilong Gui: Well-posedness and blow-up for an integrable shallow water equation with strong dispersive term. *International Journal of Nonlinear Science* 1(1): 3-13 (2006)

[3] Lixia Wang, Jiangbo Zhou, Lihong Ren: The exact solitary wave solutions for a family of BBM equation . *International Journal of Nonlinear Science.* 1(1):68-74(2006)

[4] Yunbo Zeng: New symplectic maps:integrablity and Lax representation. *China.ann.of math*.B18:457-466(1997)

[5] Engui Fan: A Liouville integrable Hamiltonian system associated with a generalized Kuap-Newell spectral problem. *Phy. A*. 301:105-113(2001)

[6] X.H. Fan, L.X. Tian, L. H. Ren: New compactons in nonlinear atomic chain equations with first-and second-neighbour interactions. *International Journal of Nonlinear Science*. 1(2): 105-110(2006)

[7] Guizhang TU: The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems. *J. Math. Phys*. 30(2): 330-338(1989)

[8] Engui Fan: Algebra-geometric solution for Hamiltonian systems. *J.math.Phy*.46:032701-21(2005)

[9] Weimin Zhang: A Loop Algebra and its Corresponding (2+1)-Dimensional Integrable Hierarchy. *International Journal of Nonlinear Science* 1(2): :88-92(2006)

[10] X.B.Hu,H.Tam:Application of bilinear method to integ rable differential-difference equation. *Glasagow Math J*.43A:43–51(2001)

[11] X.B.HU,Z.Zhu:Some new results on the Blaszak-Marciniak:Backlund transformation and nonlinear supersition formula.*J math Phys*.38:4766(1998)

[12] WenXiu Ma and B. Fuchssteiner: Algebraic structure of discrete zero curvature equations and master symmetries of discrete evolution equations. *Journal of Mathematical Physics.* 40: 2400–2418(1999)

[13] Xixiang Xu, Shifan Wang: A hierarchy of discrete generalized Hamiltonian systems and a new integrable symplectic map. *Acta mathematica scientia A* 23(3):298-305(2003)

[14] Guizhang TU: The trace identity and its applications to the theory of discrete integrable systems. *J. of Phys A: Math Gen*.23:3903-3922(1990)