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Adomian Decomposition Approach to a Filtration Model

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Abstract: This article proposes a generalized but efficient filtration model. Employing a recent modification of the Adomian Decomposition method, reliable approximations to this nonlinear model, and consequently an important special case, are obtained.

Key words: Adomian decomposition method; optimal Lagrange interpolation; filtration

1 Introduction

In a recent paper [1], Demchik discussed a model of filtration at a decreasing rate which generalizes that of Mint [2]. In this short paper, we propose a filtration model governed by the initial boundary value problem

$$\begin{cases} \frac{\partial^2 U}{\partial t \partial x} + b(t)\frac{\partial U}{\partial t} - p\sin^2 \alpha U = 0; \\ b(t) = \beta\nu_0^{-1}(1+\gamma t),\ p = \beta\nu_0^{-1}\gamma(z-1), \\ U(0,t) = C_0(1+\gamma t)^z,\ U(x,0) = C_0\exp(-\beta\nu_0^{-1}x);\ z = a_0\nu_0\gamma^{-1}, \end{cases} \tag{1}$$

together with the relation

$$\begin{cases} C(x,t) = U(1+\gamma t)^{-z};\ \frac{\partial \rho(x,t)}{\partial t} = \beta C - a(t)\rho(x,t), \\ \nu(t) = \frac{\nu_0}{1+\gamma t};\ a(t) = a_0\nu(t), \end{cases} \tag{2}$$

where $\alpha = k\vartheta(t)$, $k > 0$, $\vartheta(t) : [0,T] \to \mathbb{R}^1$ such that $\vartheta(t)$ is smooth enough over its domain of definition, and show how it can be solved for $C(x,t)$ and $\rho(x,t)$ by a recent modification of the Adomian decomposition method (ADM) which relies on optimal Lagrange interpolation.

The notation employed herein is essentially that of Demchik; basically we recall $\nu(t)$ the filtration rate, $C(x,t)$ and $\rho(x,t)$ the required concentrations of impurities suspended in the liquid and sediment, respectively, β is a kinetic coefficient, and C_0 is the impurity concentration at the filter inlet. The exception are α, k, ϑ which, respectively, have the dimensions $dmmol^{-1}$, $dmmol^{-1}sec^{-1}$, sec. It is noted that k depicts the behaviour of a second order rate constant.

We note that this filtration process concerns the purification of a liquid and solid mixture laden with impurities while $sin^2(\alpha U)$ represents a positive oscillating impurity annihilation factor. Also, we observe that model (1) generalizes that of Demchik in the limit as α goes small, and of course Mints' if in addition $\gamma = 0$. The ADM will be employed to profer solutions to (1) in the special case when $\vartheta \in \mathbb{R}^1$.

The structure of this paper is as follows. Section 1 gives the statement of the problem, the employed technique is described in section 2, the decomposition of the proposed model, its numerical algorithm and solution profiles given in section 3 while section 4 concludes the article.

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2 Description of the Method

George Adomian developed the ADM in the 1980s and showed how it can be applied to solving nonlinear differential equations. The ADM has undergone several modification in recent times, see [3–5] for instance. In this paper, we apply a recent reliable modification of the ADM by Layeni [6] relying on optimal Lagrange interpolation polynomial.

Suppose a partial differential equation of the form

$$\begin{cases} Lu + Gu + Nu = f(x); \\ u = u(x), \end{cases} \tag{3}$$

where N is a nonlinear operator, G is a linear operator of order less that of the linear invertible operator L is given. Rephrasing (3) gives

$$u = g(x) - L^{-1}(Gu) - L^{-1}(Nu), \tag{4}$$

with $g(x)$ arising from the given initial and or boundary conditions. The extension which we shall apply to the filtration problem is that of obtaining an optimal Lagrange interpolation $\mathcal{L}_n(g(x))$ of $g(x)$, over the apposite interval $[a, b]$, such that

$$g(x) \approx \mathcal{L}_n(g(x)), \tag{5}$$

and

$$\begin{aligned} \mathcal{L}_n(g(x)) &= \sum_{j=0}^{l} g(x_j) \prod_{\substack{k=0 \\ k\neq j}}^{l} \frac{x - x_k}{x_j - x_k} \\ &= \sum_{i=0}^{l} \mathsf{A}_i(x). \end{aligned} \tag{6}$$

In equation (6) the x_k's are the roots of the $l+1$st Chebyshev polynomial $\mathcal{T}_{l+1}(x)$ in $[-1, 1]$ translated onto the interval $[a, b]$ given by

$$x_i = \frac{b+a}{2} + \frac{b-a}{2} \cos\left[\frac{(2i+1)}{2(l+1)}\pi\right], \quad i = 0, \ldots, l. \tag{7}$$

It is well known that in the ADM, limits of partial sums $\lim_{l\to\infty} \sum_{i=0}^{l} u_i$ converges to the solution u, see [7, 8]. Consequently, for an Adomian approximation $\sum_{i=0}^{n} u_i$ of u, $0 \le n \le l$, the u_is are given by the scheme [9];

$$\begin{aligned} &u_0 = \mathsf{A}_0(x), \\ &u_1 = \mathsf{A}_1(x) - L^{-1}(Gu_0) - L^{-1}(\mathsf{B}_0), \\ &u_2 = \mathsf{A}_2(x) - L^{-1}(Gu_1) - L^{-1}(\mathsf{B}_1), \\ &u_3 = \mathsf{A}_3(x) - L^{-1}(Gu_2) - L^{-1}(\mathsf{B}_2), \\ &\vdots \\ &u_n = \mathsf{A}_n(x) - L^{-1}(Gu_{n-1}) - L^{-1}(\mathsf{B}_{n-1}), \\ &\vdots \\ &u_l = \mathsf{A}_l(x) - L^{-1}(Gu_{l-1}) - L^{-1}(\mathsf{B}_{l-1}), \end{aligned} \tag{8}$$

where B_i is the Adomian polynomial expansion of $Nu(x)$ given by

$$\sum_{i=0}^{\infty} \mathsf{B}_i(u_0, \ldots, u_i) = Nu(x) = \sum_{i=0}^{\infty} \frac{1}{i!} \left[\frac{d^i}{d\lambda^i}\left(N\left(\sum_{j=0}^{i} \lambda^j u_j\right)\right)\right]_{\lambda=0}. \tag{9}$$

3 Filtration Problem

This section details the Adomian decomposition of the governing equations for the proposed model together with pertinent illustration of its dynamics.

3.1 Decomposition

A direct integration of (1) gives:

$$\begin{cases} U(x,t) = & U(0,t) + U(x,0) - U(0,0) + \int_0^x \int_0^t p \sin^2 \alpha U(x,t) dt dx \\ & - \int_0^x \int_0^t b(t) \frac{\partial U(x,t)}{\partial t} dt dx, \end{cases} \tag{10}$$

which, on reflecting the boundary condition in (1) is equivalent to

$$\begin{cases} U(x,t) = & C_0(1+\gamma t)^z + C_0 \exp(-\beta \nu_0^{-1} x) - C_0 + \int_0^x \int_0^t p \sin^2 \alpha U(x,t) dt dx \\ & - \int_0^x \int_0^t b(t) \frac{\partial U(x,t)}{\partial t} dt dx. \end{cases} \tag{11}$$

Assuming a degree of accuracy, defined by partial sum $\sum_{i=0}^{l} U_i$, an application of the modified method yields;

$$\begin{cases} \sum_{i=0}^{\infty} U_i(x,t) = & \sum_{i=0}^{l} \mathsf{A}_i(x,t) + \int_0^x \int_0^t p \sum_{i=0}^{\infty} B_i dt dx \\ & - \int_0^x \int_0^t b(t) \frac{\partial}{\partial t} \sum_{i=0}^{\infty} U_i(x,t) dt dx, \end{cases} \tag{12}$$

where

$$\begin{cases} \sum_{i=0}^{l} \mathsf{A}_i(x,t) & = \mathcal{L}_i \left(C_0(1+\gamma t)^z \right) + \mathcal{L}_i \left(C_0 \exp(-\beta \nu_0^{-1} x) \right) - C_0 \\ & = \sum_{i=0}^{l} \mathsf{A}_i^{\star} x^i + \sum_{i=0}^{l} \mathsf{A}_i^{\dagger} t^i - C_0 \end{cases} \tag{13}$$

and

$$\begin{aligned} \mathsf{A}_0(x,t) &= \mathsf{A}_0^{\star} + \mathsf{A}_0^{\dagger} - C_0; \\ \mathsf{A}_1(x,t) &= \mathsf{A}_1^{\star} x + \mathsf{A}_1^{\dagger} t; \\ &\vdots \\ \mathsf{A}_n(x,t) &= \mathsf{A}_n^{\star} x^n + \mathsf{A}_n^{\dagger} t^n; \\ &\vdots \\ \mathsf{A}_l(x,t) &= \mathsf{A}_l^{\star} x^l + \mathsf{A}_l^{\dagger} t^l. \end{aligned} \tag{14}$$

We apply the modification in [5] to construct the Adomian decomposition of the source term $\sin^2 \alpha U$. The interval of convergence of its Taylor's series is $\mathbb{R}$ and $B_i^{\star}$, the Adomian decomposition of its three term Taylor's approximation $\alpha^2 U^2 - \frac{\alpha^4}{3} U^4 + \frac{2\alpha^6}{45} U^6$, is given by

$$\begin{cases} B_i^{\star}(U_0, \ldots, U_i) = \\ \frac{1}{i!} \left[\frac{d^i}{d\lambda^i} \left(\alpha^2 \sum_{i=0}^{\infty} U_i \lambda^i - \frac{\alpha^4}{3} \left(\sum_{i=0}^{\infty} U_i \lambda^i \right)^4 + \frac{2\alpha^6}{45} \left(\sum_{i=0}^{\infty} U_i \lambda^i \right)^6 \right) \right]_{\lambda=0}. \end{cases} \tag{15}$$

The first few terms of $B_i^{\star}(U_0, \ldots, U_i)$ are

$$\begin{aligned} B_0^{\star} &= \alpha^2 U_0^2 - \frac{1}{3} \alpha^4 U_0^4 + \frac{2}{45} \alpha^6 U_0^6 \\ B_1^{\star} &= 2\alpha^2 U_0 U_1 - \frac{4}{3} \alpha^4 U_0^3 U_1 + \frac{4}{15} \alpha^6 U_0^5 U_1 \\ B_2^{\star} &= \alpha^2 U_1^2 - 2\alpha^4 U_0^2 U_1^2 + \frac{2}{3} \alpha^6 U_0^4 U_1^2 \\ &\quad + 2\alpha^2 U_0 U_2 - \frac{4}{3} \alpha^4 U_0^3 U_2 + \frac{4}{15} \alpha^6 U_0^5 U_2 \\ &\vdots \quad . \end{aligned} \tag{16}$$

The terms of the approximate solution $\sum_{i=0}^{l} U_i$ to U are thus given by

$$
\begin{aligned}
&U_0 = \mathsf{A}_0(x,t);\\
&U_1 = \mathsf{A}_1(x,t) + \int_0^x \int_0^t pB_0^\star dtdx - \int_0^x \int_0^t b(t)\frac{\partial}{\partial t}U_0(x,t)dtdx;\\
&U_2 = \mathsf{A}_2(x,t) + \int_0^x \int_0^t pB_1^\star dtdx - \int_0^x \int_0^t b(t)\frac{\partial}{\partial t}U_1(x,t)dtdx;\\
&\vdots\\
&U_n = \mathsf{A}_n(x,t) + \int_0^x \int_0^t pB_{n-1}^\star dtdx - \int_0^x \int_0^t b(t)\frac{\partial}{\partial t}U_{n-1}(x,t)dtdx;\\
&\vdots\\
&U_l = \mathsf{A}_l(x,t) + \int_0^x \int_0^t pB_{l-1}^\star dtdx - \int_0^x \int_0^t b(t)\frac{\partial}{\partial t}U_{l-1}(x,t)dtdx.
\end{aligned}
\tag{17}
$$

Subsequently, the sought concentrations $C(x,t)$ and $\rho(x,t)$ are easily obtained by reflecting the $U(x,t)$ values in relation (2).

3.2 Illustration

The pertinent ADM algorithm (16) is implemented in this section for some illustrative cases employing the symbolic mathematical software *Mathematica* 5. The consequent profiles depicting the filtration dynamics i.e. concentration $C(x,t)$ and $\rho(x,t)$, obtained by reflecting $U(x,t)$ (see Appendix) in relation (2), over $(x,t) \in [0,100] \times [0,100]$ are given thus:

Case A: Profiles of impurity concentration $C(x,t)$ and $\rho(x,t)$ in, respectively, liquid and sediments over $(x,t) \in [0,100] \times [0,100]$; $C_0 = 1000$, $l = 9$, $\gamma = \frac{1}{70}$, $\alpha = 5$, $\beta = 2 \times 10^{-20}$, $\nu_0 = \frac{1}{140}$, $z = 0.5$:

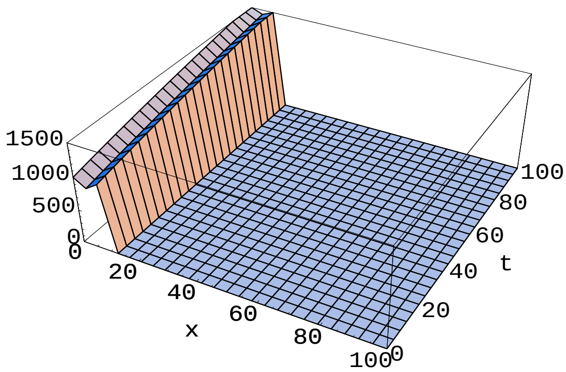

Figure 1: Plot of $C(x,t)$ over $(x,t) \in [0,100] \times [0,100]$.

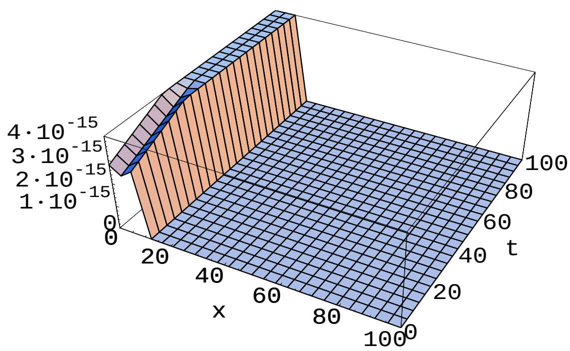

Figure 2: Plot of $\rho(x,t)$ over $(x,t) \in [0,100] \times [0,100]$.

Case B: Profiles of impurity concentration $C(x,t)$ and $\rho(x,t)$ in, respectively, liquid and sediments over $(x,t) \in [0,100] \times [0,100]$ section of the filter space-time; $C_0 = 1000$, $l = 9$, $\gamma = \frac{1}{70}$, $\alpha = 5 \times 10^{-3}$, $\beta = 2 \times 10^{-20}$, $\nu_0 = \frac{1}{140}$, $z = 0.5$:

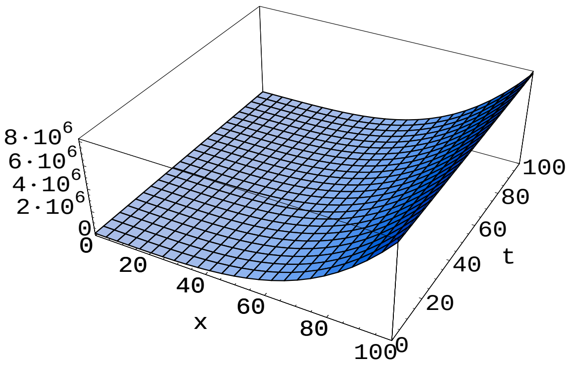

Figure 3: Plot of $C(x,t)$ over $(x,t) \in [0,100] \times [0,100]$.

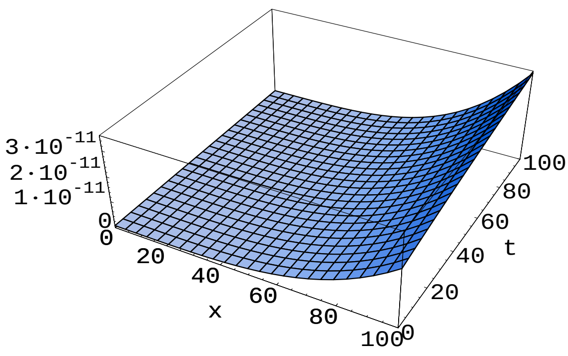

Figure 4: Plot of $\rho(x,t)$ over $(x,t) \in [0,100] \times [0,100]$.

3.3 Discussion

It is observed from the above that the purification process due to the proposed filtration model is effective; in fact, in case A the impurity concentration in the liquid and sediment is zero over the length of the filter just after 10.79464 units of time. This is not so in case B (which correspond to Demchik's case); impurity concentration increases over the filter length thereby demonstrating in a simple but effective way the decreasing rate of Demchik's model. Consequently, we conclude that provided the filter's constituent can withstand the appropriate threshold of impurity concentration without clogging, which is about $1643.38289\, mol\, dm^{-1}$ in the illustration, we have an efficient filter model. Of course, it is noted that efficiency can otherwise be obtained by using short length filters over small periods of time.

4 Conclusion

In this article, we have proposed and solved a generalized model of filtration for pertinent functions. Via the easy and efficient Adomian decomposition technique, the effectiveness of the proposed model at purification is demonstrated. Furthermore, Demchik's model of filtration at a decreasing rate and its comparison with the proposed model are shown.

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A Expressions for $U(x,t)$

Case A

$$\begin{aligned}U(x,t) &= 1000.000008354 + 7.1428403358t - 0.025504494t^2 + 0.000181443t^3 \\ &- 1.572058732 \times 10^{-6}t^4 + 1.391643590 \times 10^{-8}t^5 - 1.078514871 \times 10^{-10}t^6 \\ &+ 6.273349588 \times 10^{-13}t^7 - 2.302255376 \times 10^{-15}t^8 + 3.886871214 \times 10^{-18}t^9 \\ &+ 2.842170943 \times 10^{-13}x - 6.9444427092x^2 - 0.198412200x^3 + 0.171136325x^4 \\ &+ 0.002196490x^5 - 0.000773368x^6 - 0.000040530x^7 + 0.000010843x^8 \\ &+ 7.277060784 \times 10^{-7}x^9 - 1.671133466 \times 10^{-7}x^{10} - 1.1661264410 \times 10^{-8}x^{11} \\ &+ 1.467220425 \times 10^-9x^{12} + 1.772878185 \times 10^{-10}x^{13} - 2.005843669 \times 10^{-11}x^{14} \\ &- 7.423156873 \times 10^{-13}x^{15} + 1.049502097 \times 10^{-14}x^{16} + 1.138092482 \times 10^{-14}x^{17} \\ &- 8.054673014 \times 10^{-16}x^{18}.\end{aligned} \tag{18}$$

Case B

$$\begin{aligned}U(x,t) &= 1000.000008354 + 7.142840335t - 0.025504494t^2 + 0.000181443t^3 \\ &- 1.572058732 \times 10^{-6}t^4 + 1.391643590 \times 10^{-8}t^5 - 1.078514871 \times 10^{-10}t^6 \\ &+ 6.273349588 \times 10^{-13}t^7 - 2.302255376 \times 10^{-15}t^8 + 3.886871214 \times 10^{-18}t^9 \\ &+ 2.842170943 \times 10^{-13}x - 6.041124362 \times 10^{-14}x^2 + 6.661129407 \times 10^{-15}x^3 \\ &+ 0.125036993x^4 + 1.127569366 \times 10^{-17}x^5 - 2.171224532 \times 10^{-19}x^6 \\ &+ 1.472925518 \times^{-21} x^7 - 1.084896406 \times 10^{-23}x^8 + 2.078332739 \times 10^{-26}x^9 \\ &- 3.193404989 \times 10^{-26}x^{10} + 6.222335340 \times 10^{-47}x^{11} - 1.790766909 \times 10^{-64}x^{12} \\ &+ 6.751602048 \times 10^{-83}x^{13} - 5.932662243 \times 10^{-102}x^{14} + 8.90337337 \times 10^{-122}x^{15} \\ &- 2.311313741 \times 10^{-140}x^{16} + 2.550457019 \times 10^{-159}x^{17} - 1.001810628 \times 10^{-178}x^{18}.\end{aligned} \tag{19}$$