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Existence of Solutions to Nonlocal Neutral Functional Differential and Integrodifferential Equations

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Abstract: This paper is concerned with a class of partial nonlocal neutral functional differential and integrodifferential equations with bounded delay in Banach spaces, which are more general than those models been studied. Some existence results of mild solutions to such problems are obtained under the conditions in respect of the Hausdorff's measure of noncompactness.

Key words: neutral partial differential equation; nonlocal condition; Hausdorff measure of noncompactness; mild solution

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1 Introduction

The purpose of this paper is to study the existence of mild solutions to partial neutral functional differential equation with nonlocal condition

$$\frac{d}{dt}(x(t)+g(t,x(t),x_t)) = Ax(t)+f(t,x(t),x_t), \quad t\in[0,b], \tag{1.1}$$

$$x_0 = \phi + h(x) \tag{1.2}$$

and functional integrodifferential equation

$$\frac{d}{dt}(x(t)+g(t,x(t),x_t)) = Ax(t)+\int_0^t K(t,s)f(s,x(s),x_s)ds, t\in[0,b], \tag{1.3}$$

$$x_0 = \phi + h(x) \tag{1.4}$$

where A is the infinitesimal generator of an analytic semigroup $\{T(t) : t\geq 0\}$ of linear operators defined on a Banach space X, $x\in C([0,b];X)$, and $x_t : [-q,0]\to X$ defined by $x_t(\theta)=x(t+\theta)$ for $\theta\in[-q,0]$; $f,g:[0,b]\times X\times C([-q,0];X)\to X$, $K:[0,b]\times[0,b]\to(0,+\infty)$ and $h: C([0,b];X)\to C([-q,0];X)$ are appropriate functions; $b,q>0$ are constants.

The theory of differential and functional differential equations with nonlocal conditions has been extensively studied in the literature. Some results on the existence, uniqueness, and stability of solutions are given by Byzewski [8], Byzewski and Alca [9], Balachandran and Chandrasekaran [3] and Byszewski and Lakshmikantham [10]. Recently Bahuguna [1] obtained existence and regularity results for functional differential equation with nonlocal condition using semigroup theorey. Fan et al. [12] discussed semilinear differential equations with nonlocal condition using measure of noncompacness. Xue [24] proved the existence results for nonlinear nonlocal Cauchy problem. Tian and Li [22] considered a kind of nonlinear dispersive shallow water wave equations by using viscous approximations and prior estimates, and Ding and Gou [11] considered the quasi-linear evolution equations.

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Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last decades, see, for example, Hernández and Henríquez [17, 18], Hernández [16], Balachandran and Sakthivel [5] and references therein. Hernández [15] established the existence results for partial neutral functional differential equations with nonlocal conditions modelled as

$$\frac{d}{dt}(u(t) + F(t, u_t)) = Au(t) + G(t, u_t), \quad 0 \le t \le T,$$
$$u_0 = \phi + q(u_{t_1}, u_{t_2}, \cdots, u_{t_n}) \in \Omega,$$

Hernández, Rabello and Henríquez [19] also studied a class of impulsive neutral functional differential equations. Bahuguna and Agarwal [2] studied the approximation of solution to a partial neutral functional differential equation with nonlocal history condition

$$\frac{d}{dt}(u(t) + g(t, u(t), u(t-\tau_1)) + Au(t) = f(t, u(t), u(t-\tau_2)), \quad t > 0,$$
$$h(u) = \phi, \quad on[-\tau, 0],$$

in a separable Hilbert space, where $\tau = \max\{\tau_1, \tau_2\}$, $\tau_1, \tau_2 > 0$. Benchohra and Ntouyas [4] studied the neutral functional differential inclusion with nonlocal condition

$$\frac{d}{dt}[y(t) - f(t, y_t)] \in Ay(t) + F(t, y_t), \quad a.e.\, t \in [0, b]$$
$$y(t) + (\xi(y_{t_1}, \cdots, y_{t_p}))(t) = \phi(t), \quad t \in [-r, 0]$$

and neutral functional integrodifferential inclusion of the form

$$\frac{d}{dt}[y(t) - f(t, y_t)] \in Ay(t) + \int_0^t K(t,s)F(s, y_s)ds, t \in [0, b]$$
$$y(t) + (\xi(y_{t_1}, \cdots, y_{t_p}))(t) = \phi(t), \quad t \in [-r, 0]$$

where $\xi : [C([-r, 0]; X)]^p \to C([-r, 0]; X)$ is completely continuous and uniformly bounded and the semigroup $\{T(t) : t \ge 0\}$ generated by A is compact.

In this paper, we consider the partial neutral functional differential equations with nonlocal condition (1.1)-(1.2) and integrodifferential equations (1.3)-(1.4), which is more general than those mentioned above. We give the existence of mild solution of the system (1.1)-(1.2) and (1.3)-(1.4) under the conditions in respect of the Hausdorff's measure of noncompactness. Neither the semigroup $\{T(t) : t \ge 0\}$ nor the function f is needed to be compact in our result. Hence we extend and improve some previous results.

2 Preliminaries

Throughout this paper X will represent a Banach space with norm $\|\cdot\|$. As usual, $C([a, b]; X)$ denotes the Banach space of all continuous X-valued functions defined on $[a, b]$ with norm $\|x\|_{[a,b]} = \sup_{s\in[a,b]} \|x(s)\|$ for $x \in C([a, b]; X)$.

Let $A : D(A) \subset X \to X$ be the infinitesimal generator of an uniformly bounded analytic semigroup of linear operators $\{T(t) : t \ge 0\}$ on X such that $0 \in \rho(A)$ and we always assume that $\|T(t)\| \le M$ for every $t \in [0, b]$. Under these conditions it is possible to define the fractional power $(-A)^\alpha$, $0 < \alpha \le 1$, as a closed linear operator on its domain $D((-A)^\alpha)$. Furthermore, $D((-A)^\alpha)$ is dense in X and the expression $\|x\|_\alpha = \|(-A)^\alpha x\|$ defines a norm on $D((-A)^\alpha)$. If X_α is the space $D((-A)^\alpha)$ endowed with the norm $\|\cdot\|_\alpha$, then the following properties hold ([23], pp.74).

Lemma 2.1 *Let $0 < \alpha \le \beta \le 1$. Then the following properties hold:*

(i) X_β is a Banach space and $X_\beta \hookrightarrow X_\alpha$ is continuous.

(ii) The function $s \mapsto (-A)^\alpha T(s)$ is continuous in the uniform operator topology on $(0, \infty)$ and there exists a positive constant C_α such that $\|(-A)^\alpha T(t)\| \le \frac{C_\alpha}{t^\alpha}$ for every $t > 0$.

For more details of the semigroup theory we refer the readers to [23].

The $Hausdorff's\ measure\ of\ noncompactness\ \chi_Y$ is defined by $\chi_Y(B) = \inf\{r > 0, B$ can be covered by finite number of balls with radii $r\}$ for bounded set B in any Banach space Y.

Lemma 2.2 *([6]): Let Y be a real Banach space and $B, C \subseteq Y$ be bounded,the following properties are satisfied :*

(1). B is pre-compact if and only if $\chi_Y(B) = 0$;

(2). $\chi_Y(B) = \chi_Y(\overline{B}) = \chi_Y(convB)$ where $\overline{B}$ and $convB$ mean the closure and convex hull of B respectively;

(3). $\chi_Y(B) \leq \chi_Y(C)$ when $B \subseteq C$;

(4). $\chi_Y(B + C) \leq \chi_Y(B) + \chi_Y(C)$ where $B + C = \{x + y; x \in B, y \in C\}$;

(5). $\chi_Y(B \cup C) \leq max\{\chi_Y(B), \chi_Y(C)\}$;

(6). $\chi_Y(\lambda B) = |\lambda|\chi_Y(B)$ for any $\lambda \in R$;

(7). If the map $Q : D(Q) \subseteq Y \to Z$ is Lipschitz continuous with constant k then $\chi_Z(QB) \leq k\chi_Y(B)$ for any bounded subset $B \subseteq D(Q)$,where Z be a Banach space;

(8). $\chi_Y(B) = \inf\{d_Y(B, C); C \subseteq Y\ be\ precompact\} = \inf\{d_Y(B, C); C \subseteq Y\ be\ finite\ valued\}$, where $d_Y(B, C)$ means the nonsymmetric (or symmetric) $Hausdorff$ distance between B and C in Y.

(9). If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subsets of Y and $\lim_{n\to+\infty} \chi_Y(W_n) = 0$,then $\bigcap_{n=1}^{+\infty} W_n$ is nonempty and compact in Y.

The map $Q : W \subseteq Y \to Y$ is said to be a $\chi_Y - contraction$ if there exists a positive constant $k < 1$ such that $\chi_Y(Q(C)) \leq k\chi_Y(C)$ for any bounded closed subset $C \subseteq W$ where Y is a Banach space.

Lemma 2.3 *([6]):(Darbo-Sadovskii) If $W \subseteq Y$ is bounded closed and convex ,the continuous map $Q : W \to W$ is a χ_Y-contraction,then the map Q has at least one fixed point in W.*

In this paper we denote χ by the $Hausdorff's$ measure of noncompactness of X and denote χ_c by the $Hausdorff's$ measure of noncompactness of $C([0, b]; X)$. To discuss the existence we need the following lemmas in this paper.

Lemma 2.4 *([6]): If $W \subseteq C([0, b]; X)$ is bounded , then*

$$\chi(W(t)) \leq \chi_c(W)$$

for all $t \in [0, b]$,where $W(t) = \{u(t); u \in W\} \subseteq X$.Fouthermore if W is equicontinuous on [0,b], then $\chi(W(t))$ is continuous on $[0, b]$ and

$$\chi_c(W) = \sup\{\chi(W(t)), t \in [0, b]\}.$$

Lemma 2.5 *([14, 20]): If $\{u_n\}_{n=1}^{\infty} \subset L^1(a, b; X)$ is uniformly integrable, then $\chi(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and*

$$\chi(\{\int_a^t u_n(s)ds\}_{n=1}^{\infty}) \leq \xi \int_a^t \chi(\{u_n(s)\}_{n=1}^{\infty})ds,$$

where $\xi = 1$ if $\{u_n\}$ is equicontinuous and $\xi = 2$ if $\{u_n\}$ is not equicontinuous.

Lemma 2.6 *([6]): If $W \subseteq C([0, b]; X)$ is bounded and equicontinuous , then $\chi(W(s))$ is continuous and*

$$\chi(\int_0^t W(s)ds) \leq \int_0^t \chi(W(s))ds$$

for all $t \in [0, b]$, where $\int_0^t W(s)ds = \{\int_0^t x(s)ds : x \in W\}$.

The C_0 semigroup $T(t)$ is said to be $equicontinuous$ if $t \to \{T(t)x : x \in B\}$ is equicontinuous for $t > 0$ for all bounded set B in X. It is known that the analytic semigroup is equicontinuous. The following lemma is obvious.

Lemma 2.7 *: If the semigroup $T(t)$ is equicontinuous and $\eta \in L(0, b; R^+)$, then the set $\{\int_0^t T(t - s)u(s)ds, \|u(s)\| \leq \eta(s)$ for a.e. $s \in [0, b]\}$ is equicontinuous for $t \in [0, b]$.*

3 Existence results for neutral functional differential equations

In order to define the concept of mild solution for (1.1)-(1.2), by comparison with the abstract Cauchy initial value problem

$$\frac{d}{dt}x(t) = Ax(t) + f(t), \quad x(0) = a$$

whose properties are well known [23], we associate (1.1)-(1.2) to the integral equation

$$\begin{aligned} x(t) \;&= T(t)(\phi(0) + h(x)(0) + g(0, \phi(0) + h(x)(0), \phi + h(x))) \\ &-g(t, x(t), x_t) - \int_0^t AT(t-s)g(s, x(s), x_s)ds \\ &+ \int_0^t T(t-s)f(s, x(s), x_s)ds, \quad t \in [0, b]. \end{aligned} \tag{3.1}$$

Definition 3.1 *A continuous function* $x : [-q, b] \to X$ *is said to be a mild solution to the nonlocal neutral problem (1.1)-(1.2) if* $x_0 = \phi + h(x)$, *for each* $t \in [0, b]$ *the function* $s \mapsto AT(t-s)g(s, x(s), x_s)$ *is integrable on* $[0, t]$, *and the integral equation (3.1) is satisfied.*

In this section by using the usual techniques of the Hausdorff measure of noncompactnes and its applications in differential equations in Banach spaces (see,e.g. [6], [21]) we give some existence results of the nonlocal neutral problem (1.1)-(1.2). Here we list the following hypotheses.

(Hf)(1): $f : [0, b] \times X \times C([-q, 0]; X) \to X$ satisfies the $Carath\acute{e}odory$-type condition, i.e.,$f(\cdot, x, \phi) : [0, b] \to X$ is measurable for all $(x, \phi) \in X \times C([-q, 0]; X)$ and $f(t, \cdot) : X \times C([-q, 0]; X) \to X$ is continuous for a.e. $t \in [0, b]$;
(2): There exists an integrable function $\alpha : [0, b] \to [0, +\infty)$ and a continuous nondecreasing function $\Omega : [0, +\infty) \to [0, +\infty)$ such that

$$\|f(t, x, \phi)\| \le \alpha(t)\Omega(\|x\| + \|\phi\|_{[-q,0]})$$

for all $(t, x, \phi) \in [0, b] \times X \times C([-q, 0]; X)$;
(3): There exists an integrable function $\eta : [0, b] \to [0, +\infty)$ such that:

$$\chi(T(s)f(t, D_1, D_2)) \le \eta(t)(\chi(D_1) + \sup_{-q\le\theta\le0} \chi(D_2(\theta)))$$

for a.e. $t, s \in [0, b]$ and any bounded subset $D_1 \subset X$ and $D_2 \subset C([-q, 0]; X)$, where $D_2(\theta) = \{v(\theta) : v \in D_2\}$.

(Hg): There exists $0 < \beta < 1$ such that g is X_β-valued, $(-A)^\beta g(.)$ is continuous and there exist positive constants c_1, c_2 and L_g such that

$$\|(-A)^\beta g(t, x, \phi)\| \le c_1(\|x\| + \|\phi\|_{[-q,0]}) + c_2$$

and

$$\|(-A)^\beta g(t, x_1, \phi_1) - (-A)^\beta g(t, x_2, \phi_2)\| \le L_g(\|x_1 - x_2\| + \|\phi_1 - \phi_2\|_{[-q,0]})$$

for all $t \in [0, b]$, $x, x_1, x_2 \in X$ and $\phi, \phi_1, \phi_2 \in C([-q, 0]; X)$.

(Hh)(1): $h : C([0, b]; X) \to C([-q, 0]; X)$ is Lipschitz continuous in the following sense: there exists a positive constants L_h such that

$$\|h(x) - h(y)\|_{[-q,0]} \le L_h\|x - y\|_{[0,b]}$$

for all $x, y \in C([0, b]; X)$;
(2): h is uniformly bounded, i.e., there is a positive constant Nsuch that

$$\|h(x)\|_{[-q,0]} \le N$$

for all $x \in C([0, b]; X)$.

(Hc): $2c_1(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}) + M\int_0^b \alpha(s)ds \liminf_{k\to\infty} \frac{\Omega(2k)}{k} < 1$.

Now, we are prepared to state and prove our main theorem of this section.

Theorem 3.2 *Assume the hypotheses (Hf), (Hg), (Hh) and (Hc) are satisfied. Then for every* $\phi \in C([-q,0];X)$*, the nonlocal neutral problem (1.1)-(1.2) has at least one mild solution if*

$$L_0 + 2\int_0^b \eta(t)dt < 1,$$

where $L_0 = ML_h(1 + 2L_g\|(-A)^{-\beta}\|) + 2L_g(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta})$.

Proof. For each $k \in \mathbb{N}$ (the set of all positive integers) we denote by $B_k = B_k(C([-q,b];X)) = \{x \in C([-q,b];X) : \|x(s)\| \le k, s \in [-q,b]\}$. For each $x \in B_k$, the restriction of x on $[0,b]$ $x|_{[0,b]}$ is an element of $B_k(C([0,b];X))$. For simplicity, we also write $h(x|_{[0,b]})$ as $h(x)$.

Define $\Gamma : C([-q,b];X) \to C([-q,b];X)$ by $\Gamma = \Gamma_1 + \Gamma_2$, where

$$\Gamma_1 x(t) = \begin{cases} \phi(t) + h(x)(t), & t \in [-q,0], \\ T(t)[\phi(0) + h(x)(0) + g(0, \phi(0) + h(x)(0), \phi + h(x))] & \\ -g(t, x(t), x_t) - \int_0^t AT(t-s)g(s, x(s), x_s)ds, & t \in [0,b], \end{cases}$$

$$\Gamma_2 x(t) = \begin{cases} 0, & t \in [-q,0], \\ \int_0^t T(t-s)f(s, x(s), x_s)ds, & t \in [0,b]. \end{cases}$$

Since the function g satisfies (Hg), from the continuity of the function $s \to AT(t-s)$ in the uniform operator topology on $(0,t)$ and the estimate

$$\begin{aligned} & \|(-A)T(t-s)g(s,x(s),x_s)\| \\ = & \|(-A)^{1-\beta}T(t-s)(-A)^\beta g(s,x(s),x_s)\| \\ \le & \tfrac{C_{1-\beta}}{(t-s)^{1-\beta}}(2c_1k + c_2), \end{aligned}$$

it follows that $AT(t-s)g(s,x(s),x_s)$ is integrable on $(0,t)$, for every $t \in (0,b]$ and $x \in B_k$. Therefore Γ is well-defined and with values in $C([-q,b];X)$. In addition, Γ is continuous by the usual technique involving the hypotheses (Hf), (Hg), (Hh) and Lebesgue's dominate convergence theorem.

It is easily seen that the fixed point of Γ is the mild solution of the equation (1.1)-(1.2). We shall show that Γ is a continuous χ_c-contraction on some bounded closed convex subset $B_k \subset C([-q,b];X)$. And then by using Darbo-Sadovskii's fixed point theorem we get a fixed point of Γ.

We first show that there is a $k \in \mathbb{N}$ such that $\Gamma(B_k) \subset B_k$. Suppose contrary that for each $k \in \mathbb{N}$ there is $x^k \in B_k$ and $t^k \in [0,b]$ such that $\|\Gamma x^k(t^k)\| > k$. If $t^k \in [-q,0]$, then

$$\begin{aligned} k < \|\Gamma x^k(t^k)\| &\le \|\phi(t^k) + h(x^k)(t^k)\| \\ &\le \|\phi\|_{[-q,0]} + N \end{aligned}$$

and if $t^k \in [0,b]$ we have

$$\begin{aligned} k < \|\Gamma x^k(t^k)\| \le\ & \|\Gamma_1 x^k(t^k)\| + \|\Gamma_2 x^k(t^k)\| \\ \le\ & M[\|\phi(0)\| + \|h(x^k)(0)\| \\ & + \|g(0, \phi(0) + h(x^k)(0), \phi + h(x^k))\|] \\ & + \|g(t^k, x^k(t^k), x^k_{t^k})\| \\ & + \int_0^{t^k} \|AT(t^k - s)g(s, x^k(s), x^k_s)\|ds \\ & + \int_0^{t^k} \|T(t^k - s)f(s, x^k(s), x^k_s)\|ds \\ \le\ & M[\|\phi(0)\| + N + \|(-A)^{-\beta}\|(2c_1(\|\phi\|_{[-q,0]} + N) + c_2)] \\ & + (\|(-A)^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta})(2c_1k + c_2) \\ & + M\Omega(2k)\int_0^b \alpha(s)ds. \end{aligned}$$

Denote by L_k the right hand side of the above inequality, then we get that

$$k < \max(\|\phi\|_{[-q,0]} + N, L_k), \tag{3.2}$$

Divided by k on both sides of (3.2) and then take liminf as $k \to \infty$ we have

$$2c_1(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}) + M\int_0^b \alpha(s)ds \liminf_{k\to\infty} \frac{\Omega(2k)}{k} \geq 1,$$

which contradicts the hypotheses (Hc). Therefore, there is a $k \in \mathbb{N}$ such that $\Gamma(B_k) \subset B_k$. From now on we will restrict Γ on such B_k.

Below we will verify that Γ is a χ_c-contraction. To this end, we first show that Γ_1 is Lipschitzian with Lipschitz constant L_0. In fact, take x and y in B_k. By the hypotheses (Hg), (Hh) and Lemma 2.1 we have

$$\begin{aligned}\|\Gamma_1 x(t) - \Gamma_1 y(t)\| &\leq \|h(x)(t) - h(y)(t)\| \\ &\leq \|h(x) - h(y)\|_{[-q,0]} \\ &\leq L_h\|x - y\|_{[0,b]}\end{aligned}$$

for $t \in [-q, 0]$ and

$$\begin{aligned}& \|\Gamma_1 x(t) - \Gamma_1 y(t)\| \\ \leq\ & M(\|h(x)(0) - h(y)(0)\| \\ & +\|g(0, \phi(0) + h(x)(0), \phi + h(x)) - g(0, \phi(0) + h(y)(0), \phi + h(y))\|) \\ & +\|g(t, x(t), x_t) - g(t, y(t), y_t)\| \\ & + \textstyle\int_0^t \|AT(t - s)(g(s, x(s), x_s) - g(s, y(s), y_s)\|ds \\ \leq\ & ML_h(1 + 2\|(-A)^{-\beta}\|L_g)\|x - y\|_{[-q,b]} \\ & +2L_g\|(-A)^{-\beta}\|\|x - y\|_{[-q,b]} \\ & + \textstyle\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \cdot 2L_g\|x - y\|_{[-q,b]}ds \\ =\ & [ML_h(1 + 2L_g\|(-A)^{-\beta}\|) + 2L_g(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta})]\|x - y\|_{[-q,b]} \\ =\ & L_0\|x - y\|_{[-q,b]}\end{aligned}$$

for $t \in [0, b]$. It follows that

$$\|\Gamma_1 x - \Gamma_1 y\|_{[-q,b]} \leq L_0\|x - y\|_{[-q,b]}.$$

This means that Γ_1 is Lipschitzian with Lipschitz constant L_0.

On the other hand, since the analytic semigroup $\{T(t) : t \geq 0\}$ generated by A is equicontinuous, from Lemma 2.4, 2.6, 2.7 and (Hf)(3) we have

$$\begin{aligned}\chi_c(\Gamma_2 W) &= \sup_{-q\leq t\leq b} \chi(\Gamma_2 W(t)) \\ &= \sup_{0\leq t\leq b} \chi(\Gamma_2 W(t)) \\ &\leq \sup_{0\leq t\leq b} \chi(\textstyle\int_0^t T(t - s)f(s, W(s), W_s)ds) \\ &\leq \sup_{0\leq t\leq b} \textstyle\int_0^t \chi(T(t - s)f(s, W(s), W_s))ds \\ &\leq \sup_{0\leq t\leq b} \textstyle\int_0^t \eta(s)(\chi(W(s)) + \sup_{q\leq t\leq b} \chi(W(s + \theta)))ds \\ &\leq \sup_{0\leq t\leq b} \textstyle\int_0^t \eta(s)(\chi(W(s)) + \sup_{0\leq\tau\leq s} \chi(W(\tau)))ds \\ &\leq 2\chi_c(W) \textstyle\int_0^b \eta(s)ds\end{aligned}$$

for every bounded subset $W \subset C([-q, b]; X)$, where $W(t) = \{x(t) : x \in W\} \subset X$ and $W_t = \{x_t : x \in W\} \subset C([-q, 0]; X)$.

From Lemma 2.2 we obtain that

$$\begin{aligned}\chi_c(\Gamma W) &\leq \chi_c(\Gamma_1 W) + \chi_c(\Gamma_2 W) \\ &\leq (L_0 + 2\textstyle\int_0^b \eta(s)ds)\chi_c(W)\end{aligned}$$

for each bounded subset $W \subset C([-q,b];X)$. Since $L_0+2\int_0^b \eta(s)ds < 1$, we get that Γ is a χ_c-contraction. By Lemma 2.3 (Darbo-Sadovskii's fixed point theorem), there is a fixed point x of Γ on B_k, which is a mild solution of the equation (1.1)-(1.2), and the proof is complete. ■

Remark 3.3 *If the semigroup $\{T(t) : t \geq 0\}$ or the function f is compact (see, e.g.,[4], [13] and [25]), then (Hf)(3) is automatically satisfied.*

Remark 3.4 *In [17] and [19], the authors suppose that $\{T(t) : t \geq 0\}$ and f satisfies the conditions which is similar to the following condition (a):*

(a) For every $r > 0$ and all $\varepsilon > 0$, there is a compact set $U_{\varepsilon,r} \subset X$ such that $T(\varepsilon)f(s,x,\phi) \subset U_{\varepsilon,r}$ for every $(s,x,\phi) \in [0,b] \times B_r(X) \times B_r(C([-q,0];X))$.

Since for each $t \in (0,b]$ and $\varepsilon \in (0,t)$, $T(t) = T(\varepsilon)T(t-\varepsilon)$, it is easily to see that condition (a) is a special case of (Hf)(3).

If we replace the condition (Hf)(3) of Theorem 3.2 by

$(\mathrm{Hf})(3)^{'}$: There is an integrable function $\eta : [0,b] \to [0,+\infty)$ such that

$$\chi(f(t,D_1,D_2)) \leq \eta(t)(\chi(D_1) + \sup_{-q\leq\theta\leq 0} \chi(D_2(\theta)))$$

for a.e. $t \in [0,b]$ and any bounded subset $D_1 \subset X$ and $D_2 \subset C([-q,0];X)$,

then we can get the obvious result:

Theorem 3.5 *Assume the hypotheses (Hf)(1), (2) $(3)^{'}$, (Hg), (Hh) and (Hc) are satisfied. Then for every $\phi \in C([-q,0];X)$, the nonlocal neutral problem (1.1)-(1.2) has at least one mild solution if*

$$L_0 + 2M\int_0^b \eta(t)dt < 1.$$

In some of the early related results in references and the two results above, it is supposed that the map h is uniformly bounded. We indicate here that this condition can be released. Indeed, the fact that h is Lipschitzian implies that h is bounded on bounded subset. Thus we have

Theorem 3.6 *Assume the hypotheses (Hf), (Hg), and (Hh)(1) are satisfied. Then for every $\phi \in C([-q,0];X)$, the nonlocal neutral problem (1.1)-(1.2) has at least one mild solution if*

$$L_0 + 2\int_0^b \eta(t)dt < 1,$$

and

$$2c_1(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}) + M\liminf_{k\to\infty}(\int_0^b \alpha(s)ds\frac{\Omega(2k)}{k} + (1+c_1\|(-A)^{-\beta}\|)\frac{\gamma(k)}{k}) < 1,$$

where $\gamma(r) = \sup\{\|h(x)\|_{[-q,0]} : x \in C([0,b];X), \|x\|_{[0,b]} \leq r\}$.

Proof. Proceeding as in the proof of Theorem 3.1 we only need to prove that there is a $k \in \mathbb{N}$ such that $\Gamma(B_k) \subset B_k$. To do this we need to modify the estimate of $\|\Gamma_1 x^k(t^k)\|$. In fact, in this case,

$$\begin{aligned}\|\Gamma_1 x^k(t^k)\| &\leq M[\|\phi(0)\| + \|h(x)\|_{[-q,0]} \\ &\quad +\|(-A)^{-\beta}\|(c_1(\|\phi\|_{[-q,0]} + \|h(x)\|_{[-q,0]}) + c_2)] \\ &\quad +\|(-A)^{-\beta}\|(2c_1\|x\|_{[0,b]} + c_2) \\ &\leq M(1+c_1\|(-A)^{-\beta}\|)\gamma(k) \\ &\quad +M[\|\phi\|_{[-q,0]}(1+c_1\|(-A)^{-\beta}\|) + c_2] \\ &\quad +\|(-A)^{-\beta}\|(2c_1 k + c_2).\end{aligned}$$

Then we can complete the proof similar to the proof of Theorem 3.2. ■

4 Existence results for neutral functional integrodifferential equations

In this section we consider the existence results of the problem (1.3)-(1.4). We define the mild solution for problem (1.3)-(1.4) by the integral equation

$$\begin{aligned} x(t) &= T(t)(\phi(0) + h(x)(0) + g(0, \phi(0) + h(x)(0), \phi + h(x))) \\ &\quad -g(t, x(t), x_t) - \int_0^t AT(t-s)g(s, x(s), x_s)ds \\ &\quad + \int_0^t T(t-s)\int_0^s K(s,r)f(r, x(r), x_r)drds, \quad t \in [0, b]. \end{aligned} \tag{4.1}$$

Definition 4.1 *A continuous function $x : [-q, b] \to X$ is said to be a mild solution to the nonlocal neutral problem (1.3)-(1.4) if $x_0 = \phi + h(x)$, for each $t \in [0, b]$ the function $s \mapsto AT(t-s)g(s, x(s), x_s)$ is integrable on $[0, t]$, and the integral equation (4.1) is satisfied.*

Let us list the following hypotheses:

(HK): (1) For each $t \in (0, b]$, $K(t, \cdot)$ is measurable on $[0, t]$, and

$$K(t) = ess\sup\{|K(t, s)| : 0 \le s \le t\}$$

is bounded on $[0, b]$.

(2) The map $t \mapsto K_t$ is continuous from $[0, b]$ to $L^\infty(0, b; \mathbb{R}^+)$, here $K_t(s) = K(t, s)$.

(Hc)$'$: $2c_1(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}) + bKM\int_0^b \alpha(s)ds \liminf_{k\to\infty} \frac{\Omega(2k)}{k} < 1$, where $K = \sup_{0\le t\le b} K(t)$.

Now we are in the position to state our main result of this section.

Theorem 4.2 *Assume that the hypotheses (Hf)(1), (2), $(3)'$, (Hg), (Hh), (HK) and (Hc)$'$ are satisfied. Then for every $\phi \in C([-q, 0]; X)$, the problem (1.3)-(1.4) has at least one mild solution provided*

$$L_0 + 8bMK\int_0^b \eta(s)ds < 1.$$

Proof. Consider the map $\Gamma : C([-q, b]; X) \to C([-q, b]; X)$ defined by $\Gamma = \Gamma_1 + \Gamma_2$, where

$$\Gamma_1 x(t) = \begin{cases} \phi(t) + h(x)(t), & t \in [-q, 0], \\ T(t)[\phi(0) + h(x)(0) + g(0, \phi(0) + h(x)(0), \phi + h(x))] & \\ -g(t, x(t), x_t) - \int_0^t AT(t-s)g(s, x(s), x_s)ds, & t \in [0, b], \end{cases}$$

$$\Gamma_2 x(t) = \begin{cases} 0, & t \in [-q, 0], \\ \int_0^t T(t-s)\int_0^s K(s,r)f(r, x(r), x_r)drds, & t \in [0, b]. \end{cases}$$

As in the proof of Theorem 3.2, we can verify (with some obvious modifications) that Γ is a continuous χ_c-contraction, and that there is a $k \in \mathbb{N}$ such that Γ maps B_k into itself. Thus Darbo-Sadovskii's fixed point theorem can be used to get a fixed point of Γ, which is a mild solution of (1.3)-(1.4).

Here we only need to prove that there is a $k \in \mathbb{N}$ such that $\Gamma(B_k) \subset B_K$ and to estimate $\chi_c\Gamma_2(W)$ for every bounded subset $W \subset C([-q, b]; X)$.

Suppose for each $k \in \mathbb{N}$ there is $x^k \in B_k$ and $t^k \in [-q, b]$ such that $\|\Gamma x^k(t^k)\| > k$, then if $t^k \in [-q, 0]$ we have

$$\begin{aligned} k \le \|\Gamma x^k(t^k)\| &\le \|\phi(t^k) + h(x^k)(t^k)\| \\ &\le \|\phi\|_{[-q,0]} + N \end{aligned} \tag{4.2}$$

and if $t^k \in [0,b]$ we have

$$\begin{aligned} k < \|\Gamma x^k(t^k)\| &\le \|\Gamma_1 x^k(t^k)\| + \|\Gamma_2 x^k(t^k)\| \\ &\le M[\|\phi(0)\| + \|h(x^k)(0)\| \\ &\quad + g(0, \phi(0) + h(x^k)(0), \phi + h(x^k))\|] \\ &\quad + \|g(t^k, x^k(t^k), x^k_{t^k})\| \\ &\quad + \int_0^{t^k} \|AT(t^k - s)g(s, x^k(s), x^k_s)\| ds \\ &\quad + \int_0^{t^k} M \int_0^s |K(s)| \|f(r, x^k(r), x^k_r)\| drds \\ &\le M[\|\phi(0)\| + N + \|(-A)^{-\beta}\|(2c_1(\|\phi\|_{[-q,0]} + N) + c_2)] \\ &\quad + (\|(-A)^{-\beta}\| \frac{C_{1-\beta}b^\beta}{\beta})(2c_1 k + c_2) \\ &\quad + MK \int_0^t \int_0^s \alpha(r)\Omega(2k) drds \\ &\le M[\|\phi(0)\| + N + \|(-A)^{-\beta}\|(2c_1(\|\phi\|_{[-q,0]} + N) + c_2)] \\ &\quad + (\|(-A)^{-\beta}\| \frac{C_{1-\beta}b^\beta}{\beta})(2c_1 k + c_2) \\ &\quad + bMK \int_0^t \alpha(s) ds \cdot \Omega(2k). \end{aligned}$$

Denote by L_k the right hand side of the above inequality, then we have

$$k < \|\Gamma(x^k(t^k)\| \le \max(\|\phi\|_{[-q,0]} + N, L_k),$$

Divided by k on both sides of inequality (4.2) and then take liminf as $k \to \infty$ we have

$$2c_1(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}) + bKM \int_0^b \alpha(s) ds \liminf_{k\to\infty} \frac{\Omega(2k)}{k} \ge 1,$$

which contradicts the hypotheses (Hc)$'$. Hence there is a $k \in \mathbb{N}$ such that $\Gamma(B_k) \subset B_K$.

Now, for every bounded subset $W \subset C([-q,b];X)$ and any $\varepsilon > 0$, we can take a sequence $\{x_n\}_{n=0}^\infty \subset W$ such that $\chi_c(W) \le 2\chi_c(\{x_n\}_{n=0}^\infty) + \varepsilon$ (see, e.g., [7] pp.125). From (Hf)(2) and (HK) we know that $\{K(s,\cdot)f(\cdot, x_n(\cdot), x_{n\cdot})\}_{n=1}^\infty$ is uniformly integrable on $[0,s]$ for $s \in (0,b]$. By using Lemma 2.2, Lemma 2.4-2.7, (Hf)(3)$'$ and (HK), we have

$$\begin{aligned} \chi_c(\Gamma_2 W) &\le 2\chi_c(\Gamma_2\{x_n\}) + \varepsilon \\ &= 2 \sup_{-q\le t\le b} \chi(\Gamma_2\{x_n(t)\}) + \varepsilon \\ &= 2 \sup_{0\le t\le b} \chi(\Gamma_2\{x_n(t)\}) + \varepsilon \\ &= 2 \sup_{0\le t\le b} \chi(\{\int_0^t T(t-s) \int_0^s K(s,r) f(r, x_n(r), x_{nr}) drds\}) + \varepsilon \\ &\le 2M \sup_{0\le t\le b} \int_0^t \chi(\{\int_0^s K(s,r) f(r, x_n(r), x_{nr}) dr\}) ds + \varepsilon \end{aligned}$$

$$\begin{aligned}
&\leq 4M\sup_{0\leq t\leq b}\int_0^t\int_0^s|K(s,r)|\chi(\{f(r,x_n(r),x_{nr})\})drds+\varepsilon\\
&\leq 4MK\int_0^b\int_0^s\eta(r)(\chi(\{x_n(r)\})+\sup_{-q\leq\theta\leq 0}\chi(\{x_n(r+\theta)\}))drds+\varepsilon\\
&\leq 4MK\int_0^b\int_0^s\eta(r)\cdot 2\sup_{-q\leq\tau\leq b}\chi(\{x_n(\tau)\})drds+\varepsilon\\
&\leq 8MK\chi_c(\{x_n\})\int_0^b\int_0^s\eta(r)drds+\varepsilon\\
&\leq 8bMK\chi_c(W)\int_0^b\eta(s)ds+\varepsilon.
\end{aligned}$$

As ε is an arbitrary positive number, we get that

$$\chi_c(\Gamma_2 W)\leq 8bMK\chi_c(W)\int_0^b\eta(s)ds.$$

Since $L_0+8bMK\int_0^b\eta(s)ds<1$ and $\chi_c(\Gamma W)\leq\chi_c(\Gamma_1 W)+\chi_c(\Gamma_2 W)\leq(L_0+8bMK\int_0^b\eta(s)ds)\chi_c(W)$, we obtain that Γ is a χ_c-contraction. Using Lemma2.2, we get a fixed point x of Γ, which is a mild solution of (1.3)-(1.4). The proof is complete. ■

From the proof Theorem 4.2, we can see that the condition (HK)(1) can be replaced by

(HK)$(1)^{'}$: *For each* $t\in(0,b]$, $K(t,\cdot)$ *is measurable on* $[0,t]$, *and*

$$K(t)=ess\sup\{|K(t,s)|:0\leq s\leq t\}$$

is integrable on $[0,b]$,

which is slightly weaker than (HK)(1). Denote by $K_1=\int_0^b K(t)dt$, then we get the following obvious result:

Theorem 4.3 *Assume that the hypotheses (Hf)(1), (2),* $(3)^{'}$*, (Hg), (Hh), and (HK)*$(1)^{'}$*, (2) are satisfied. Then for every* $\phi\in C([-q,0];X)$*, the problem (1.3)-(1.4) has at least one mild solution if*

$$L_0+8MK_1\int_0^b\eta(s)ds<1.$$

and

$$2c_1(\|(-A)^{-\beta}\|+\frac{C_{1-\beta}b^\beta}{\beta})+K_1M\int_0^b\alpha(s)ds\liminf_{k\to\infty}\frac{\Omega(2k)}{k}<1.$$

We can also remove the restriction that the map h is uniformly bounded.

Theorem 4.4 *Assume the hypotheses (Hf)(1), (2),* $(3)^{'}$*, (Hg), (HK) and (Hh)(1) are satisfied. Then for every* $\phi\in C([-q,0];X)$*, the nonlocal neutral problem (1.3)-(1.4) has at least one mild solution if*

$$L_0+8bMK\int_0^b\eta(t)dt<1,$$

and

$$2c_1(\|(-A)^{-\beta}\|+\frac{C_{1-\beta}b^\beta}{\beta})+M\liminf_{k\to\infty}(\int_0^b\alpha(s)ds\frac{\Omega(2k)}{k}+(1+c_1\|(-A)^{-\beta}\|)\frac{\gamma(k)}{k})<1.$$

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