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Well-posedness of the Solution to D-P Equation with Dispersive Term

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Abstract: In this paper,we study the D-P equation with dispersive term.By applying Kato's semigroup approach, one can obtain the local well-posedness of the equation in Sobolev space $(H^s, s > \frac{3}{2})$. By using the prior estimates we can obtain the existence of global smooth solutions under the initial value $u_0 \in H^s\left(R\right), s > \frac{3}{2}$.

Keywords:well-posedness; Kato's theory; infinitesimal generator; global existence

1 Introduction

In [1], Degasperis and Procesi studied the following family of third order dispersive PDE conservation laws

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x \tag{1}$$

They found that there are at least four equations that satisfy the completely integrability condition within this family:KdV equation,C-H equation,DGH equation and D-P equation.

With $\alpha = c_2 = c_3 = 0$, it becomes the well-known Korteweg-de Veris equation.The KdV equation is completely integrable and its solitary waves are solitions.

For $c_1 = -\frac{3}{2} c_3 / \alpha^2, c_2 = c_3/2$, it becomes the Camassa-Holm equation, which has a bi-Hamilti-onian structure and is completely integrable. Tian,etc.in [2] discussed the traveling wave solutions and double soliton solutions of Camassa-Holm equation.Tian,Song,Yin in [3] considered the generalized Camassa-Holm equation and derived some new exact peakon and compacton.

Dullin,Gottwald,Holm in [4] discussed the following 1+1 quadratically nonlinear equation in this class for a unidirectional water wave with fluid velocity $u(x,t)$

$$m_t + c_0 u_x + u m_x + 2 m u_x = -\gamma u_{xxx}, \quad x \in R, \quad t \in R, \tag{2}$$

Equation (2) has a bi- Hamiltonian and a Lax pair formulation. Many researches carried out on the DGH equation. Tian and Fang in [5] considered the well-posedness and blowup for an integrable shallow water equation with dispersive term.

With $c_1 = -2 {}^{c_3}/_{\alpha^2}, c_2 = c_3$ in Eq.(1) ,we find the Degasperis-Procesi equation of the form

$$u_t - u_{txx} + 4 u u_x = 3 u_x u_{xx} + u u_{xxx} \quad t > 0\,, \quad x \in R \tag{3}$$

Degasperis,Holm and Hone ([6]) proved the integrability of Eq. (3) by constructing a Lax pair.They also showed that Eq. (3) had bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admited exact peakon solutions which are analogous to the Camassa-Holm equation.

Eq .(3) with a strong dispersive term,we get

$$u_t - u_{txx} + 4 u u_x = 3 u_x u_{xx} + u u_{xxx} + \gamma\left(u - u_{xx}\right)_x \quad t > 0, x \in R \tag{4}$$

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The conservation laws of Eq. (4) is the same with Degasperis-Procesi equation.And it has Lax pair

$$\left(1-\partial_x^2\right)\psi_x=\mu m\psi,\quad \psi_t+\frac{1}{\mu}\psi_{xx}+\left(u+\gamma\right)\psi_x-u_x\psi=0.$$

With $m=u-u_{xx}$,Eq.(4) takes the form of a quasi-linear evolution equation of hyperbolic type

$$\begin{cases} m_t+um_x+3u_xm+\gamma m_x=0, & t>0,x\in R,\\ m\left(0,x\right)=u_0\left(x\right)-u_{0,xx}\left(x\right) & x\in R\end{cases}\tag{5}$$

2 Local Well-posedness

For convenience, we state here Kato's theorem in a form suitable for our purpose. Consider the abstract quasi-linear evolution equation:

$$\frac{dv}{dt}+A\left(v\right)v=f\left(v\right),t\geq 0,v\left(0\right)=v_0\tag{6}$$

Theorem 1 (Kato[7]). *Assume that (i), (ii), and (iii) hold(see[5]). Given $v_0\in Y$, there is a maximal $T>0$ depending only on $\|v_0\|_Y$, and a unique solution v to Eq.(6) such that*

$$v=v\left(\cdot,v_0\right)\in C\left([0,T);Y\right)\cap C^1\left([0,T);X\right).$$

Moreover,the map $v_0\to v\left(\cdot,v_0\right)$ is a continuous map from Y to $C\left([0,T);Y\right)\cap C^1\left([0,T);X\right)$.

Eq.(4) is a bi-Hamiltonian system, it can be written as two accessible Hamiltonian style:

$$m_t=-B_2\frac{\delta E}{\delta m}=-B_1\frac{\delta F}{\delta m}$$

$$E=-\frac{1}{6}\int u^3dx,\quad F=-\frac{9}{2}\int mdx$$

$$B_1=m^{2/3}\partial_x m^{1/3}\left(\partial_x-\partial_x^3\right)^{-1}m^{1/3}\partial_x m^{2/3}+\frac{2}{9}\partial_x\left(1-\partial_x^2\right)u\left(1-\partial_x^2\right)$$

$$B_2=\partial_x\left(1-\partial_x^2\right)\left(4-\partial_x^2\right)+2\gamma\partial_x\left(1-\partial_x^2\right)u^{-1}\left(1-\partial_x^2\right)\tag{7}$$

Note that if $p\left(x\right):=\frac{1}{2}e^{-|x|}$, $x\in$ R, then $\left(1-\partial_x^2\right)^{-1}f=p*f$ for all $f\in L^2\left(R\right)$, and $p*m=u$. Using this identity, we can rewrite Eq.(4) as

$$\begin{cases} u_t+uu_x+\gamma u_x=-\partial_x p*\left(\frac{3}{2}u^2\right), & t>0,x\in R\\ u\left(0,x\right)=u_0\left(x\right), & x\in R.\end{cases}\tag{8}$$

Theorem 2 *Given $u_0\in H^s\left(R\right),\left(s>\frac{3}{2}\right)$, there exists a maximal value $T=T\left(\alpha,c_0,\gamma,u_0\right)>0$, and a unique solution u to Eq.(2), such that*

$$u=u\left(\cdot,u_0\right)\in C\left([0,T);H^s\right)\cap C^1\left([0,T);L^2\right)$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$u_0\to u\left(\cdot,u_0\right):H^s\to C\left([0,T);H^s\right)\cap C^1\left([0,T);L^2\right)$$

is continuous.

Let $A(u)=(u+\gamma)\partial_x$, $f\left(u\right)=-\partial_x\left(1-\partial_x^2\right)^{-1}\left(\frac{3}{2}u^2\right),Y=H^s,X=H^{s-1},Q=\wedge=\left(1-\partial_X^2\right)^{\frac{1}{2}}$. Obviously, Q is an isomorphism of H^s onto H^{s-1}. Thus, in order to derive Theorem 2 from Theorem 1, we only need to verify that $A(u)$ and $f(u)$ satisfy the conditions(i),(ii),(iii).we introduce some useful lemmas.

Lemma 1 ([7]) Let γ, t be real numbers such that $-\gamma < t \leq \gamma$. Then,

$$\|fg\|_t \leq c\,\|f\|_\gamma \,\|g\|_t \quad \gamma > \frac{1}{2}, \quad \|fg\|_{\gamma+t-\frac{1}{2}} \leq c\,\|f\|_\gamma \,\|g\|_t \quad \gamma < \frac{1}{2}$$

where c is a positive constant depending on γ and t.

Lemma 2 ([8]) Let $f \in H^s, s > \frac{3}{2}$. Then $\left\|\wedge^{-\gamma}\left[\wedge^{\gamma+t+1}, Mf\right]\wedge^{-t}\right\|_{L(L^2)} \leq c\,\|f\|_s \quad |\gamma|, |t| \leq s-1$ where M_f is the operator of multiplication by f.

Lemma 3 Let X and Y be two Banach spaces and Y be continuously and densely embedded in X. Let $-A$ be the infinitesimal generator of the C_0-semigroup $T(t)$ on X and let S be an isomorphism from Y onto X. Then Y is $-A$-admissible if and only if $-A_1 = -SAS^{-1}$ is the infinitesimal generator of the C_0-semigroup $T_1(t) = ST(t)S^{-1}$ on X. Moreover, if Y is $-A$ -admissible, then the part of $-A$ in Y is the infinitesimal generator of the restriction of $T(t)$ to Y.

Lemma 4 The operator $A(u) = (u+\gamma)\partial_x$, with $u \in H^s, s > \frac{3}{2}$, belongs to $G\left(L^2, 1, \beta\right)$.

Lemma 5 The operator $\mathrm{A(u)} = (\mathrm{u}+\gamma)\partial_{\mathrm{x}}$, with $u \in H^s, s > \frac{3}{2}$, belongs to $G(H^{s-1}, 1, \beta)$.

The proof of Lemma 4 and Lemma 5 are similar with the proof of Lemma2.4 and Lemma2.5 in [5].

Lemma 6 The operator $A(u) = (u+\gamma)\partial_x, u \in H^s, s > \frac{3}{2}$, then $A(u) \in L\left(H^s, H^{s-1}\right)$,

$$\|(A(u) - A(z))\,w\|_{s-1} \leq \mu_1 \|u-z\|_{s-1} \|w\|_s \quad u, z, w \in H^s$$

Taking $z = 0$ in the above inequality,we obtain $A(u) \in L\left(H^s, H^{s-1}\right)$.This completes the proof of Lemma 6.

Lemma 7 We have $B(u) = \left[\wedge^1, u\partial_x\right]\wedge^{-1} \in L\left(H^{s-1}\right)$, for $u \in H^s$. Moreover,

$$\|(B(u) - B(z))\,w\|_{s-1} \leq \mu_2 \|u-z\|_s \|w\|_{s-1}$$

Proof. Let $u, z \in H^s, s > \frac{2}{3}$, and $w \in L^2$. Then

$$\begin{aligned}\|(B(u) - B(z))\,w\|_{s-1} &= \left\|\wedge^{s-1}\left[\wedge^1, (u-v)\,\partial_x\right]\wedge^{-1} w\right\|_0 \\ &\leq \left\|\wedge^{s-1}\left[\wedge, (u-v)\right]\wedge^{1-s}\right\|_{L(L^2)} \left\|\wedge^{s-2}\partial_x w\right\|_0 \\ &\leq \mu_2 \|y-z\|_s \|w\|_{s-1}\end{aligned}$$

where we applied Lemma 2 with $\gamma = 1-s, t = s-1$. Taking $z = 0$ in the above inequality, we obtain $B(u) \in L\left(H^{s-1}\right)$. This completes the proof of Lemma 7. □

Lemma 8 Let $f(u) = -\partial_x(1-\partial_x^2)^{-1}(\frac{3}{2}u^2)$, then it is bounded on bounded sets in H^s, and satisfies

$$(a)\ \|f(y) - f(z)\|_s \leq \mu_3 \|y-z\|_s \quad y, z \in H^s,$$

$$(b)\ \|f(y) - f(z)\|_{s-1} \leq \mu_4 \|y-z\|_{s-1} \quad y, z \in H^s$$

Proof. Let $y, z \in H^s, s > \frac{3}{2}$, and note that H^{s-1} is a Banach algebra. Then we have

$$\begin{aligned}\|f(y) - f(z)\|_s &= \left\|-\partial_x\left(1-\partial_x^2\right)^{-1}\left(\frac{3}{2}\left(y^2 - z^2\right)\right)\right\|_s \\ &\leq C\,\|(y-z)(y+z)\|_{s-1} \\ &\leq C\,\|y-z\|_s \|y+z\|_s \\ &\leq C\left(\|y\|_s + \|z\|_s\right)\|y-z\|_s\,.\end{aligned}$$

This proves (a).Taking $z = 0$ in the above inequality, we obtain that f is bounded on bounded sets in H^s.

Next, we prove (b). Let $y, z \in H^s, s > \frac{3}{2}$, and define H^{s-1} as above.
Then we have

$$\|f(y) - f(z)\|_{s-1} \leq C\left(\|y\|_{s-1} + \|z\|_{s-1}\right)\|y-z\|_{s-1}\,.$$

where we applied Lemma 1 with $\gamma = 1-s, t = s-1$. This completes the proof of Lemma 8.□
Proof of Theorem 3. The result follows by combining Theorem 2 and Lemmas 5-8.

Theorem 3 *The maximal T in Theorem 2 may be chosen independent of s in the following sense. If*

$$u = u(\cdot, u_0) \in C([0,T); H^s) \cap C^1\left([0,T); H^{s-1}\right),$$

is a solution to Eq.(2), and if $u_0 \in H^{s'}$ for some $s' \neq s, s' > \frac{3}{2}$, then

$$u \in C\left([0,T); H^{s'}\right) \cap C^1\left([0,T); H^{s'-1}\right),$$

with the same value of T. In particular, if $u_0 \in H^\infty = \bigcap_{s\geq 0} H^s$, then $u \in C([0,T); H^\infty)$.

Proof. If $s' < s$, the result follows the uniqueness of the solution guaranteed by Theorem 2, so it suffices to consider the case $s' > s$. To this end, we return to Eq.(6).

Setting $m(t) = \wedge^2 u(t) = u - \alpha^2 u_{xx}$,we have

$$\frac{dm}{dt} + A(t)m + B(t)m = f(t) \quad m(0) = \wedge^2 u(0) \tag{9}$$

where $A(t)m = \partial_x((u+\gamma)m)$,$B(t)m = 3u_x m$ and $f(t) = u_x m$. Because $u \in C([0,T); H^s)$ and $u_0 \in H^{s'}$,we have $m \in C\left([0,T); H^{s-2}\right)$ and $m(0) = \left(1 - \alpha^2\partial_x^2\right)u(0) \in C\left([0,T); H^{s'}\right)$.We will show that $m \in C\left([0,T); H^{s'-2}\right)$, which implies $u \in C\left([0,T); H^{s'}\right)$ since $\left(1-\alpha^2\partial_x^2\right)$ is an isomorphism from $H^{s'}$ to $H^{s'-2}$. This will complete the proof of Theorem 3.

Since $u \in C([0,T); H^s)$,$u_x \in H^{s-1}$, we have $B(t) \in L\left(H^{s-1}\right), f(t) \in C\left([0,T); H^{s-1}\right)$.

We first prove that the family $A(t)$ has a unique evolution operator $\{U(t,\tau)\}$ associated with the spaces $X = H^h$ and $Y = H^k$, where $-s \leq h \leq s-2$; $1-s \leq k \leq s-1$,and $k \geq h+1$. To this end, we need to verify the following three conditions:

(i)$A(t) \in G\left(H^h, 1, \beta\right), \forall m \in H^s$.

(ii)$\wedge^h \partial_x\left[\wedge^{k-h}, u\right]\wedge^{-k}$ is uniformly bounded on L^2.

(iii)$A(t) \in L\left(H^k, H^h\right)$ is strongly continuous in t.

Let us first show (i). Since H^h is a Hilbert space,we have $A(t) \in G\left(H^h, 1, \beta\right)$ for some real number β if and only if the following conditions hold:

(a) $(A(t)m, m)_h \geq -\beta\|m\|_h^2$,

(b) $\quad -A(t)$is the infinitesimal generator of a C_0-semigroup on H^h,for some (or all) $\lambda > \beta$.

To prove (a), take $m \in H^h$ and note that

$$\wedge^h\partial_x((u+\gamma)m) = -\wedge^h\partial_x\left[\wedge^{-h}, u\right]\wedge^h m + \partial_x\left(u\wedge^h m\right) + \gamma\wedge^h\partial_x m$$

$$(A(t)m, m)_h = (\,-\wedge^h\partial_x\left[\wedge^{-h}, u\right]\wedge^h m + \partial_x\left(u\wedge^h m\right) + \gamma\wedge^h\partial_x m\wedge^h m\,)_0 \leq c\|u\|_s\|m\|_h^2$$

where we have used Lemma 2. with $\gamma = -(h+1)$ and $t = 0$. Setting $\beta = c\|u\|_s$, we obtain $(A(t)m, m)_h \geq -\beta\|m\|_h^2$ as claimed.

Next, we prove (b). Let $S = \wedge^{s-1-h}$, and note that S is an isomorphism of H^{s-1} onto H^h and that H^{s-1} is continuously and densely embedded in H^h as $-s \leq h \leq s-2$. Define $A_1(t) := SA(t)S^{-1} = \wedge^{s-1-h}A(t)\wedge^{h+1-s}, B_1(t) = A_1(t) - A(t) = [S, A(t)]S^{-1}$.
Let $m \in H^h$ and $u \in H^s, s > \frac{3}{2}$. Then

$$\|B_1(t)m\|_h \leq \|\wedge^h\partial_x[\wedge^{s-1-h}, u]\wedge^{1-s}\|_{L(L^2)}\|\wedge^h m\|_0$$

on applying Lemma 2 with $\gamma = -(h+1), t = s-1$.

By applying Lemma 5 and a perturbation theorem for semigroups, we see that H^{s-1} is A(t)-admissible. Further, applying Lemma 8 with $Y = H^{s-1}$,$X = H^h$ and $S = \wedge^{s-1-h}$, we obtain that $-A_1(t)$ is the infinitesimal generator of a C_0-semigroup on H^h. Since $A_1(t) = A(t) + B_1(t)$ and $B_1(t) \in L\left(H^h\right)$, by a perturbation theorem for semigroups, it follows that $-A(t)$ is the infinitesimal generator of a C_0 -semigroup on H^h. This proves (b).

Next, we verify (ii). For $m \in L^2$ we have $\left\|\wedge^h \partial_x \left[\wedge^{k-h}, u\right] \wedge^{-k} m\right\|_0 \leq c \|u\|_s \|m\|_0$, by Lemma 2 with $\gamma = -(h+1), t = k$. This proves (ii).

Finally, we verify (iii). Take $m \in H^h$. Then

$$\begin{aligned}
\|(A(t+\tau) - A(t)) m\|_h &= \|\partial_x (u(t+\tau) - u(t)) m + (u(t+\tau) - u(t)) \partial_x m\|_h \\
&= \|\partial_x ((u(t+\tau) - u(t)) m)\|_h \leq c \|u(t+\tau) - u(t)\|_{s-1} \|m\|_{h+1} \\
&\leq c \|u(t+\tau) - u(t)\|_s \|m\|_k
\end{aligned}$$

by Lemma 2 with $\gamma = s-1, t = h+1$. The continuity of u now yields (iii).

Thus, the above conditions (i)-(iii) imply the existence and uniqueness of an evolution operator $U(t,\tau)$ for the family $A(t)$. In particular, for $-s \leq \gamma \leq s-1$, $U(t,\tau)$ maps H^γ into itself.

Let $Y = H^{s-2}$,$X = H^{s-3}$,and note that $m \in C\left([0,T); H^{s-1}\right) \cap C^1\left([0,T); H^{s-2}\right)$.

Using the properties of the evolution operator $U(t,\tau)$, we obtain

$$\frac{d}{d\tau}(U(t,\tau) m(\tau)) = U(t,\tau)(-B(\tau) m(\tau) + f(\tau)).$$

An integration over $\tau \in [0,t]$ gives

$$m(t) = U(t,0) m(0) + \int_0^t U(t,\tau)(-B(\tau) m(\tau) + f(\tau)) d\tau \tag{10}$$

If $s < s' \leq s+1$, then $f(t) \in C\left([0,T); H^{s-1}\right) \cap C^1\left([0,T); H^{s'-2}\right)$, $B_1(t) = u_x(t) \in L\left(H^{s'-2}\right)$ is strongly continuous on $[0,t)$, and $H^{s-1} H^{s'-2} \subset H^{s'-2}$ since $s-1 > \frac{1}{2}$. Since $-s < s-2 < s'-2 \leq s-1$, the family $\{U(t,\tau)\}$ is a strongly continuous map from the space $H^{s'-2}$ into itself. Noting that $m(0) \in H^{s'-2}$,and regarding Eq.(10) as an integral equation of Volterra type that can be solved for m by successive approximation, we then obtain the assertion of Theorem 3 for the case $s < s' \leq s+1$.

In the case $s' > s+1$, the result follows by a repeated application of the above argument. This completes the proof of Theorem 3. □

3 Global Existence

In this part,we begin with the conservation law of Eq.(8),using this conservation law,we can obtain the L^∞priori estimate of the strong solution .

Lemma 9 If $u_0 \in H^s(R), (s > \frac{3}{2})$,then as long as the solution $u(t,x)$exists ,we have

$$\int_R m(t,x) v(t,x) dx = \int_R m_0(x) v_0(x) dx,$$

where $m(t,x) = u(t,x) - u_{xx}(t,x), v = \left(4 - \partial_x^2\right)^{-1} u$. Moreover, we have $\|u(t)\|_{L^2}^2 \leq 4 \|u_0\|_{L^2}^2$.

Proof. Applying theorem 1 and a simple density argument ,we only need to show that the above Lemma with some $s > \frac{3}{2}$.Here we assume $s = 3$ to proof the above Lemma. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(8) with the initial data $u_0 \in H^3(R)$, such that $u \in C([0,T); H^3(R)) \cap C^1([0,T); H^2(R))$, we have

$$\frac{1}{2}\frac{d}{dt}\int_R mv dx == -\int_R v(mu)_x dx - 2\int_R v m u_x dx - \gamma \int_R v m_x dx$$

Using the relations $m = u - u_{xx}, 4v - v_{xx} = u$.ie.$m = 4v - 5v_{xx} + v_{xxxx}$,it yields that

$$\begin{aligned}
\int_R v(mu)_x dx &= -\int_R v_x m u dx = -\int_R v_x u^2 dx + \int_R v_x u u_{xx} dx \\
&= \int_R v_x u^2 dx - \int_R v_x u_x^2 dx
\end{aligned}$$

$$2\int_R vmu_x dx = -\int_R v_x u^2 dx + \int_R v_x u_x^2 dx, \gamma \int_R vm_x dx = 0$$

Combing the above relations,we deduce that $\frac{1}{2}\frac{d}{dt}\int_R mvdx = 0$. Consequently, this implies the desired conserved quantity.In view of the above conservation law,it then follows that

$$\begin{aligned}\|u(t)\|_{L^2}^2 &= \sum_{n=-\infty}^{\infty} \left|\hat{u}_n\right|^2 \le 4\sum_{n=-\infty}^{\infty} \frac{1+4\pi^2 n^2}{4+4\pi^2 n^2} \left|\hat{u}_n\right|^2 \\ &= 4(m(t), v(t)) = 4(m_0, v_0) = 4\sum_{n=-\infty}^{\infty} \frac{1+4\pi^2 n^2}{4+4\pi^2 n^2}\left|\left(\hat{u}_0\right)_n\right|^2 \\ &\le \sum_{n=-\infty}^{\infty}\left|\left(\hat{u}_0\right)_n\right|^2 = 4\|u_0\|_{L^2}^2\end{aligned}$$

This completes the proof of Lemma 9. □

Lemma 10 Assume $u_0 \in H^s(R), s > \frac{3}{2}, m_0 \ge 0$,where $m_0 = \left(1-\partial_x^2\right)u_0$. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(8).Then we have

$$\|u(t,x)\|_{L^\infty} \le 3\|u_0(x)\|_{L^2}^2 t + \|u_0(x)\|_{L^\infty}, \quad \forall t \in [0,T]$$

Proof. Assume $u(t,x), t \in [0,t)$is the solution of Eq.(3).Consider the equation

$$\begin{cases} \xi_t(t,x) = u(t,\xi)(t,x)) + \gamma, & t \ge 0, x \in R \\ \xi(0,x) = x, & x \in R \end{cases} \tag{11}$$

In accordance with the Sobolev imbedding theorem and property of $u(t,x)$, one can see that $u(t,\xi)$ satisfies Lipschitz condition.And combining with ordinary differential equation theory, we see that there exists the unique solution $\xi(t,x)$ of Eq.(6) in $C([0,T))$ for any real x,we have

$$\begin{cases} \xi_{tx}(t,x) = u_x(t,\xi(t,x))\xi_x(t,x) & t \ge 0, x \in R \\ \xi_x(0,x) = 1, & x \in R \end{cases} \tag{12}$$

From the above Cauchy problem (12),we obtain

$$\xi_x(t,x) = \exp\left(\int_0^t u_x(z,\xi(z,x))dz\right) > 0 \tag{13}$$

Combing with Eq.(4)we obtain

$$\partial_t(m(t,\xi(t,x))\xi_x^3(t,x)) = \xi_x^3(3mu_x + m_t + m_x u + \gamma m_x) = 0$$

As $\xi(0,x) = x, x \in R$, so $m(t,\xi(t,x))\xi_x^3(t,x) = m_0$. Therefore,if the initial potential satisfies $m_0 \ge 0$, then this inequality will hold under the flow of Eq.(3),$m \ge 0$ for every $t \in [0,t)$.

Applying Theorem 2 and a simple density argument ,it suffices to consider $s = 3$ to prove the above theorem .Let $T > 0$ be the maximal time of existence of the solution u to Eq.(8)with the initial data $u_0 \in H^3(R)$ such that $u \in C([0,T); H^3(R)) \cap C^1([0,T); H^2))$,we have

$$u_t + uu_x + \gamma u_x = -\partial_x p * \left(\frac{3}{2}u^2\right) = -3p*(uu_x) \tag{14}$$

$$-3p^*(uu_x) = -\frac{3}{2}\int_{-\infty}^{+\infty} e^{-|x-\eta|}uu_\eta d\eta = \frac{3}{4}\int_{-\infty}^{x} e^{-|x-\eta|}u^2 d\eta - \frac{3}{4}\int_x^{-\infty} e^{-|x-\eta|}u^2 d\eta$$

$$\frac{du(t,q(t,x))}{dt} = u_t(t,\xi(t,x)) + u_x(t,\xi(t,x))$$

$$\frac{d\xi(t,x)}{dt} = (u_t + uu_x + \gamma u_x)(t,\xi(t,x))$$

$$\left|\frac{du\left(t,\xi\left(t,x\right)\right)}{dt}\right| \leq \frac{3}{4}\int_{-\infty}^{+\infty} e^{-\left|\xi(t,x)-\eta\right|}u^2 d\eta \leq \frac{3}{4}\int_{-\infty}^{+\infty} u^2\left(t,\eta\right)d\eta$$

in view of Lemma 9,we have $-3\left\|u_0\right\|_{L^2}^2 \leq \frac{du(t,\xi(t,x))}{dt} \leq 3\left\|u_0\right\|_{L^2}^2$.

Integrating the above inequality with respect to $t < T$ on $[0,t]$ yields

$$-3\left\|u_0\right\|_{L^2}^2 t + u_0\left(x\right) \leq u\left(t,\xi\left(t,x\right)\right) \leq 3\left\|u_0\right\|_{L^2}^2 t + u_0\left(x\right)$$

Thus

$$\left|u\left(t,\xi\left(t,x\right)\right)\right| \leq \left\|u\left(t,\xi\left(t,x\right)\right)\right\|_{L^\infty} \leq 3\left\|u_0\right\|_{L^2}^2 t + \left\|u_0\right\|_{L^\infty} \tag{15}$$

Using the Sobolev embedding(the periodicity in the spatial variable) to ensure the uniform boundedness of $u_x(s,\eta)$ with $t\in[0,T)$, we get for every $t\in[0,T]$ a constant $C\left(t\right)>0$, such that $e^{-C(t)} \leq \xi_x\left(t,x\right) \leq e^{C(t)}$, $\quad x\in R$ we deduce from the above equation that the function ξ(t, .)is strictly increasing on R with $\lim\limits_{t\to\pm\infty}\xi(t,x)=\pm\infty$as long as $t\in[0,T)$.Thus by Eq.(15) we can obtain

$$\left\|u\left(t,x\right)\right\|_{L^\infty} = \left\|u\left(t,\xi\left(t,x\right)\right)\right\|_{L^\infty} \leq 3\left\|u_0\right\|_{L^2}^2 t + \left\|u_0\right\|_{L^\infty}. \ \square$$

Theorem 4 *Assume $u_0\in H^s\left(R\right), s>\frac{3}{2}$,if $m_0=\left(1-\partial_x^2\right)u_0$ does not change sign on R ,then Eq.(8)has a global strong solution*

$$u=u\left(\cdot,u_0\right)\in C\left(\left[0,T\right);H^s\right)\cap C^1\left(\left[0,T\right);H^{s-1}\right).$$

Moreover,$E_2\left(u\right)=\int_R mvdx$is a conservation law ,where$m=\left(1-\partial_x^2\right)u$,$v=\left(4-\partial_x^2\right)^{-1}u$, and we have for all $t\in R_+$,

$$\left\|u\right\|_1^2 \leq 6\left\|u_0\right\|_{L^2}^2\left\|u_0\right\|_{L^2}^2 t^2 + 4\left\|u_0\right\|_{L^2}^2\left\|u_0\right\|_{L^\infty} t + \left\|u_0\right\|_1^2$$

Proof. we only need to show that the above theorem with $s=3$.Multiplying Eq. (3)with u,integrating by parts ,we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_R(u^2+u_x^2)dx &= -4\int_R u^2u_xdx + 3\int_R uu_xu_{xx}dx + \int_R u^2u_{xxx}dx + \gamma\int_R u(u-u_{xx})_xdx\\ &= \frac{1}{2}\int_R u_x^3dx \leq \frac{1}{2}\int_R u^3dx \leq \frac{1}{2}\|u\|_{L^\infty}\int_R u^2dx\\ &= \frac{1}{2}\left(12\left\|u_0\right\|_{L^2}^2 t + 4\left\|u_0\right\|_{L^\infty}\right)\int_R u_0^2dx\end{aligned}$$

Integrating the above inequality with respect to $t<T$ on $[0,t]$,we have

$$\left\|u\right\|_1^2 \leq 6\left\|u_0\right\|_{L^2}^2\left\|u_0\right\|_{L^2}^2 t^2 + 4\left\|u_0\right\|_{L^2}^2\left\|u_0\right\|_{L^\infty} t + \left\|u_0\right\|_1^2 \ \square$$

Theorem 5 *Assume $u_0\in H^s\left(R\right), s>\frac{3}{2}$,and there exists $x_0\in R$,such that*

$$\begin{cases} m_0\left(x\right)\leq 0 & x\leq x_0\\ m_0\left(x\right)\geq 0 & x\geq x_0\end{cases}$$

Then Eq.(7)has a unique global strong solution $u=u(\cdot,u_0)\in C([0,\infty);H^s)\cap C^1([0,\infty);H^{s-1})$

Moreover ,$E_2\left(u\right)=\int_R mvdx$ is a conservation law,where $m=\left(1-\partial_x^2\right)u$,$v=\left(4-\partial_x^2\right)^{-1}u$, and for all $t\in R_+$,we have

$$\left\|u\right\|_1^2 \leq 6\left\|u_0\right\|_{L^2}^2\left\|u_0\right\|_{L^2}^2 t^2 + 4\left\|u_0\right\|_{L^2}^2\left\|u_0\right\|_{L^\infty} t + \left\|u_0\right\|_1^2.$$

Proof. we only need to show that the above theorem with $s=3$.

Multiplying Eq. (3)with u ,integrating by parts ,we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_R(u^2+u_x^2)dx &= -4\int_R u^2u_xdx + 3\int_R uu_xu_{xx}dx + \int_R u^2u_{xxx}dx + \gamma\int_R u(u-u_{xx})_xdx\\ &= -\frac{1}{2}\int_R u_x^3dx \leq \frac{1}{2}\int_R u^3dx \leq \frac{1}{2}\|u\|_{L^\infty}\int_R u^2dx\\ &= \frac{1}{2}\left(12\left\|u_0\right\|_{L^2}^2 t + 4\left\|u_0\right\|_{L^\infty}\right)\int_R u_0^2dx\end{aligned}$$

Integrating the above inequality with respect to $t<T$on $[0,t]$,we have

$$\left\|u\right\|_1^2 \leq 6\left\|u_0\right\|_{L^2}^2\left\|u_0\right\|_{L^2}^2 t^2 + 4\left\|u_0\right\|_{L^2}^2\left\|u_0\right\|_{L^\infty} t + \left\|u_0\right\|_1^2. \ \square$$

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