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The First Integral Method for Solving a System of Nonlinear Partial Differential Equations

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Abstract: We apply the first integral method to study the solutions of the variant Boussinesq and the nonlinear Drinfeld-Sokolov systems. This method is based on the theory of commutative algebra. The new idea in this paper is to find the solution of a system of nonlinear partial differential equations using the first integral method.

Key words: theory of commutative algebra; the first integral method; the variant Boussinesq system; the Drinfeld-Sokolov system

1 Introduction

The study of the solutions of partial differential equations (PDEs) has enjoyed an intense period of activity over the last forty years from both theoretical and numerical points of view. Improvements in numerical techniques, together with the rapid advance in computer technology, have meant that many of the PDEs arising from engineering and scientific applications, which were previously intractable, can now be routinely solved. In finite difference methods differential operators are approximated and difference equations are solved [1]. In the finite element method [2] the continuous domain is represented as a collection of a finite number N of subdomains known as elements. The collection of elements is called the finite element mesh. The differential equations for time dependent problems are approximated by the finite element method to obtain a set of ordinary differential equations (ODEs) in time. These differential equations are solved approximately by finite difference methods. In all finite difference and finite element methods it is necessary to have boundary and initial conditions. However, the Adomian decomposition method, which has been developed by George Adomian [3], depends only on the initial conditions and obtains a solution in series which converges to the exact solution of the problem. In recent years, ansatz methods have been developed, such as the tanh-function method [4-6], extended tanh-function method [7, 8], the modified extended tanh-function method [9, 10], the generalized hyperbolic function [11, 12]. Other methods are the variable separation method [13, 14] and the first integral method [15-21]. Wu and He [22] solved the variant Boussinesq system using the Exp-function. He and Abdou [23] derived the solutions of the nonlinear Drinfeld-Sokolov system using the Exp-function. Yomba [24] proposed the extended Fan's sub-equation method to solve the variant Boussinesq system. Wazwaz [25] used the Sine-Cosine and tanh-function methods to obtain the solutions of the nonlinear Drinfeld-Sokolov system. The purpose of this paper is to propose a new approach by applying the theory of commutative algebra to study the solutions of the variant Boussinesq and the nonlinear Drinfeld-Sokolov systems using the first integral method.

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2 The first integral method

Consider the nonlinear PDE:

$$F(u,u_t,u_x,u_{xx},u_{xt},...)=0, \tag{1}$$

where $u(x,t)$is the solution of equation (1). We use the transformations

$$u(x,t)=f(\xi), \qquad \xi=x-c\,t, \tag{2}$$

where c is constant. Using the chain rule we obtain

$$\frac{\partial}{\partial t}(.)=-c\,\frac{d}{d\xi}(.)\,,\quad \frac{\partial}{\partial x}(.)=\frac{d}{d\xi}(.)\,,\quad \frac{\partial^2}{\partial x^2}(.)=\frac{d^2}{d\xi^2}(.)\,,\ ... \tag{3}$$

We use (3) to change the PDE (1) to ODE:

$$G(f,\ f_\xi,f_{\xi\xi},...)=0. \tag{4}$$

Next, we introduce new independent variables

$$X(\xi)=f(\xi), \quad Y=f_\xi(\xi). \tag{5}$$

This yields a system of ODEs

$$\begin{aligned} &X_\xi(\xi)=Y(\xi),\\ &Y_\xi(\xi)=F_1(X(\xi),Y(\xi)). \end{aligned} \tag{6}$$

If we can find the integrals to equation (6) under the same conditions of the qualitative theory of ordinary differential equations [26], then the general solutions to (6) can be solved directly. However, in general, it is very difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. So, we apply the Division Theorem to obtain one first integral to (6) which reduces (4) to a first order integrable ODE. Then, an exact solution to (1) is obtained by solving this equation. Now, let us recall the Division Theorem:

Division Theorem: Suppose that $P(\omega,z)$ and $Q(\omega,z)$ are polynomials in C$[\omega,z]$ and $P(\omega,z)$ is irreducible in C$[\omega,z]$. If $Q(\omega,z)$ vanishes at all zero points of $P(\omega,z)$, then there exists a polynomial $G(\omega,z)$ in C$[\omega,z]$ such that

$$Q[\omega,z]=P[\omega,z]\,G[\omega,z].$$

3 Applications

In order to illustrate the solution procedure, we consider the following two systems, the variant Boussinesq, and the nonlinear Drinfeld-Sokolov systems. All calculations in this article have been done using the aid of the MATHEMATICA software package.

Example 1: We start with the variant Boussinesq system [25]

$$u_t(x,t)+v_x(x,t)+u(x,t)\ u_x(x,t)=0, \tag{7a}$$

$$v_t(x,t)+(u(x,t)\,v(x,t))_x+u_{xxx}(x,t)=0. \tag{7b}$$

Introducing the following transformations

$$\begin{aligned} &u(x,t)=f(\xi),\\ &v(x,t)=g(\xi), \end{aligned} \tag{8}$$

where, $\xi=x-c\,t$, the system (7) becomes

$$-c\frac{df(\xi)}{d\xi}+\frac{dg(\xi)}{d\xi}+f(\xi)\frac{df(\xi)}{d\xi}=0\,, \tag{9a}$$

$$-c\frac{dg(\xi)}{d\xi}+\frac{d(f(\xi)\,g(\xi))}{d\xi}+\frac{d^3f(\xi)}{d\xi^3}=0. \tag{9b}$$

Integrating equation (9a), we obtain $g(\xi)$ as

$$g(\xi)=c\,f(\xi)-\frac{1}{2}\,(f(\xi))^2+\alpha, \tag{10}$$

where α is an arbitrary integration constant. Integrating equation (9b) and substitute $g(\xi)$ we get

$$\frac{d^2f(\xi)}{d\,\xi^2}=(\beta+\alpha\,c)+(c^2-\alpha)\,f(\xi)-\frac{3\,c}{2}\,(f(\xi))^2+\frac{1}{2}\,(f(\xi))^3. \tag{11}$$

Let $X=f(\xi),\quad Y=\frac{df}{d\xi}$, and then equation (11) is equivalent to

$$\dot{X}(\xi)=Y(\xi), \tag{12a}$$

$$\dot{Y}(\xi)=(\beta+\alpha\,c)+(c^2-\alpha)\,X(\xi)-\frac{3\,c}{2}\,(X(\xi))^2+\frac{1}{2}\,(X(\xi))^3. \tag{12b}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (12), and $q(X,Y)=\sum_{i=0}^{m}a_i(X)\,Y^i=0$ is an irreducible polynomial in the complex domain $\mathrm{C}[X,Y]$ such that

$$q[X(\xi),Y(\xi)]=\sum_{i=0}^{m}a_i(X)\,Y^i=0, \tag{13}$$

where $a_i(X)(i=0,\ 1,\ \ldots,\ m)$ are polynomials of X and $a_m(X)\neq 0$. Equation (13) is called the first integral to (12), due to the Division Theorem, there exists a polynomial $g(X)+h(X)\,Y$ in the complex domain $\mathrm{C}[X,Y]$ such that

$$\frac{dq}{d\,\xi}=\frac{\partial\,q}{\partial\,X}\frac{\partial\,X}{\partial\,\xi}+\frac{\partial\,q}{\partial\,Y}\frac{\partial\,Y}{\partial\,\xi}=(g(X)+h(X)\,Y)\,\sum_{i=0}^{m}a_i(X)\,Y^i. \tag{14}$$

In this example, we take two different cases, assuming that $m=1$ and $m=2$ in equation (13).

Case I: Suppose that $m=1$, by equating the coefficients of Y^i $(i=2,\ 1,\ 0)$ on both sides of equation (14), we have

$$\dot{a}_1(X)=h(X)\,a_1(X), \tag{15a}$$

$$\dot{a}_0(X)=\ g(X)+h(X)\,a_0(X), \tag{15b}$$

$$a_1(X)((\beta+\alpha c)+(c^2-\alpha)X(\xi)-\frac{3c}{2}(X(\xi))^2+\frac{1}{2}(X(\xi))^3)=g(X)a_0(X). \tag{15c}$$

Since $a_i(X)(i=0,1)$ are polynomials, then from (15a) we deduce that $a_1(X)$ is constant and $h(X)=0$. For simplicity, take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that deg $(g(X))=1$ only. Suppose that $g(X)=A_1X+B_0$, and $A_1\neq 0$, then we find $a_0(X)$

$$a_0(X)=A_0+B_0\,X+\frac{1}{2}A_1X^2. \tag{16}$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in equation (15c) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$\beta=0,\qquad \alpha=-\,A_0,\qquad A_1=1,\qquad c=-B_0, \tag{17a}$$

$$\beta=0,\qquad \alpha=\ A_0,\qquad A_1=-1,\qquad c=B_0. \tag{17b}$$

Using the conditions (17a) in equation (13), we obtain

$$Y=\frac{-2\,A_0-2\,B_0\,X-X^2}{2}. \tag{18}$$

Combining (18) with (12), we obtain the exact solution to (11) and then the exact solutions to the variant Boussinesq system (7) can be written as:

$$\begin{aligned} u(x,t) &= -B_0 - \sqrt{2\,A_0 - B_0^2}\,\tan[\tfrac{\sqrt{2\,A_0-B_0^2}}{2}(x + B_0\,t + \xi_0)], \\ v(x,t) &= \tfrac{2A_0-B_0^2}{-2}\,(\sec[\tfrac{\sqrt{2\,A_0-B_0^2}}{2}(x + B_0\,t + \xi_0)])^2. \end{aligned} \tag{19}$$

where ξ_0 is an arbitrary constant.

Similarly, in the case of (17b), from equation (13) we obtain

$$Y = \frac{-2\,A_0 - 2\,B_0\,X + X^2}{2}, \tag{20}$$

and the exact solutions to the variant Boussinesq system (7) are given by:

$$\begin{aligned} u(x,t) &= B_0 + \sqrt{-2\,A_0 - B_0^2}\,\tan[\tfrac{\sqrt{-2\,A_0-B_0^2}}{2}(x - B_0\,t + \xi_0)], \\ v(x,t) &= \tfrac{2A_0+B_0^2}{2}\,(\sec[\tfrac{\sqrt{-2\,A_0-B_0^2}}{2}(x - B_0\,t + \xi_0)])^2. \end{aligned} \tag{21}$$

Case II: Suppose that $m = 2$, by equating the coefficients of $Y^i\,(i = 3,\ 2,\ 1,\ 0)$ on both sides of equation (14), we have

$$\dot{a}_2(X) = h(X)\,a_2(X), \tag{22a}$$

$$\dot{a}_1(X) = g(X)\,a_2(X) + h(X)\,a_1(X), \tag{22b}$$

$$\begin{aligned} \dot{a}_0(X) = &-2\,a_2(X)\,((\beta + \alpha\,c) + (c^2 - \alpha)\,X(\xi) - \tfrac{3\,c}{2}\,(X(\xi))^2 + \tfrac{1}{2}\,(X(\xi))^3) \\ &+g(X)\,a_1(X) + h(X)\,a_0(X), \end{aligned} \tag{22c}$$

$$a_1(X)\,((\beta + \alpha\,c) + (c^2 - \alpha)\,X(\xi) - \frac{3\,c}{2}\,(X(\xi))^2 + \frac{1}{2}\,(X(\xi))^3) = g(X)\,a_0(X). \tag{22d}$$

Since $a_2(X)$ is a polynomial of X , from (22a), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_2(X) = 1$, and hence (22) can be rewritten as

$$a_2(X) = 1, \tag{23a}$$

$$\dot{a}_1(X) = g(X), \tag{23b}$$

$$\begin{aligned} \dot{a}_0(X) = &-2\,((\beta + \alpha\,c) + (c^2 - \alpha)\,X(\xi) - \tfrac{3\,c}{2}\,(X(\xi))^2 + \tfrac{1}{2}\,(X(\xi))^3) \\ &+g(X)\,a_1(X) + h(X)\,, \end{aligned} \tag{23c}$$

$$a_1(X)\,((\beta + \alpha\,c) + (c^2 - \alpha)\,X(\xi) - \frac{3\,c}{2}\,(X(\xi))^2 + \frac{1}{2}\,(X(\xi))^3) = g(X)\,a_0(X). \tag{23d}$$

Balancing the degrees of $g(X),\ a_1(X)$ and $a_0(X)$, we conclude that deg $(g(X)) = 1$ only. Now we discuss the case if deg $(g(X)) = 1$, suppose that $g(X) = A_1\,X + B_0$, and $A_1 \neq 0$, then we obtain $a_1(X)$ and $a_0(X)$ as

$$a_1(X) = A_0 + B_0\,X + \frac{1}{2}\,A_1\,X^2, \tag{24}$$

$$\begin{aligned} a_0(X) = d &+ \tfrac{1}{2}(A_1\,B_0 + 2\,c)\,X^3 + \tfrac{1}{8}(A_1^2 - 2)\,X^4 + \tfrac{1}{2}X^2(A_0A_1+ \\ &B_0^2 - 2\,c^2 + 2\,\alpha) + X\,(A_0B_0 - 2\,c\,\alpha - 2\,\beta). \end{aligned} \tag{25}$$

where A_0 and d are arbitrary integration constants.

Substituting $a_0(X)$, $a_1(X)$, $a_2(X)$ and $g(X)$ in (23d) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we get

$$d = \frac{A_0^2}{4},\ \beta = 0\,, \alpha = \frac{-A_0}{2},\ c = \frac{-B_0}{2}\ ,\ A_1 = 2, \tag{26a}$$

$$d = \frac{A_0^2}{4},\ \beta = 0\,, \alpha = \frac{A_0}{2},\ c = \frac{B_0}{2}\ ,\ A_1 = -2. \tag{26b}$$

Using the conditions (26a) in equation (13), we obtain

$$Y = \frac{-A_0 - B_0\,X - X^2}{2}. \tag{27}$$

Expression (27) is the first integral of (12). Combining equation (12) with equation (27), we obtain the exact solution to equation (11) as follows:

$$f(\xi) = \frac{1}{2}(-B_0 - \sqrt{4\,A_0 - B_0^2}\,\tan(\frac{\sqrt{4\,A_0 - B_0^2}\,(\xi + \xi_0)}{4})). \tag{28}$$

where ξ_0 is an arbitrary integration constant. Then the exact solutions to the variant Boussinesq system (7) can be written as

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}(-B_0 - \sqrt{4\,A_0 - B_0^2}\,\tan(\tfrac{\sqrt{4\,A_0 - B_0^2}\,(x + \frac{B_0\,t}{2} + \xi_0)}{4})), \\ v(x,t) &= -\tfrac{A_0}{2} - \tfrac{B_0}{4}(-B_0 - \sqrt{4\,A_0 - B_0^2}\,\tan(\tfrac{\sqrt{4\,A_0 - B_0^2}\,(x + \frac{B_0\,t}{2} + \xi_0)}{4})) \\ &\quad - \tfrac{1}{8}(-B_0 - \sqrt{4\,A_0 - B_0^2}\,\tan(\tfrac{\sqrt{4\,A_0 - B_0^2}\,(x + \frac{B_0\,t}{2} + \xi_0)}{4}))^2. \end{aligned} \tag{29}$$

Similarly, in the case of (26b), from equation (13) we get

$$Y = \frac{-A_0 - B_0\,X + X^2}{2}, \tag{30}$$

and the exact solutions to the variant Boussinesq system (7) are given respectively by

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}(B_0 + \sqrt{-4\,A_0 - B_0^2}\,\tan(\tfrac{\sqrt{-4\,A_0 - B_0^2}\,(x - \frac{B_0\,t}{2} + \xi_0)}{4})), \\ v(x,t) &= \tfrac{A_0}{2} + \tfrac{B_0}{4}(B_0 + \sqrt{-4\,A_0 - B_0^2}\,\tan(\tfrac{\sqrt{-4\,A_0 - B_0^2}\,(x - \frac{B_0\,t}{2} + \xi_0)}{4})) \\ &\quad - \tfrac{1}{8}(B_0 + \sqrt{-4\,A_0 - B_0^2}\,\tan(\tfrac{\sqrt{-4\,A_0 - B_0^2}\,(x - \frac{B_0\,t}{2} + \xi_0)}{4}))^2. \end{aligned} \tag{31}$$

All these solutions are new exact solutions.

Example 2: We consider the nonlinear Drinfeld – Sokolov system [25] in the form

$$u_t(x,t) + v_x^2(x,t) = 0, \tag{32a}$$

$$v_t(x,t) - v_{xxx}(x,t) + (3\,u(x,t)\,v(x,t))_x = 0. \tag{32b}$$

Introducing the following transformations

$$\begin{aligned} u(x,t) &= f(\xi), \\ v(x,t) &= g(\xi), \end{aligned} \tag{33}$$

where $\xi = x - c\,t$, the system (32) reduces to

$$-c\frac{df(\xi)}{d\xi} + \frac{d(g(\xi)^2)}{d\xi} = 0\,, \tag{34a}$$

$$-c\frac{dg(\xi)}{d\xi} - \frac{d^3g(\xi)}{d\xi^3} + \frac{d(3\,f(\xi)\,g(\xi))}{d\xi} = 0. \tag{34b}$$

Integrating equation (34a), we obtain $f(\xi)$ as

$$f(\xi) = \frac{(g(\xi))^2 - \alpha}{c}, \tag{35}$$

where α is an arbitrary integration constant. Substituting $f(\xi)$ into equation (34b) yields

$$\frac{d^2g(\xi)}{d\,\xi^2} = \frac{3\,(g(\xi))^3}{c} + (\frac{-3\,\alpha}{c} - c)\,g(\xi) - \beta. \tag{36}$$

Using the first integral method we get the system of ODEs

$$\dot{X}(\xi) = Y(\xi), \tag{37a}$$

$$\dot{Y}(\xi) = \frac{3}{c}\,(X(\xi))^3 + (\frac{-3\alpha}{c} - c)\,X(\xi) - \beta. \tag{37b}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are the nontrivial solutions of (37), and $q(X,Y) = \sum_{i=0}^{m} a_i(X)\,Y^i = 0$ is an irreducible polynomial in the complex domain such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)\,Y^i = 0, \tag{38}$$

where $a_i(X)(i = 0,\ 1,\ \ldots,\ m)$ are polynomials of X and $a_m(X) \neq 0$. Equation (38) is called the first integral to (37), due to the Division Theorem, there exists a polynomial $g(X) + h(X)\,Y$ in the complex domain $\mathrm{C}[X,Y]$ such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = (g(X) + h(X)\,Y)\sum_{i=0}^{m} a_i(X)\,Y^i. \tag{39}$$

In this example we discuss two different values of m assuming that $m = 1$ and $m = 2$ in equation (38).

Case I: Suppose that $m = 1$, by equating the coefficients of Y^i $(i = 2, 1, 0)$ on both sides of equation (39), we have

$$\dot{a}_1(X) = h(X)\,a_1(X), \tag{40a}$$

$$\dot{a}_0(X) = g(X) + h(X)\,a_0(X), \tag{40b}$$

$$a_1(X)\,(\frac{3}{c}\,(X(\xi))^3 + (\frac{-3\alpha}{c} - c)\,X(\xi) - \beta) = g(X)\,a_0(X). \tag{40c}$$

Since $a_1(X)$ is a polynomial of X, then from equation (40a), we deduce that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X), a_1(X)$ and $a_0(X)$, we conclude that deg $g(X) = 1$ only. Suppose that $g(X) = A_1X + B_0$, and $A_1 \neq 0$, then we find $a_0(X)$.

$$a_0(X) = A_0 + B_0\,X + \frac{1}{2}A_1X^2. \tag{41}$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in equation (40c) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$\beta = 0,\ \alpha = \frac{-\sqrt{6\,c}\,A_0 - c^2}{3},\ A_1 = \sqrt{\frac{6}{c}},\ B_0 = 0, \tag{42a}$$

$$\beta = 0,\ \alpha = \frac{\sqrt{6\,c}\,A_0 - c^2}{3},\ A_1 = -\sqrt{\frac{6}{c}},\ B_0 = 0. \tag{42b}$$

Using the conditions (42a) in equation (38), we obtain

$$Y = -A_0 + \sqrt{\frac{3}{2\,c}}X. \tag{43}$$

Expression (43) is the first integral of (37). Combining equation (37) with equation (43), we obtain the exact solution to (36). Hence the exact solutions to the Drinfeld-Sokolov system (32) can be expressed as:

$$\begin{aligned} v(x,t) &= -\ \sqrt[4]{\tfrac{2\,c}{3}}\,\sqrt{A_0}\ \tan[\sqrt[4]{\tfrac{3}{2\,c}}\,\sqrt{A_0}\,(x - c\,t + \xi_0)], \\ u(x,t) &= \frac{\sqrt{6}\,A_0 + \sqrt{c^3} + \sqrt{6}\,A_0\,(\tan[\sqrt[4]{\frac{3}{2\,c}}\,\sqrt{A_0}\,(x - c\,t + \xi_0)])^2}{3\,\sqrt{c}}. \end{aligned} \tag{44}$$

Similarly, in the case of (42b), from equation (38) we obtain

$$Y = -A_0 + \sqrt{\frac{3}{2\,c}}X^2, \tag{45}$$

and the exact solution to the Drinfeld-Sokolov system (32) can be written as:

$$\begin{aligned} v(x,t) &= -\sqrt[4]{\tfrac{2\,c}{3}}\,\sqrt{A_0}\,\tanh[\sqrt[4]{\tfrac{3}{2\,c}}\,\sqrt{A_0}\,(x-c\,t+\xi_0)], \\ u(x,t) &= \tfrac{-\sqrt{6}\,A_0+\sqrt{c^3}+\sqrt{6}\,A_0\,(\tanh[\sqrt[4]{\frac{3}{2\,c}}\,\sqrt{A_0}\,(x-c\,t+\xi_0)])^2}{3\,\sqrt{c}}. \end{aligned} \tag{46}$$

Case II: Suppose that $m = 2$, by equating the coefficients of $Y^i\,(i = 3,2,1,0)$ on both sides of equation (39), we have

$$\dot{a}_2(X) = h(X)\,a_2(X), \tag{47a}$$

$$\dot{a}_1(X) = g(X)\,a_2(X) + h(X)\,a_1(X), \tag{47b}$$

$$\dot{a}_0(X) = -2\,a_2(X)\,(\frac{3}{c}\,(X(\xi))^3 + (\frac{-3\alpha}{c} - c)\,X(\xi) - \beta) + g(X)\,a_1(X) + h(X)\,a_0(X), \tag{47c}$$

$$a_1(X)\,(\frac{3}{c}\,(X(\xi))^3 + (\frac{-3\alpha}{c} - c)\,X(\xi) - \beta) = g(X)\,a_0(X). \tag{47d}$$

Since $a_2(X)$ is a polynomial of X , from (47a), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_2(X) = 1$. Balancing the degrees of $g(X)$, $a_1(X)$ and $a_0(X)$, we conclude that deg $g(X) = 1$ only. Now we discuss this case: if deg $g(X) = 1$, suppose that $g(X) = A_1\,X + B_0$, then we find $a_1(X)$, and $a_0(X)$.

$$a_1(X) = A_0 + B_0\,X + \frac{1}{2}\,A_1\,X^2, \tag{48a}$$

$$\begin{aligned} a_0(X) = d &+ \tfrac{1}{2}A_1\,B_0\,X^3 + \tfrac{(A_1^2\,c-12)\,X^4}{8\,c} + \tfrac{X^2(A_0A_1c+B_0^2\,c+2\,c^2+6\,\alpha)}{2\,c} \\ &+X\,(A_0B_0 + 2\,\beta), \end{aligned} \tag{48b}$$

where A_0 and d are arbitrary integration constants.

Substituting $a_0(X)$, $a_1(X)$ $a_2(X)$ and $g(X)$ in (47d) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$d = \frac{A_0^2}{4},\, \beta = 0\,, \alpha = \frac{-\sqrt{6\,c}A_0 - 2\,c^2}{6},\;\; A_1 = \frac{2\,\sqrt{6}}{\sqrt{c}},\, B_0 = 0, \tag{49a}$$

$$d = \frac{A_0^2}{4},\, \beta = 0\,, \alpha = \frac{\sqrt{6\,c}A_0 - 2\,c^2}{6},\;\; A_1 = \frac{-2\,\sqrt{6}}{\sqrt{c}},\, B_0 = 0. \tag{49b}$$

Using the conditions (49a) in equation (38), we obtain

$$Y = \frac{-A_0\,\sqrt{c} - \sqrt{6}X^2}{2\,\sqrt{c}}. \tag{50}$$

Expression (50) is the first integral of (37). Combining (37) with (50), we obtain the exact solution to equation (36) as follows:

$$g(\xi) = -\frac{\sqrt{A_0}\,\sqrt[4]{c}}{\sqrt[4]{6}}\,\tan(\frac{\sqrt[4]{3}\sqrt{A_0}\,(\xi+\xi_0)}{\sqrt[4]{8\,c}}), \tag{51}$$

where ξ_0 is an arbitrary integration constant. Then the exact solutions to the nonlinear Drinfeld – Sokolov system (32) can be written as

$$\begin{aligned} v(x,t) &= -\tfrac{\sqrt{A_0}\,\sqrt[4]{c}}{\sqrt[4]{6}}\,\tan(\tfrac{\sqrt[4]{3}\sqrt{A_0}\,(x-c\,t+\xi_0)}{\sqrt[4]{8\,c}}), \\ u(x,t) &= -\tfrac{\sqrt{6\,c}\,A_0-2\,c^2}{6\,c} + \tfrac{A_0}{\sqrt[4]{6}}(\tan(\tfrac{\sqrt[4]{3}\sqrt{A_0}\,(x-c\,t+\xi_0)}{\sqrt{6\,c}}))^2. \end{aligned} \tag{52}$$

Similarly, in the case of (49b), from equation (38) we obtain

$$Y = \frac{-A_0\sqrt{c} + \sqrt{6}X^2}{2\sqrt{c}}, \tag{53}$$

and the exact solutions to the nonlinear Drinfeld – Sokolov system (32) are given respectively by

$$\begin{aligned} v(x,t) &= -\frac{\sqrt{A_0}\,\sqrt[4]{c}}{\sqrt[4]{6}}\tanh\left(\frac{\sqrt[4]{3}\sqrt{A_0}\,(x-c\,t+\xi_0)}{\sqrt[4]{8\,c}}\right),\\ u(x,t) &= -\frac{\sqrt{6\,c}\,A_0-2\,c^2}{6\,c} + \frac{A_0}{\sqrt{6\,c}}\left(\tanh\left(\frac{\sqrt[4]{3}\sqrt{A_0}\,(x-c\,t+\xi_0)}{\sqrt[4]{8\,c}}\right)\right)^2. \end{aligned} \tag{54}$$

All these solutions are new exact solutions.

4 Conclusion

The first integral method is applied successfully for solving the system of nonlinear partial differential equations which are the variant Boussinesq and the nonlinear Drinfeld-Sokolov systems exactly. Thus, we deduce that the proposed method can be extended to solve many systems of nonlinear partial differential equations which are arising in the theory of solitons and other areas. The exact solution of the general system of nonlinear partial differential equations using the first integral method is still an open point of research.

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