ACADIMIC
World Academic Union

ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.5(2008) No.2,pp.106-110

The Equivalence between a Complementarity Control System and a Differential Inclusion [*]

Xiaoxiao Dong[†],Yan Gao [‡]
School of Management, University of Shanghai for Science and Technology,
516 Jungong Road, Shanghai 200093, China

(Received 30 July 2007, accepted 27 December 2007)

Abstract: This paper is devoted to the equivalence between a complementarity control system with subdifferentiable constraint function and a differential inclusion. We mainly use tools of convex analysis to verify the equivalence through constructing the tangent cone and the normal cone.

Key words:complementarity system; differential inclusion; tangent cone; normal cone; subdifferentiable function

1 Introduction

In recent years, the control systems with complementarity constraints are widely studied, see for instance [3] and references therein. It goes without saying that transforming a constrained control system into an unconstrained system is meaningful. In [3], the following systems are considered:

$$\begin{aligned} -\dot{x}(t) &= f(x(t)) + g(t) + \nabla h(x(t))^T \lambda(t), \quad a.e. \\ 0 \geq h(x(t)) &\perp \lambda(t) \geq 0, \quad a.e. \end{aligned} \tag{1}$$

and

$$-\dot{x}(t) \in f(x(t)) + g(t) + N^h(x(t)), \quad a.e. \tag{2}$$

where $f : \Re^n \to \Re^n$ and $g : \Re_+ \to \Re^n$ are measurable, $\lambda : \Re_+ \to \Re^m$, $h(x) = (h_1(x), \ldots, h_m(x))^T$, $h_i : \Re^n \to \Re$ for $i = 1, \ldots, m$ are smooth , and $N^h(x(t))$ is a cone (see [3]). The equivalence between (1) and (2) is obtained. In the engineering systems, the complementarity systems are the subject of many recent years. Indeed, if we convert a original control system into a complementarity control system, then the stability and reliability of the system can be improved. Furthermore, if we convert a control system with complementarity constraints into a control system without constraints, we can solve the problem easily. The method used in this paper can be applied to some problems in [5,8,10,11].

In previous papers, see for instance [3], the constraint function h is smooth. The present paper will generalize the related work, nonsmooth constraint function h is considered. We will suppose that h is subdifferentiable.

2 Preliminaries

In this section, the notions about a subdifferentiable function and a quasiconvex function are given.

---

[*] This work was supported by National Science Foundation of China (under grant: 10671126), key project for fundamental research of STCSM(under grant: 06JC14057) and Shanghai leading discipline project(under grant: T0502).

[†] E-mail: dongxiaoxiao1983@163.com

[‡] **Corresponding author**. *E-mail address*:gaoyan1962@263.net

Let $h : \Re^n \to \Re^n$ be directional differentiable. As in [4], if there exists a convex and compact set $\partial h(x)$, such that the directional derivative of h can be written as

$$h'(x;d) = \lim_{t\to 0^+} \frac{1}{t}(h(x+td) - h(x)) = \max_{\xi\in\partial h(x)} \xi^T d, \quad \forall d \in \Re^n,$$

then, we call that h is subdifferentiable, $\partial h(x)$ is called the subdifferential of h.

Subdifferentiable function is widely used in control and optimization. Continuous function, convex function, and maximum function are subdifferentiable. Next, we give an example of a subdifferentiable function. Let

$$\psi(x) = G(\max_{j\in J_1} h_{1j}(x), \dots, \max_{j\in J_m} h_{mj}(x)),$$

where G and h_{ij} are continuous and differentiable on $\Re^m$ and $\Re^n$,respectively, J_i for $i = 1, \dots, m$ are finite index sets. Denote

$$J_i(x) = \{k \in J_i | h_{kj}(x) = \max_{j\in J_i} h_{ij}(x)\}.$$

As in [6], the subdifferential of ψ is written by

$$\partial\psi(x) = \mathrm{co}\{\zeta \in \Re^n | \zeta = \sum_{i=1}^{m} \frac{\partial G(y_1, \dots, y_m)}{\partial y_i}|_{y_i=\max_{j\in J_i} h_{ij}(x)} \nabla h_{j_i}(x), j_i \in J_i\}.$$

Evidently, $\partial\psi(x)$ is a convex hull of finite number of points.

If the function $h_i, i \in I$ are subdifferentiable, then $h(x) = \max_{i\in I} h_i(x)$ also is subdifferentiable with the subdifferential

$$\partial h(x) = \mathrm{co}\{\bigcup_{i\in I(x)} \partial h_i(x)\},$$

where $I(x) = \{i \in I | h_i(x) = h(x)\}$, see [4]. Therefore if $\partial h_i(x)$ is a convex hull of finite number of points, then $\partial h(x)$ is a convex hull of finite number of points also .

Definition 1 *([2]). A function $h : \Re^n \to \Re$ is called quasiconvex, if $h(q_1x^1 + q_2x^2) \le \max[h(x^1), h(x^2)]$, $\forall x^1, x^2 \in \Re^n$ and $q_1, q_2 \ge 0$ with $q_1 + q_2 = 1$.*

Theorem 2 *([2]). A function $h : \Re^n \to \Re$ is called quasiconvex, if and only if the level set of h is convex for all $\alpha \in \Re$.*

Suppose $h_i, i \in I$ are a quasiconvex function. In what follows we show that the maximum function $h(x) = \max_{i\in I} h_i(x)$ is a quasiconvex function, too.

By deducing, we have

$$\begin{aligned} h(q_1x^1 + q_2x^2) &\le \max_{i\in I} h_i(q_1x^1 + q_2x^2) \le \max_{i\in I} \max[h_i(x^1), h_i(x^2)] \\ &= \max[\max_{i\in I} h_i(x^1), \max_{i\in I} h_i(x^2)] = \max[h(x^1), h(x^2)], \\ &\qquad \forall x^1, x^2 \in \Re^n, q_1, q_2 \ge 0, q_1 + q_2 = 1. \end{aligned}$$

According to Definition 1 , $h(x) = \max_{i\in I} h_i(x)$ is a quasiconvex function.

The normal cone to a nonempty closed convex set $K \subset \Re^n$ at $x \in K$ is defined as

$$N_K(x) := \{s \in \Re^n | \langle s, y - s\rangle \le 0, \quad \forall y \in K\},$$

while the tangent cone is the polar of the normal cone, which means

$$T_K(x) := [N_K(x)]^\circ := \{d \in \Re^n | \langle s, d\rangle \le 0, \quad \forall s \in N_K(x)\}.$$

3 An Equivalence

If there are multiconstraints, for instance $h_j(x) \leq 0, j = 1, \ldots, m$. Then, we reformulate the constraints by $h(x) = \max_{1 \leq j \leq m} h_j(x) \leq 0$. In Section 2, we know that if h_j for $j = 1, \ldots, m$ are subdifferentiable, quasiconvex and the subdifferential of h_j is a convex hull of finite number of points, then the maximum function h also is subdifferentiable, quasiconvex and its subdifferential is a convex hull of finite number of points. Therefore, without loss of generalization, we consider only one constraint function. That is, the constraint function is a maximum function $h(x) = \max_{1 \leq j \leq m} h_j(x)$.

Throughout this section, we suppose that h_j for $j = 1, \ldots, m$ are subdifferentiable, quasiconvex and $\partial h_j(x)$ for $j = 1, \ldots, m$ are a convex hull of finite number of points. Therefore, h also is subdifferentiable. Since the subdifferential of h_j is a convex hull of finite number of points, we have

$$\partial h(x) = \mathrm{co}\{v_1, \ldots, v_p\}.$$

Since h_j is quasiconvex, h also is quasiconvex. By virtue of Theorem 1, $C := \{x \in \Re^n | h(x) \leq 0\}$ is convex.

Constraint qualifications (CQ for short) of set C at x are given as follows:

CQ 1 ([1]). There exists $y_0 \in \Re^n$, such that $h'(x; y_0) < 0$.

CQ 2 ([9]). $\mathrm{cl}\gamma(x) = \Gamma(x)$,where cl is a closure,

$$\gamma(x) = \{y \in \Re^n | h'(x; y) < 0\},$$

$$\Gamma(x) = \{y \in \Re^n | h'(x; y) \leq 0\}.$$

Proposition 1 *([9]). If CQ1 or CQ2 is satisfied at $x \in C$, then $T_C(x) = \Gamma(x)$.*

Proposition 2 *If CQ1 or CQ2 is satisfied, then*

$$N_C(x) = [T_C(x)]^\circ = \{\sum_{i \in I_0} \lambda_i v_i | \lambda_i \geq 0, i \in I_0\},$$

where $I_0 = \{i \in \{1, \ldots, p\} | {v_i}^T y = 0\}$.

Proof. According to [7], we obtain

$$T_C = \{y \in \Re^n | {v_i}^T y \leq 0, 1 \leq i \leq p\}.$$

Let

$$D = \{\sum_{i \in I_0} \lambda_i v_i | \lambda_i \geq 0, i \in I_0\}.$$

We will verify

$$D^\circ = \{y \in \Re^n | {v_i}^T y \leq 0, 1 \leq i \leq p\}.$$

If ${v_i}^T y \leq 0$ for all i , then $x^T y \leq 0$ for all $x \in D$. Therefore, $y \in D^\circ$ and

$$D^\circ \supset \{y \in \Re^n | {v_i}^T y \leq 0, 1 \leq i \leq p\}.$$

Conversely, if $y \in D^\circ$, i.e., $x^T y \leq 0$ for all $x \in D$, and since $v_i \in D$ for all i , then we have ${v_i}^T y \leq 0$ for all i. Thus, it follows that

$$D^\circ \subset \{y \in \Re^n | {v_i}^T y \leq 0, 1 \leq i \leq p\}.$$

Since D is closed and convex, we have $(D^\circ)^\circ = D$, in other words,

$$[T_C(x)]^\circ = \{\sum_{i \in I_0} \lambda_i v_i | \lambda_i \geq 0, i \in I_0\}.$$

Furthermore, the convexity and closeness of C implies

$$N_C(x) = [T_C(x)]^\circ = \{\sum_{i \in I_0} \lambda_i v_i | \lambda_i \geq 0, i \in I_0\}.$$

■

Consider the following complementarity system

$$\begin{aligned} -\dot{x}(t) = f(x(t)) + g(t) + \lambda(t)V, \quad a.e. \\ (v_i, y)\lambda_i = 0, \quad (v_i, y) \le 0, \quad \lambda_i \ge 0, \quad a.e. \end{aligned} \tag{3}$$

where $V = (v_1, \dots, v_p)^T, \lambda(t) = (\lambda_1, \dots, \lambda_p)$ and $y \in T_C(x)$, and the differential inclusion

$$-\dot{x}(t) \in f(x(t)) + g(t) + N_C(x(t)). \quad a.e. \tag{4}$$

Proposition 3 *$x = x(t)$ and $\dot{x} = \dot{x}(t)$ in $\Re^n$ satisfy (3) if and only if they satisfy (4).*

Proof. If $x(t) \notin C, x(t)$ does not solve (3) or (4). We suppose $x(t) \in C.$ $(v_i, y)\lambda_i = 0, (v_i, y) \le 0, \lambda_i \ge 0$ mean that if $(v_i, y) < 0$, i.e. if $i \notin I_0$, then $\lambda_i = 0$. From the definition of $N_C(x)$, (3) is equivalent to (4). ■

Remark 1 *When the constraint function $h : \Re^n \to \Re$ is smooth, the system (3) degenerates to the form of (1). In fact, (3) is written as*

$$\begin{aligned} -\dot{x}(t) \in f(x(t)) + g(t) + \lambda(t)\partial h(x(t)), \quad a.e. \\ (v_i, y)\lambda_i = 0, (v_i, y) \le 0, \lambda_i \ge 0. \quad a.e. \end{aligned} \tag{5}$$

We know that $v_i \in \partial h(x(t))$ for $i = 1, \dots, p$. Thus, v_i has the form

$$v_i = (v_i^1, v_i^2, \dots, v_i^n)^T.$$

Obviously, if h is smooth, then $\partial h(x) = \{\nabla h(x)\}$ and $p = 1$. Therefore, (5) can be rewritten by

$$\begin{aligned} -\dot{x}(t) = f(x(t)) + g(t) + \lambda(t)\nabla h(x(t)), \quad a.e. \\ (\nabla h(x), y)\lambda = 0, (\nabla h(x), y) \le 0, \lambda \ge 0, \quad a.e. \end{aligned} \tag{6}$$

where $h : \Re^n \to \Re$ is a smooth function. At the same time, (4) can be rewritten by

$$-\dot{x}(t) \in f(x(t)) + g(t) + N_C(x(t)), \quad a.e. \tag{7}$$

where $N_C = \{\lambda\nabla h(x) | \lambda \ge 0, (\nabla h(x), y) = 0, y \in T_C\}$ or $N_C = \emptyset$ (according to proposition 3).

If one of the qualifications (CQ1-4) is satisfied in [3], then according to proposition 3 of [3], we have (7) and (2) are equivalent. Thus, (6) and (1) are equivalent. Namely, (1) is the degeneration of (3).

Example 1 *In the set C , let $h(x) = \max\{-x_1, -x_2, x_1^2 + x_2^2 - 1\}, x = (x_1, x_2)^T$. Evidently, h is subdifferentiable and C is convex and closed. Consider $x_0 = (0, 1)^T$. According to the definition and operation of differential, we have*

$$\partial h(x_0) = \text{co}\{(0, -1)^T, (2, 0)^T\}.$$

Denoting

$$I_0 = \{i \in \{1, 2\} | {v_i}^T y = 0, \},$$

since

$$\begin{aligned} T_C &= \{y \in \Re^2 | \max_{1 \le i \le 2} {v_i}^T y \le 0\} \\ &= \{y \in \Re^2 | {v_i}^T y \le 0, 1 \le i \le 2\}, \end{aligned}$$

we obtain

$$N_C(x) = [T_C(x)]^\circ = \{\sum_{i \in I_0} \lambda_i v_i | \lambda_i \ge 0, i \in I_0\}.$$

Setting $v_1 = (0, -1)^T, v_2 = (2, 0)^T$, it follows that

$$T_C = \{y \in \Re^2 | y_1 \le 0, y_2 \ge 0\}.$$

Therefore, it is obtained that

if $y = (y_1 \neq 0, y_2 \neq 0)^T$, then $I_0 = \emptyset, \lambda_1 = \lambda_2 = 0$;
if $y = (y_1 \neq 0, y_2 = 0)^T$, then $I_0 = \{1\}, \lambda_1 \ge 0, \lambda_2 = 0$;
if $y = (y_1 = 0, y_2 = 0)^T$, then $I_0 = \{1, 2\}, \lambda_1 \ge 0, \lambda_2 \ge 0$;
if $y = (y_1 = 0, y_2 \neq 0)^T$, then $I_0 = \{2\}, \lambda_1 = 0, \lambda_2 \ge 0$.

Similarly, we can get the equivalence between (3) and (4) at other points. Therefore, the complementarity system (3) and the differential inclusion (4) are equivalent.

4 Conclusions

In nonlinear control systems, constraint functions usually are nonsmooth. However, subdifferentiable function is a wide class of nonsmooth functions. It is meaningful to consider the equivalence between a complementarity system with subdifferentiable constraint function and a differential inclusion .

References

[1] Aubin J P: Viability Theory. *Birkhäuser: Boston*(1991)

[2] Avriel M: Nonlinear Programming Analysis and Methods. *Prentice-Hall: New Jersey*(1976)

[3] Brogliato B, Daniilidis A, Lemarécha C, Acary V: On the equivalence between complementarity systems, projected systems and differential inclusion. *System and Control Letters.* 55:45-51(2006)

[4] Demyanov V F, Rubinov A M: Constructive Nonsmooth Analysis. *Peter Lang: Frankfurt am Main*(1995)

[5] Fan Z B, Dong Q X, Li G: Semilinear differential equations with nonlocal conditions in Banach spaces. *International Journal of Nonlinear Science* . 2(3):131-139(2006)

[6] Gao Y: Demyanov difference of two sets and optimality conditions of Lagrange multiplier type for constrained quasidifferentiable optimization. *Optimization Theory and Applications* . 104:377-394(2000)

[7] Gao Y: Determing the viability for a class of nonlinear control systems on a region with nonsmooth boundary (in Chinese). *Control and Decision* . 21:923-925(2006)

[8] Li Z L, Chen Z Q, Yuan Z Z: The stability analysis and control of nonminimum phase nonlinear systems. *International Journal of Nonlinear Science* . 3(2):103-110(2007)

[9] Quincampoix M, Seube N: Stabilization of uncertain control systems through piecewise constant feedback. *Mathematics Analysis and Applications* . 218:240-255(1998)

[10] Sun M, Tian L X, Xu J: Time-delayed feedback control of the energy resource chaotic system. *International Journal of Nonlinear Science* . 1(3):172-177(2006)

[11] Wang X D, Tian L X, Yu L Q: Linear feedback controlling and synchronization of the Chen's chaotic system. *International Journal of Nonlinear Science* . 2(1):43-49(2006)