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New Exact Solutions of Some Nonlinear Partial Differential Equations

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Abstract: The modified extended tanh-function (METF) method for finding solitary wave solutions to nonlinear physical problems is described. We consider four kinds of non-linear partial differential equations such as modified Korteweg-de Vries equation (MKdV for short), two-dimensional Korteweg-de Vries equation (2DKdV-Burgers for short), variant nonlinear water wave equation and variant Boussinesq equations. Their solitary wave solutions are constructed as well. The method is usually tedious to use by hand, therefore we use the Maple package to solve the algebraic system and consequently to get the exact solutions.

Key words: MKdV equation, 2DKdV-Burgers' equation, variant nonlinear water wave equation, variant Boussinesq equations, METF method

1 Introduction

Explicit solutions to the mathematical modelling of physical problems are of fundamental importance. There are many methods in literature to solve the nonlinear equations. Among them are Backlund transformation [1,2], Darboux transformation[3], Inverse scattering method[4], Hirota's bilinear method[5], the tanh method [6], the sine-cosine method [7,10], the homogeneous balance method[8,9], variational iteration method [11-16], and the Riccati expansion method with constant coefficients[17,18]. The third-order KdV equation has been the focus of considerable recent studies by [19-21]. Gardner [21] developed a variational and its Hamiltonian formulation to handle this problem and also Gardner et. al. [22] introduced various methods for explicit solutions for the KdV equation. Soliman [23] used the collocation solution of the KdV equation using septic splines as element shape function. Ma [24] applied a special solution of square Hopf-Cole type to an ordinary equation to obtain the bounded travelling wave solution to the two-dimensional Kortewege-de Vries-Burgers equation. Kaya [25] applied the Adomian decomposition method for solving the variant non-linear water wave equation. The Adomian decomposition method [26] is used to obtain the solitary wave solution of (2+1) dimensional Boussinesq equation.

Recently the modified extended tanh-function method and symbolic computation have been suggested for solving the system of of nonlinear partial differential equations [27-29] and nonlinear equations of special interest in physics namely, the Broer-Kaup-Kupershmidt, nonlinear coupled plasma, and coupled-nonlinear reaction-diffusion equations[30].

The aim of this paper is to extend the modified extended tanh-function (METF) method to solve four different types such as the modified Korteweg-de Vries, two-dimensional Korteweg-de Vries equation, variant water wave and the variant Boussinesq equations.

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2 Modified extended tanh-function method

To illustrate the basic concepts of the modified extended tanh-function method, consider a given PDE in two indpendent variables given by

$$F(u, u_x, u_t, u_{xx}, ...) = 0 \tag{1}$$

We first consider its travelling solutions $u(x,t) = u(\xi)$, $\xi = x + \lambda t$ or $\xi = x - \lambda t$, then Eq.(1) becomes an ordinary differential equation. In order to seek the travelling wave solutions of Eq.(1), we introduce the following ansatze

$$u(\xi) = a_0 + \sum_{i=1}^{M} a_i\phi^i + b_i\phi^{-i}, \tag{2}$$

$$\phi' = b + \phi^2, \tag{3}$$

where b is a parameter to be determined, $\phi = \phi(\xi)$, $\phi' = \frac{d\phi}{d\xi}$. The parameter M can be found by balancing the highest-order linear term with the nonlinear terms [31]. Inserting (2) and (3) into the ordinary differential equation will yield a system of algebraic equations with respect to a_i, b_i, b and λ (where $i = 1...M$) because all the coefficients of ϕ^i have to vanish. Then we can determine a_0, a_i, b_i, b, and λ. The Riccati equation (3) has the general solutions:

(I) If $b < 0$

$$\phi = -\sqrt{-b}tanh(\sqrt{-b}\xi)$$

$$\phi = -\sqrt{-b}coth(\sqrt{-b}\xi)$$

(II) If $b > 0$

$$\phi = \sqrt{b}tan(\sqrt{b}\xi)$$

$$\phi = -\sqrt{b}cot(\sqrt{b}\xi)$$

(III) If $b = 0$

$$\phi = \frac{-1}{\xi}$$

To illustrate the above-mentioned, four examples of special interest of important nonlinear partial differential equations such as modified Korteweg-de Vries equation, two-dimensional Korteweg-de Vries equation, variant non-linear water wave equation and variant Boussinesq equations are discussed.

3 Applications

3.1 MKdV equation

Let us first consider the MKdV equation which has the form [32]

$$u_t + u^2\, u_x + u_{xxx} = 0. \tag{4}$$

In order to solve Eq. (4) by the modified extended tanh-function method, we use the wave transformation $u(x,t) = U(\xi)$ with wave variable $\xi = x - \lambda t$, thus Eq. (4) takes the form of an ordinary differential equation such as

$$-\lambda U' + \frac{1}{3}(U^3)' + U''' = 0, \tag{5}$$

integrating Eq. (5) once with respect to ξ and setting the constant of integration to be zero, we obtain

$$-\lambda U + \frac{1}{3}(U^3) + U'' = 0. \tag{6}$$

Balancing the order of U^3 with the order of U'' in Eq. (6), we find $M = 1$. So the solution takes the form

$$U(\xi) = a_0 + a_1\phi(\xi) + b_1\phi(\xi)^{-1} \tag{7}$$

Inserting Eq. (7) into Eq. (6), making use of Eq. (3), and by using Maple Package, we get a system of algebraic equations, for a_0, a_1, b_1 and b in the form

$$\begin{gathered}
\frac{1}{3}a_1^3 + 2a_1 = 0, \\
a_0 a_1^2 = 0, \\
\begin{aligned}
-\lambda a_1 + a_0^2 a_1 + a_1^2 b_1 + 2a_1 b &= 0, \\
-\lambda a_0 + \frac{1}{3}a_0^3 + 2a_0 a_1 b_1 &= 0, \\
-\lambda b_1 + a_0^2 b_1 + a_1 b_1^2 + 2b_1 b &= 0,
\end{aligned} \\
a_0 b_1^2 = 0, \\
2ab_1b_2 - 2eb_2b + 2cb_1b^2 = 0, \\
\frac{1}{3}b_1^3 + 2b_1 b^2 = 0.
\end{gathered} \tag{8}$$

These equations lead to the following cases:

Case (1) :

$$a_0 = 0, a_1 = 0, b = \frac{1}{6}I\sqrt{6}b_1, \lambda = \frac{1}{6}I\sqrt{6}b_1 \tag{9}$$

Substituting (9) into (7) and using (3), for b_1 being arbitrary constant, the solution of Eq. (4) is given by:

$$u(x,t) = -I\sqrt{I\sqrt{6}b_1}\cot\Big(\frac{\sqrt{6}}{6}\sqrt{Ib_1\sqrt{6}}\big(x + \frac{1}{3}Ib_1\sqrt{6}t\big)\Big),$$

if $b_1 > 0$, and

$$u(x,t) = -I\sqrt{-I\sqrt{6}b_1}\coth\Big(\frac{\sqrt{6}}{6}\sqrt{-Ib_1\sqrt{6}}\big(x + \frac{1}{3}Ib_1\sqrt{6}t\big)\Big),$$

if $b_1 < 0$.

Case (2) :

$$a_1 = I\sqrt{6},\ a_0 = 0,\ b_1 = 0,\ b = b, \lambda = 2b \tag{10}$$

Substituting (10) into (7) and using (3), the solution of Eq. (4) is given by:

$$u(x,t) = I\sqrt{6b}\tan(\sqrt{b}(x + 2bt)),$$

if $b > 0$, and

$$u(x,t) = -I\sqrt{-6b}\tanh(\sqrt{-b}(x + 2bt)),$$

if $b < 0$

Case (3) :

$$a_0 = 0, a_1 = I\sqrt{6},\ b_1 = b1,\ b = \frac{\sqrt{6}}{6}Ib_1, \lambda = \frac{4}{3}I\sqrt{6}b_1 \tag{11}$$

Substituting (11) into (7) and using (3), for b_1 being arbitrary constant, the solution of Eq. (4) is given by:

$$u(x,t) = I\sqrt{Ib_1\sqrt{6}}\left[\begin{array}{c} \tan(\frac{\sqrt{6}}{6}\sqrt{Ib_1\sqrt{6}}(x + \frac{4}{3}Ib_1\sqrt{6}t)) \\ -\cot(\frac{\sqrt{6}}{6}\sqrt{Ib_1\sqrt{6}}(x + \frac{4}{3}Ib_1\sqrt{6}t)) \end{array}\right],$$

if $b_1 > 0$, and

$$u(x,t) = -I\sqrt{-Ib_1\sqrt{6}}\left[\begin{array}{c} \tanh(\frac{\sqrt{6}}{6}\sqrt{-Ib_1\sqrt{6}}(x + \frac{4}{3}Ib_1\sqrt{6}t)) \\ +\coth(\frac{\sqrt{6}}{6}\sqrt{-Ib_1\sqrt{6}}(x + \frac{4}{3}Ib_1\sqrt{6}t)) \end{array}\right],$$

if $b_1 < 0$.

3.2 2DKdVB equation

A second important example is the two-dimensional KdV-Burgers' equation which can be written as [24]

$$(u_t + 2au\,u_x + eu_{xx} + cu_{xxx})_x + du_{yy} = 0,$$

or in the form

$$u_{tx} + 2a(u\,u_x)_x + eu_{xxx} + cu_{xxxx} + du_{yy} = 0, \tag{12}$$

where a, e, c and d are arbitrary constants. In order to solve Eq. (12) by the modified extended tanh-function method, we use the wave transformation $u(x,t) = U(\xi)$ with wave variable $\xi = x + y - \lambda t$, so Eq. (12) takes the form of ordinary differential equation as

$$(-\lambda + d)U'' + 2a(UU')' + eU''' + cU'''' = 0 \tag{13}$$

Integrating Eq. (13) twice with respect to ξ, so we find the special form of exact solutions for simplicity purpose, we take the constants of integration to be zeros and setting the constant of integration to be zero, we obtain

$$(-\lambda + d)U + aU^2 + eU' + cU'' = 0 \tag{14}$$

Balancing the order of U^2 with the order of U'' in Eq. (22), we find $M = 2$. So the solution take the form

$$U(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2 + b_1\phi(\xi)^{-1} + b_2\phi(\xi)^{-2} \tag{15}$$

Inserting Eq. (15) into Eq. (14), making use of Eq. (3), and using Maple Package, we get a system of algebraic equations, for a_0, a_1, a_2, b_1, b_2 and b in the form

$$aa_2^2 + 6ca_2 = 0$$

$$2aa_1a_2 + 2ea_2 + 2ca_1 = 0$$

$$\begin{aligned} 2aa_0a_2 + 8ca_2b - \lambda a_2 + da_2 + aa_1^2 + ea_1 &= 0 \\ 2aa_0a_1 + 2ab_1a_2 + 2ea_2b + 2ca_1b - \lambda a_1 + da_1 &= 0 \end{aligned}$$

$$-\lambda a_0 + aa_0^2 + 2aa_1b_1 + 2aa_2b_2 + ea_1b + 2ca_2b^2 + da_0 - eb_1 + 2cb_2 = 0$$

$$\begin{aligned} 2aa_0b_1 + 2aa_1b_2 + 2cb_1b - \lambda b_1 + db_1 - 2eb_2 &= 0 \\ 2aa_0b_2 - eb_1b + 8cb_2b - \lambda b_2 + db_2 + ab_1^2 &= 0 \end{aligned}$$

$$\begin{aligned} 2ab_1b_2 - 2eb_2b + 2cb_1b^2 &= 0 \\ ab_2^2 + 6cb_2b^2 &= 0 \end{aligned}$$

We select three cases of solutions for this system as

Case (1)

$$a_0 = \frac{9e^2}{50ac}, a_1 = a_2 = 0, b_1 = -\frac{3e^3}{250c^2a}, b_2 = \frac{-3e^4}{5000c^3a}, \lambda = \frac{6e^2 + 25dc}{25c}, b = \frac{-e^2}{100c^2}. \tag{16}$$

Substituting (16) into (15) and using (3), the solution of Eq. (12) is given by

$$u(x,y,t) = \frac{9e^2}{50ac} + \frac{3e^2}{25ac}\coth\left(\frac{e}{10c}(x + y + \frac{6e^2 + 25dc}{25c}t)\right) - \frac{3e^2}{50ac}\coth^2\left(\frac{e}{10c}(x + y + \frac{6e^2 + 25dc}{25c}t)\right)$$

Case (2)

$$a_0 = \frac{9e^2}{50ac}, a_1 = \frac{-6e}{5a}, a_2 = -6\frac{c}{a}, b_1 = 0, b_2 = 0, \lambda = \frac{6e^2 + 25dc}{25c}, b = \frac{-e^2}{100c^2}. \tag{17}$$

Substituting (17) into (15) and using (3), the solution of Eq. (12) is given by

$$\begin{aligned} u(x,y,t) &= \frac{9e^2}{50ac}+\frac{3e^2}{25ac}\tanh(\frac{e}{10c}(x+y+\frac{6e^2+25dc}{25c}t)) \\ &\quad -\frac{3e^2}{50ac}\tanh^2(\frac{e}{10c}(x+y+\frac{6e^2+25dc}{25c}t)) \end{aligned}$$

Case (3)

$$a_0=\frac{3e^2}{20ac}, a_1=\frac{-6e}{5a}, a_2=-6\frac{c}{a}, b_1=\frac{-3e^3}{1000c^2a}, b_2=\frac{-3e^4}{8000c^3a}, \lambda=\frac{6e^2+25dc}{25c}, b=\frac{-e^2}{400c^2}. \tag{18}$$

Substituting (18) into (15) and using (3), the solution of Eq. (12) is given by

$$\begin{aligned} u(x,y,t) &= \frac{3e^2}{20ac}+\frac{3e^2}{50ac}\left[\begin{array}{c} \tanh(\frac{e}{20c}(x+y+\frac{6e^2+25dc}{25c}t)) \\ +\coth(\frac{e}{20c}(x+y+\frac{6e^2+25dc}{25c}t)) \end{array}\right] \\ &\quad -\frac{e^2}{400ac}\left[\begin{array}{c} \tanh^2(\frac{e}{20c}(x+y+\frac{6e^2+25dc}{25c}t)) \\ +\coth^2(\frac{e}{20c}(x+y+\frac{6e^2+25dc}{25c}t)) \end{array}\right] \end{aligned}$$

3.3 The variant nonlinear water wave equation

An instructive third example as an illustration of the modified extended tanh-method for solving the variant nonlinear water wave equation [25], can be in the form

$$u_t+u_x+u\,u_x+u_{xxx}+u_x\,u_{xx}+u\,u_{xxx}+u_{xxxxx}=0, \tag{19}$$

In order to solve Eq. (19) by the modified extended tanh-function method, we use the wave transformation $u(x,t)=U(\xi)$ with wave variable $\xi=x+\lambda t$, so Eq. (19) takes the form of an ordinary differential equation as

$$(\lambda+1)U'+\frac{1}{2}(U^2)'+U'''+(UU'')'+U'''''=0, \tag{20}$$

Integrating Eq. (20) once with respect to ξ and setting the constant of integration to be zero, we obtain

$$(\lambda+1)U+\frac{1}{2}U^2+U''+UU''+U''''=0, \tag{21}$$

Balancing the order of UU'' with the order of U'''' in Eq. (21), we find $M=2$. So the solution takes the form

$$U(\xi)=a_0+a_1\phi(\xi)+a_2\phi(\xi)^2+b_1\phi(\xi)^{-1}+b_2\phi(\xi)^{-2} \tag{22}$$

Inserting Eq. (22) into Eq. (21), making use of Eq. (3), and using Maple Package, we get a system of algebraic equations, for $a_0, a_1, b_1, a_2, b_2, \lambda$ and b in the form

$$\begin{aligned} 6a_2^2+120a_2 &= 0 \\ 8a_1a_2+24a_1 &= 0 \end{aligned}$$

$$\begin{aligned} 6a_2+\frac{1}{2}a_2^2+2a_1^2+6a_0a_2+8a_2^2b+240a_2b &= 0 \\ 10a_1a_2b+2a_1+a_1a_2+2a_0a_1+6b_1a_2+40a_1b &= 0 \end{aligned}$$

$$\begin{aligned} &a_2+8a_0a_2b+\frac{1}{2}a_1^2+8a_2b+\lambda a_2+a_0a_2+2a_1b_1 \\ &+8a_2b_2+2a_2^2b^2+2a_1^2b+136a_2b^2=0 \end{aligned}$$

$$a_1 + 2a_1a_2b^2 + 10b_1a_2b + 2a_0a_1b + 2a_1b + \lambda a_1 + a_0a_1 + b_1a_2 + 4a_1b_2 + 16a_1b^2 = 0$$

$$2a_0a_2b^2 + 4a_1b_1b + 16a_2b_2b + a_0 + \frac{1}{2}a_0^2 + 2b_2$$

$$+2a_2b^2 + \lambda a_0 + a_1b_1 + a_2b_2 + 2a_0b_2 + 16b_2b + 16a_2b^3 = 0$$

$$b_1 + 2a_0b_1b + 10a_1b_2b + 4b_1a_2b^2 + 2b_1b + \lambda b_1$$

$$+a_0b_1 + a_1b_2 + 2b_1b_2 + 16b_1b^2 = 0$$

$$b_2 + 8a_0b_2b + 2a_1b_1b^2 + 8a_2b_2b^2 + \frac{1}{2}b_1^2 + 2b_2^2 + 8b_2b + \lambda b_2$$

$$+a_0b_2 + 2b_1^2b + 136b_2b^2 = 0$$

$$\begin{aligned} b_1b_2 + 2a_0b_1b^2 + 6a_1b_2b^2 + 10b_1b_2b + 2b_1b^2 + 40b_1b^3 &= 0 \\ 6a_0b_2b^2 + \frac{1}{2}b_2^2 + 6b_2b^2 + 2b_1^2b^2 + 8b_2^2b + 240b_2b^3 &= 0 \end{aligned}$$

$$\begin{aligned} 8b_1b_2b^2 + 24b_1b^4 &= 0 \\ 6b_2^2b^2 + 120b_2b^4 &= 0 \end{aligned}$$

These equation leads to the following cases:

Case(1)

$$\begin{aligned} a_0 &= \frac{11}{4} \mp \frac{5}{12}\sqrt{57}, a_1 = 0, a_2 = -20, \lambda = -\frac{13}{16} \mp \frac{5}{48}\sqrt{57}, \\ b_1 &= 0, b_2 = 0, b = -\frac{5}{32} \pm \frac{1}{32}\sqrt{57}. \end{aligned} \tag{23}$$

From this case we can choose the values

$$\begin{aligned} a_0 &= \frac{11}{4} - \frac{5}{12}\sqrt{57}, a_1 = 0, a_2 = -20, \lambda = -\frac{13}{16} - \frac{5}{48}\sqrt{57}, \\ b_1 &= 0, b_2 = 0, b = -\frac{5}{32} + \frac{1}{32}\sqrt{57}. \end{aligned}$$

Substituting (19) into (23) and using (3), in this case $(b > 0)$, the solution of Eq. (19) is given by,

$$u(x,t) = \frac{11}{4} - \frac{5}{12}\sqrt{57} - \frac{5}{16}(-10 + 2\sqrt{57})\tan\left(\frac{1}{8}\sqrt{-10 + 2\sqrt{57}}\left[x + \left(-\frac{13}{16} - \frac{5}{48}\sqrt{57}\right)t\right]\right).$$

Case(2)

$$a_0 = 2, a_1 = 0, a_2 = -20, \lambda = -\frac{39}{25}, b_1 = 0, b_2 = 0, b = -\frac{1}{10}. \tag{24}$$

Substituting (24) into (22) and using (3), in this case $(b < 0)$, the solution for Eq. (19) is given by,

$$u(x,t) = 2 - 2\tanh^2\left(\frac{\sqrt{10}}{10}\left[x - \frac{39}{25}t\right]\right)$$

Case (3)

$$\begin{aligned} a_0 &= \frac{19}{16} \mp \frac{5}{48}\sqrt{57}, a_1 = 0, a_2 = -20, \lambda = -\frac{13}{16} \mp \frac{5}{48}\sqrt{57} \\ b_1 &= 0, b_2 = -\frac{5}{128} \pm \frac{1}{128}\sqrt{57}, b = -\frac{5}{128} \pm \frac{1}{128}\sqrt{57} \end{aligned}$$

From this case we can choose the values

$$\begin{aligned} a_0 &= \frac{19}{16}+\frac{5}{48}\sqrt{57}, a_1=0, a_2=-20, \lambda=-\frac{13}{16}+\frac{5}{48}\sqrt{57} \\ b_1 &= 0, b_2=-\frac{5}{128}-\frac{1}{128}\sqrt{57}, b=-\frac{5}{128}-\frac{1}{128}\sqrt{57} \end{aligned} \tag{25}$$

Substituting (25) into (22) and using (3), in this case $(b<0)$, the solution for Eq. (19) is given by,

$$\begin{aligned} u(x,t) &= \frac{19}{16}+\frac{5}{48}\sqrt{57}+\frac{1}{256}(10+2\sqrt{57}) \\ &\left[\begin{array}{c} -20\tanh^2\left(\frac{1}{16}\sqrt{10+2\sqrt{57}}\left(x+\left(-\frac{13}{16}+\frac{5}{48}\sqrt{57}\right)t\right)\right)+ \\ \frac{56636\ -\frac{5}{128}-\frac{\sqrt{57}}{128}}{(10+2\sqrt{57})^2}\coth^2\left(\frac{1}{16}\sqrt{10+2\sqrt{57}}\left(x+\left(-\frac{13}{16}+\frac{5}{48}\sqrt{57}\right)t\right)\right) \end{array}\right] \end{aligned}$$

3.4 The variant Boussinesq equations

Finally, we consider a very important example as an illustration of the modified extended tanh-method for solving the variant Boussinesq equations [32-34], we will consider the following system of equations

$$u_t+v_x+u\,u_x = 0, \tag{26}$$

$$v_t+(uv)_x+u_{xxx} = 0. \tag{27}$$

To solve the system of Eqs. (26,27) by means of the modified extended tanh-function method, we use the wave transformation $u(x,t)=U(\xi)$ and $v(x,t)=V(\xi)$ with wave variable $\xi=x+y+\lambda t$. Therefore, system (26,27) is reduced to the ordinary differential equations in the form

$$\begin{aligned} \lambda U'+V'+\frac{1}{2}(U^2)' &= 0, \\ \lambda V'+(UV)'+U''' &= 0, \end{aligned}$$

Integrating both equations once leads to:

$$C_1-\lambda U-\frac{1}{2}U^2 = V, \tag{28}$$

$$\lambda V+UV+U'' = C_2, \tag{29}$$

where C_1 and C_2 are integrating constants, so as to we find the special forms of the exact solutions, for simplicity purpose, we take $C_1=C_2=0$. Substituting (28) into (29) gives

$$U''-\frac{1}{2}U^3-\frac{3\lambda}{2}U^2-\lambda^2U=0, \tag{30}$$

By balancing U'' with U^3 in Eq. (30), we find $M=1$. So the solutions take the form

$$U(\xi) = a_0+a_1\phi(\xi)+b_1\phi(\xi)^{-1} \tag{31}$$

$$V(\xi) = -\lambda[a_0+a_1\phi(\xi)+b_1\phi(\xi)^{-1}]-\frac{1}{2}[a_0+a_1\phi(\xi)+b_1\phi(\xi)^{-1}]^2 \tag{32}$$

Inserting Eqs. (31,32) into Eq. (29), making use of Eq. (3), and by using Maple Package, we get a system of algebraic equations, for a_0, a_1, b_1, λ and b in the form

$$\begin{aligned} -\frac{1}{2}a_1^3+2a_1 &= 0 \\ -\frac{3}{2}a_0a_1^2-\frac{3}{2}a_1^2 &= 0 \end{aligned}$$

$$\begin{aligned}
-\lambda^2 a_1 - \frac{3}{2}a_0^2 a_1 - \frac{3}{2}a_1^2 b_1 - 3a_0 a_1 + 2a_1 b &= 0\\
-\lambda^2 a_0 - \frac{1}{2}a_0^3 - 3a_0 a_1 b_1 - \frac{3}{2}a_0^2 - 3a_1 b_1 &= 0\\
-\lambda^2 b_1 - \frac{3}{2}a_0^2 b_1 - \frac{3}{2}a_1 b_1^2 - 3a_0 b_1 + 2b_1 b &= 0
\end{aligned}$$

$$\begin{aligned}
-\frac{3}{2}a_0 b_1^2 - \frac{3}{2}b_1^2 &= 0\\
2ab_1 b_2 - 2eb_2 b + 2cb_1 b^2 &= 0\\
-\frac{1}{2}b_1^3 + 2b_1 b^2 &= 0
\end{aligned}$$

We can select three cases of solutions as:

Case (1) :

$$a_0 = -1, a_1 = 0, b_1 = \frac{-1}{2}, b = \frac{-1}{4}, \lambda = 1 \tag{33}$$

Substituting (33) into (31,32) and using (3), in this case ($b < 0$), the solutions for Eqs. (26,27) are given by

$$\begin{aligned}
u(x,t) &= -1 + \coth(\frac{x+t}{2}),\\
v(x,t) &= 1 - \coth(\frac{x+t}{2}) - \frac{1}{2}[-1 + \coth(\frac{x+t}{2})]^2.
\end{aligned}$$

Case (2):

$$a_0 = -1, a_1 = 2, b_1 = \frac{1}{8}, b = \frac{-1}{16}, \lambda = 1. \tag{34}$$

Substituting (34) into (31,32) and using (3), in this case ($b < 0$), the solutions for Eqs. (26,27) are given by

$$\begin{aligned}
u(x,t) &= -1 - \frac{1}{2}\left[\tanh(\frac{x+t}{4}) + \coth(\frac{x+t}{4})\right],\\
v(x,t) &= 1 + \frac{1}{2}\left[\tanh(\frac{x+t}{4}) + \coth(\frac{x+t}{4})\right] -\\
&\quad \frac{1}{2}\left[-1 - \frac{1}{2}[\tanh(\frac{x+t}{4}) + \coth(\frac{x+t}{4})]\right]^2.
\end{aligned}$$

Case (3) $b > 0$:

$$a_0 = -1, a_1 = 2, b_1 = \frac{1}{4}, b = \frac{1}{8}, \lambda = 1. \tag{35}$$

Substituting (35) into (31,32) and using (3), in this case ($b < 0$), the solutions for Eqs. (26,27) are given by

$$\begin{aligned}
u(x,t) &= -1 - \frac{\sqrt{2}}{2}\left[\tan(\frac{\sqrt{2}}{4}(x+t)) + \cot(\frac{\sqrt{2}}{4}(x+t))\right],\\
v(x,t) &= -1 - \frac{\sqrt{2}}{2}\left[\tan(\frac{\sqrt{2}}{4}(x+t)) + \cot(\frac{\sqrt{2}}{4}(x+t))\right]\\
&\quad -\frac{1}{2}\left[-1 - \frac{\sqrt{2}}{2}[\tan(\frac{\sqrt{2}}{4}(x+t)) + \cot(\frac{\sqrt{2}}{4}(x+t))]\right]^2
\end{aligned}$$

4 Conclusions

In this paper, we have applied the modified extended tanh-function (METF) method for constructing exact travelling wave solutions for the four kinds of non-linear partial differential equations. The validity of the method has been tested by applying it successfully to the modified Korteweg-de Vries, two-dimensional Korteweg-de Vries, variant water wave and the variant Boussinesq equations. It was proved that the modified extended tanh-function method is a powerful mathematical technique for investigating and finding the exact solutions for the partial differential equations. As the method is usually tedious to use by hand, we have used the Maple package to solve the algebraic system. As a result we got the exact solutions.

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