

Nonsymmetrical Kink Solution of Generalized KdV Equation with Variable Coefficients

Shuimeng Yu, Lixin Tian *

Nonlinear Science Research Center, Faculty of Science, Jiangsu University
Zhenjiang, Jiangsu, 212013, P.R.China

(Received 6 November 2007, accepted 8 January 2008)

Abstract: Using the auxiliary equation method, we consider the generalized KdV equation with variable coefficients and present some new soliton-like solutions. These solutions include nonsymmetrical kink solutions, compacton solutions, solitary pattern solutions, triangular function solution, Jacobi and Weierstrass elliptic function solutions. Furthermore, it provides a guideline to classify the various types of the traveling wave solutions according to the values of some parameters.

Key words: nonsymmetrical kink solution; generalized KdV equation; variable coefficient

1 Introduction

Nonlinear wave phenomena exist in many fields, such as fluid mechanics, plasma physics, biology, hydrodynamics, solid state physics and optical fibers, etc. In order to better understand these nonlinear phenomena, it is important to seek their exact solutions. They can help to analyze the stability of these solutions and the movement role of the wave by making the graphs of the exact solutions. In the past decades, many effective methods such as the inverse scattering method[1], Darboux transformation[2], the Hirota bilinear method[3], the homogeneous balance method[4] and the Tanh method[5] have been developed. However, to our knowledge, most of aforementioned methods are related to the constant-coefficient models. Recently, the study of the variable-coefficient nonlinear equations has attracted much attention[6] because most of real nonlinear physical equations possess variable coefficients.

The research on KdV equation attracted the interest of many scientists. Khare and Sukhatme[7] found that the KdV equation possess periodic traveling wave solutions involving Jacobi elliptic functions. The multi-compacton solutions and its stability were studied by Tian and Yin[8]. Zhao, Tang and Wang[9] obtained soliton-like solutions for KdV equation with variable coefficient by using Jacobi elliptic function expansion method. In this paper, a new auxiliary equation method with symbolic computation will be used to obtain the various exact solutions of generalized KdV equation with variable coefficients[10].

$$u_t + 2\beta(t)u + [\alpha(t) + \beta(t)x]u_x - 3A\gamma(t)uu_x + \gamma(t)u_{xxx} = 0 \quad (1)$$

The rest of this paper is organized as follows: In Section 2, the auxiliary equation method is described in detail. In Section 3, this method is applied to generalized KdV equation with variable coefficients for constructing new exact solutions including nonsymmetrical kink solution, compacton solutions, rational solutions, triangular, Jacobi and Weierstrass elliptic function solutions. A conclusion is given in Section 4.

*Corresponding author. E-mail address: tianlx@ujs.edu.cn

2 The introduction of the auxiliary equation method

The following is a given nonlinear differential equation with two variables x and t

$$F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0 \quad (2)$$

where F is a polynomial function with respect to the indicated variables or some functions which can be reduced to a polynomial function by using some transformations.

Step 1: Assume that Eq.(2) has the following formal solution:

$$u(\xi) = \sum_{i=0}^n f_i(t)\varphi^i(\xi) \quad (3)$$

with the variable φ satisfying

$$\left(\frac{d\varphi}{d\xi}\right)^2 = c_0 + c_1\varphi(\xi) + c_2\varphi^2(\xi) + c_3\varphi^3(\xi) + c_4\varphi^4(\xi) \quad (4)$$

$$\xi(x, t) = p(t)x + q(t) \quad (5)$$

where $c_i (i = 0, 1, 2, 3, 4)$ are constants.

Step 2: Balancing the highest order derivative term and the highest order nonlinear term of Eq.(2) with homogeneous balance method, the parameters n in (3) can be determined.

Step 3: Substituting (3), (4) and (5) into Eq.(2) and collecting coefficients of $\varphi^k \varphi^{(l)}$, then setting coefficients equal zero, we will obtain a set of over-determined equations for $f_i(t)$, $p(t)$, $q(t)$ and c_i . By solving the system, we may determine these parameters.

Step 4: Substituting $f_i(t)$, $p(t)$, $q(t)$, c_i and $\varphi(\xi)$ obtained in Step 3 into Eq. (3), we can derive the solutions of Eq.(2).

3 Exact solutions of generalized KdV equation with variable coefficients

In this section, we apply the method introduced in section 2 to generalized KdV equation with variable coefficients Eq.(1). The importance of Eq.(1) is well known. When $\gamma(t) = -K_0(t)$, $A = -2$, $\beta(t) = h(t)$, $\alpha(t) = -4K_1(t)$, Eq.(1) can be degenerated to the nonisospectral KdV equation with variable coefficients studied in [11]:

$$u_t = K_0(t)(u_{xxx} + 6uu_x) + 4K_1(t)u_x - h(t)(2u + xu_x)$$

When $\gamma(t) = 1$, $A = -2$, $\alpha(t) = c_0$, Eq. (1) can be degenerated to the KdV equation with variable coefficients discussed in [12]:

$$u_{xxx} + 6uu_x + [(c_0 + \beta(t)x)u]_x + \beta(t)u + u_t = 0$$

According to the above method in section 2, we consider homogeneous balance between uu_x and u_{xxx} in Eq.(1), which give $n = 2$. We suppose the solution of Eq.(1) is of the form

$$u = f_0(t) + f_1(t)\varphi(\xi) + f_2(t)\varphi^2(t) \quad (6)$$

$$\xi(x, t) = p(t)x + q(t) \quad (7)$$

$$\left(\frac{d\varphi}{d\xi}\right)^2 = c_0 + c_1\varphi(\xi) + c_2\varphi^2(\xi) + c_3\varphi^3(\xi) + c_4\varphi^4(\xi) \quad (8)$$

Substituting (6), (7) and (8) into Eq.(1) and collecting coefficients of $\varphi^k \varphi^{(l)}$, $k = 0, 1, 2, 3$, $l = 0, 1$, and letting each coefficient equals zero, we obtain a set of over-determined equations:

$$\begin{cases} 2\beta f_0 + f'_0 = 0 \\ 2\beta f_1 + f'_1 = 0 \\ 2\beta f_2 + f'_2 = 0 \\ 4c_4 f_2 p^3 \gamma - A f_2^2 p \gamma = 0 \\ f_1 p [\alpha(t) + \beta(t)x] - 3A f_0 f_1 p \gamma + f_1 [p'(t)x + q'(t)] \\ + c_2 f_1 p^3 \gamma + 3c_1 f_2 p^3 \gamma = 0 \\ 2f_2 p [\alpha(t) + \beta(t)x] - 3A f_1^2 p \gamma + 2f_2 [p'(t)x + q'(t)] \\ - 6A f_0 f_2 p \gamma + 3c_3 f_1 p^3 \gamma + 8c_2 f_2 p^3 \gamma = 0 \\ 2c_4 f_1 p^3 \gamma + 5c_3 f_2 p^3 \gamma - 3A f_1 f_2 p \gamma = 0 \end{cases}$$

With mathematic software MATHEMATICA, we obtain the parameters of $f_i(t)$, $p(t)$, $q(t)$ and c_i via symbolic computation[13].

Case 1

a) When $c_3 = c_1 = 0$, $c_0 = 0$, Eq.(8) possesses a bell-shaped solitary wave solution

$$\varphi_1 = \sqrt{-\frac{c_2}{c_4}} \operatorname{sech}(\sqrt{c_2} \xi) \quad c_2 > 0 \quad c_4 < 0 \tag{9}$$

a triangular solution

$$\varphi_2 = \sqrt{-\frac{c_2}{c_4}} \operatorname{sec}(\sqrt{-c_2} \xi) \quad c_2 < 0 \quad c_4 > 0 \tag{10}$$

and a rational solution

$$\varphi_3 = -\frac{1}{\sqrt{c_4} \xi} \quad c_2 = 0 \quad c_4 > 0 \tag{11}$$

By solving the above over-determined equations under the condition $c_3 = c_1 = c_0 = 0$, the parameters $f_i(t)$, $p(t)$, $q(t)$ and c_i can be derived.

$$f_0 = k_1 e^{-2 \int \beta dt} \quad f_1 = k_2 e^{-2 \int \beta dt} \quad f_2 = k_3 e^{-2 \int \beta dt} \tag{12a}$$

$$p = k_4 e^{-\int \beta dt} \quad c_4 = \frac{A k_3}{4 k_4^2} \quad c_2 = \frac{A k_2^2}{4 k_3 k_4^2} \tag{12b}$$

$$q = \int (3A \gamma k_1 k_4 e^{-3 \int \beta dt} - \alpha k_4 e^{-\int \beta dt} - 4 \gamma k_4^3 c e^{-3 \int \beta dt}) dt \tag{12c}$$

From (6), (7) and (9)-(12), we obtain the following solutions of Eq.(1).

$$u_1 = k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\sqrt{-\frac{c_2}{c_4}} \operatorname{sech}(\sqrt{c_2} \xi) \right] + k_3 e^{-2 \int \beta dt} \left[\sqrt{-\frac{c_2}{c_4}} \operatorname{sech}(\sqrt{c_2} \xi) \right]^2 \tag{13}$$

The shape of solution (13) is shown in Fig.1, with $\beta(t) = 0$, $c_2 = -c_4 = 1$.

$$u_2 = k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\sqrt{-\frac{c_2}{c_4}} \operatorname{sec}(\sqrt{-c_2} \xi) \right] + k_3 e^{-2 \int \beta dt} \left[\sqrt{-\frac{c_2}{c_4}} \operatorname{sec}(\sqrt{-c_2} \xi) \right]^2 \tag{14}$$

The shape of solution (14) is shown in Fig. 2, with $\beta(t) = 0$, $c_2 = -c_4 = -1$.

$$u_3 = k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[-\frac{1}{\sqrt{c_4} \xi} \right] + k_3 e^{-2 \int \beta dt} \left[-\frac{1}{\sqrt{c_4} \xi} \right]^2$$

b)When $c_3 = c_1 = 0$, $c_0 = \frac{c_2^2}{4c_4}$, Eq.(8) possesses kink-shaped solitary wave solution

$$\varphi_4 = \sqrt{-\frac{c_2}{2c_4}} \tanh \left(\sqrt{\frac{-c_2}{2}} \xi \right) \quad c_2 < 0 \quad c_4 > 0 \tag{15}$$

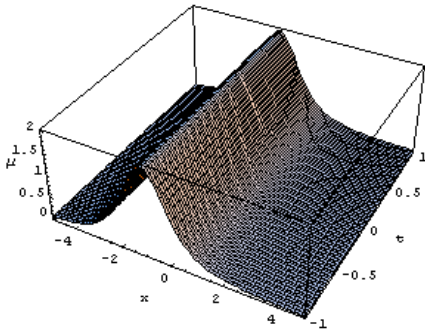


Figure 1: solitary-pattern solution.

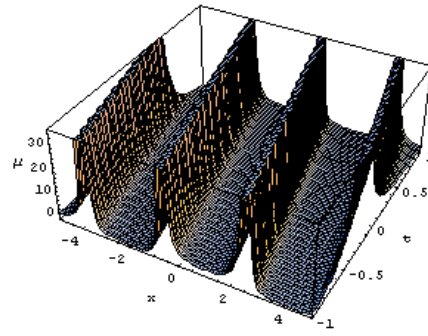


Figure 2: periodic wave solution.

The shape of solution (15) is shown in Fig. 3, with $\beta(t) = 0$, $c_2 = -2$. Eq.(8) also possesses a triangular solution

$$\varphi_5 = \sqrt{\frac{c_2}{c_4}} \tan\left(\sqrt{\frac{c_2}{2}} \xi\right) \quad c_2 > 0 \quad c_4 > 0 \quad (16)$$

We obtain the following solutions of Eq.(1) from (6), (7), (12) and (15), (16)

$$u_4 = k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\sqrt{\frac{-c_2}{2c_4}} \tanh\left(\sqrt{\frac{-c_2}{2}} \xi\right) \right] + k_3 e^{-2 \int \beta dt} \left[\sqrt{\frac{-c_2}{2c_4}} \tanh\left(\sqrt{\frac{-c_2}{2}} \xi\right) \right]^2 \quad (17)$$

This solution is a new type of nonsymmetrical kink solution, The shape of solution (17) is shown in Fig. 4, with $\beta(t) = 0$, $c_2 = -c_4 = -2$. Compared with kink solution (Fig.3), we can observe the difference between Fig.3 and Fig.4. The solution in Fig.4 is nonsymmetrical kink solution.

$$u_5 = k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\sqrt{\frac{c_2}{c_4}} \tan\left(\sqrt{\frac{c_2}{2}} \xi\right) \right] + k_3 e^{-2 \int \beta dt} \left[\sqrt{\frac{c_2}{c_4}} \tan\left(\sqrt{\frac{c_2}{2}} \xi\right) \right]^2$$

where $\xi = k_4 e^{-\int \beta dt} x + \int (3A\gamma k_1 k_4 e^{-3 \int \beta dt} - \alpha k_4 e^{-\int \beta dt} - 4\gamma k_4^3 c_2 e^{-3 \int \beta dt}) dt$, k_1 and k_2 are arbitrary constants, $k_3 \neq 0$, $k_4 \neq 0$.

c) When $c_3 = c_1 = 0$, Eq.(8) admits three Jacobi elliptic function solutions

$$\varphi_6 = \sqrt{\frac{-c_2 m^2}{c_4(2m^2 - 1)}} \operatorname{cn}\left(\sqrt{\frac{c_2}{2m^2 - 1}} \xi\right) \quad c_2 > 0 \quad c_0 = \frac{c_2^2 m^2 (m^2 - 1)}{c_4(2m^2 - 1)^2} \quad (18)$$

$$\varphi_7 = \sqrt{\frac{-c_2}{c_4(2 - m^2)}} \operatorname{dn}\left(\sqrt{\frac{c_2}{2 - m^2}} \xi\right) \quad c_2 > 0 \quad c_0 = \frac{c_2^2 (1 - m^2)}{c_4(2 - m^2)^2} \quad (19)$$

and

$$\varphi_8 = \sqrt{\frac{-c_2 m^2}{c_4(m^2 + 1)}} \operatorname{sn}\left(\sqrt{\frac{-c_2}{m^2 + 1}} \xi\right) \quad c_2 < 0 \quad c_0 = \frac{c_2^2 m^2}{c_4(m^2 + 1)^2} \quad (20)$$

So the solutions of Eq.(1) are shown as follows:

$$u_6 = k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\sqrt{\frac{-c_2 m^2}{c_4(2m^2 - 1)}} \operatorname{cn}\left(\sqrt{\frac{c_2}{2m^2 - 1}} \xi\right) \right] + k_3 e^{-2 \int \beta dt} \left[\sqrt{\frac{-c_2 m^2}{c_4(2m^2 - 1)}} \operatorname{cn}\left(\sqrt{\frac{c_2}{2m^2 - 1}} \xi\right) \right]^2$$

$$\begin{aligned}
 u_7 &= k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\sqrt{\frac{-c_2}{c_4(2-m^2)}} dn \left(\sqrt{\frac{c_2}{2-m^2}} \xi \right) \right] \\
 &\quad + k_3 e^{-2 \int \beta dt} \left[\sqrt{\frac{-c_2}{c_4(2-m^2)}} dn \left(\sqrt{\frac{c_2}{2-m^2}} \xi \right) \right]^2 \\
 u_8 &= k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\sqrt{\frac{-c_2 m^2}{c_4(m^2+1)}} sn \left(\sqrt{\frac{-c_2}{m^2+1}} \xi \right) \right] \\
 &\quad + k_3 e^{-2 \int \beta dt} \left[\sqrt{\frac{-c_2 m^2}{c_4(m^2+1)}} sn \left(\sqrt{\frac{-c_2}{m^2+1}} \xi \right) \right]^2
 \end{aligned}$$

where $\xi = k_4 e^{-\int \beta dt} x + \int (3A\gamma k_1 k_4 e^{-3 \int \beta dt} - \alpha k_4 e^{-\int \beta dt} - 4\gamma k_4^3 c_2 e^{-3 \int \beta dt}) dt$, k_1 and k_2 are arbitrary constants, $k_4 \neq 0$.

Remark Using the transformations $c_0 = \frac{c_2^2 m^2}{c_4(m^2+1)^2}$, $\bar{\varphi} = \sqrt{\frac{c_4(m^2+1)}{-c_2 m^2}} \varphi$, $\bar{\xi} = \sqrt{\frac{-c_2}{m^2+1}} \xi$. Eq.(8) is reduced to $\varphi' = \sqrt{1 - (m^2 + 1)\varphi^2 + m^2\varphi^4}$, where m is a modulus of Jacobi elliptic functions. We can obtain the solution (20). Then, by using the relations among Jacobi elliptic functions $sn \xi$, $cn \xi$ and $dn \xi$, we also get solutions (18) and (19). As $m \rightarrow 1$ the Jacobi elliptic solutions degenerate to the soliton solutions.

Case 2 When $c_4 = c_1 = c_0 = 0$, Eq.(8) possesses a bell-shaped solitary wave solution

$$\varphi_9 = -\frac{c_2}{c_3} \sec h^2 \left(\frac{\sqrt{c_2}}{2} \xi \right) \quad c_2 > 0$$

and a triangular solution

$$\varphi_{10} = -\frac{c_2}{c_3} \sec^2 \left(\frac{\sqrt{-c_2}}{2} \xi \right) \quad c_2 < 0$$

and a rational solution

$$\varphi_{11} = \frac{1}{c_3 \xi^2} \quad c_2 = 0$$

By Solving relevant over-determined equations under the condition $c_4 = c_1 = c_0 = 0$, the parameters $f_i(t)$, $p(t)$, $q(t)$ and c_i can be derived, and we obtain the following solutions for Eq.(1).

$$\begin{aligned}
 u_9 &= k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[-\frac{c_2}{c_3} \sec h^2 \left(\frac{\sqrt{c_2}}{2} \xi \right) \right] \\
 u_{10} &= k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[-\frac{c_2}{c_3} \sec^2 \left(\frac{\sqrt{-c_2}}{2} \xi \right) \right] \\
 u_{11} &= k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\frac{1}{c_3 \xi^2} \right]
 \end{aligned}$$

where $\xi = k_4 e^{-\int \beta dt} x + \int (3A\gamma k_1 k_4 e^{-3 \int \beta dt} - \alpha k_4 e^{-\int \beta dt} - 4\gamma k_4^3 c_2 e^{-3 \int \beta dt}) dt$, k_1 and k_2 are arbitrary constants, $k_4 \neq 0$.

Case 3 When $c_4 = c_2 = 0$, $c_3 > 0$, Eq.(8) possesses a Weierstrass elliptic function solution

$$\varphi_{12} = \wp \left(\frac{\sqrt{c_3}}{2} \xi, g_2, g_3 \right)$$

where $g_2 = -4c_1/c_3$ and $g_3 = -4c_0/c_3$ are called invariants of Weierstrass elliptic function. The solutions of Eq.(1) are shown as follows:

$$u_{12} = k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\wp \left(\frac{\sqrt{c_3}}{2} \xi, g_2, g_3 \right) \right]$$

where $\xi = k_4 e^{-\int \beta dt} x + \int (3A\gamma k_1 k_4 e^{-3 \int \beta dt} - \alpha k_4 e^{-\int \beta dt}) dt$, k_1 and k_4 are arbitrary constants, $k_4 \neq 0$.

Remark In the case when $c_4 = c_2 = 0$, $c_3 > 0$, by using the transformations

$$\bar{\xi} = \frac{\sqrt{c_3}}{2}\xi \quad c_0 = -\frac{1}{4}c_3g_3 \quad c_1 = -\frac{1}{4}c_3g_2$$

Eq.(8) is reduced to $\varphi' = \sqrt{-g_3 - g_2\varphi + 4\varphi^3}$, which the Weierstrass elliptic function satisfies the equation.

Case 4 When $c_4 = c_3 = 0$, Eq.(8) possesses a compacton solution

$$\varphi_{13} = \sqrt{\frac{c_1^2 - 4c_0c_2}{4c_2^2}} \sin(\sqrt{-c_2}\xi + \xi_0) - \frac{c_1}{2c_2} \quad c_2 < 0 \quad (21)$$

and the hyperbolic function solutions

$$\varphi_{14} = \sqrt{\frac{4c_0c_2 - c_1^2}{4c_2^2}} \sinh(\sqrt{c_2}\xi) - \frac{c_1}{2c_2} \quad c_2 > 0 \quad (22)$$

$$\varphi_{15} = \sqrt{\frac{c_1^2 - 4c_0c_2}{4c_2^2}} \cosh(\sqrt{c_2}\xi) - \frac{c_1}{2c_2} \quad c_2 > 0 \quad (23)$$

Using the same method, we substitute (21), (22) and (23) into (6), and obtain the solutions of Eq.(1) under the condition $c_4 = c_3 = 0$

$$u_{13} = k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\sqrt{\frac{c_1^2 - 4c_0c_2}{4c_2^2}} \sin(\sqrt{-c_2}\xi + \xi_0) - \frac{c_1}{2c_2} \right]$$

$$u_{14} = k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\sqrt{\frac{4c_0c_2 - c_1^2}{4c_2^2}} \sinh(\sqrt{c_2}\xi) - \frac{c_1}{2c_2} \right]$$

$$u_{15} = k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\sqrt{\frac{c_1^2 - 4c_0c_2}{4c_2^2}} \cosh(\sqrt{c_2}\xi) - \frac{c_1}{2c_2} \right]$$

where $\xi = k_4 e^{-\int \beta dt} x + \int (3A\gamma k_1 k_4 e^{-3 \int \beta dt} - \alpha k_4 e^{-\int \beta dt} - 4\gamma k_4^3 c_2 e^{-3 \int \beta dt}) dt$, k_1 and k_4 are arbitrary constants, $k_4 \neq 0$.

Case 5 When $c_0 = c_1 = 0$, Eq.(8) possesses solitary wave solutions

$$\varphi_{16} = \frac{c_2 c_3 \sec h^2\left(\frac{\sqrt{c_2}}{2}\xi\right)}{c_3^2 - c_2 c_4 \left[1 - \tanh\left(\frac{\sqrt{c_2}}{2}\xi\right)\right]^2}$$

$$\varphi_{17} = \frac{-8C c_2 c_3 \sec h^2(\sqrt{c_2}\xi)}{4c_3^2 C \sec h^2(\sqrt{c_2}\xi) + [4c_3^2 + C^2 (4c_2 c_4 - c_3^2)] \tanh(\sqrt{c_2}\xi) + 4c_3^2 - C^2 (4c_2 c_4 - c_3^2)}$$

where C is a constant.

The solutions of Eq.(1) under the condition $c_0 = c_1 = 0$ are as follows:

$$u_{16} = k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} \left[\frac{c_2 c_3 \sec h^2\left(\frac{\sqrt{c_2}}{2}\xi\right)}{c_3^2 - c_2 c_4 \left[1 - \tanh\left(\frac{\sqrt{c_2}}{2}\xi\right)\right]^2} \right] + k_3 e^{-2 \int \beta dt} \left[\frac{c_2 c_3 \sec h^2\left(\frac{\sqrt{c_2}}{2}\xi\right)}{c_3^2 - c_2 c_4 \left[1 - \tanh\left(\frac{\sqrt{c_2}}{2}\xi\right)\right]^2} \right]^2$$

where $\xi = k_4 e^{-\int \beta dt} x + \int \frac{1}{4k_3} (12A\gamma k_1 k_4 e^{-3 \int \beta dt} - A\gamma k_2^2 k_4 e^{-3 \int \beta dt} - 4\alpha k_3 k_4 e^{-3 \int \beta dt}) dt$, k_1 and k_2 are arbitrary constants, $k_3 \neq 0$, $k_4 \neq 0$.

$$u_{17} = k_1 e^{-2 \int \beta dt} + k_2 e^{-2 \int \beta dt} [\varphi] + k_3 e^{-2 \int \beta dt} [\varphi]^2 \quad (24)$$

where

$$\varphi = \frac{-8C c_2 c_3 \sec h^2(\sqrt{c_2}\xi)}{4c_3^2 C \sec h^2(\sqrt{c_2}\xi) + [4c_3^2 + C^2 (4c_2 c_4 - c_3^2)] \tanh(\sqrt{c_2}\xi) + 4c_3^2 - C^2 (4c_2 c_4 - c_3^2)}$$

When $\beta(t) = 0$, $c_2 = -c_4 = -2$, $k_1 = 0$, $k_2 = k_3 = 1$, (24) is transformed as follows

$$u_{17'} = \frac{-2 \sec h^2(x+t)}{\sec h(x+t)^2 + \tanh(x+t)} + \left(\frac{-2 \sec h^2(x+t)}{\sec h(x+t)^2 + \tanh(x+t)} \right)^2$$

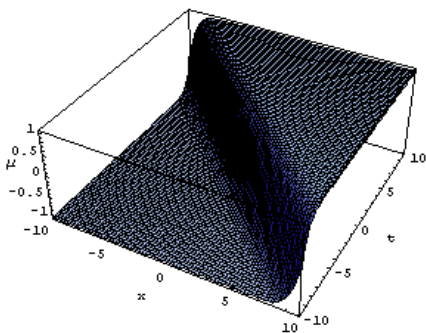


Figure 3: kink solution.

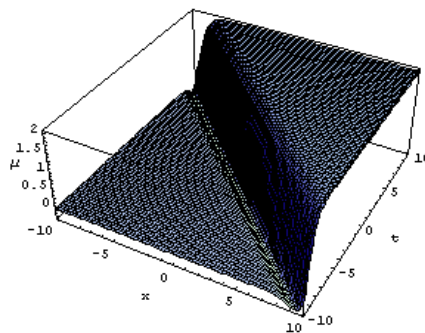


Figure 4: non-symmetric kink solution.

4 Conclusions

The auxiliary equation method is powerful to solve the exact solutions of the Eq.(1). When $c_1 = c_3 = 0$, $c_0 = 1$, $c_2 = -2$, $c_4 = 1$, Eq.(8) reduces to the Tanh method. When $c_1 = c_3 = 0$, $c_0 = b^2$, $c_2 = 2b$, $c_4 = 1$, Eq.(8) degenerates to the Riccati equation method[14] When φ is replaced by Jacobi elliptic function, Eq.(8) degenerates to Jacobi elliptic function method. On the other hand, the auxiliary equation method not only constructs various types of solutions, such as solitary pattern solutions, solitary wave solutions, triangular function solution, Jacobi and Weierstrass elliptic function solutions, but also provides a guideline to classify these solutions according to the values of some parameters. This method is applicable to a large number of nonlinear equations with variable coefficients.

Acknowledgements

Research was supported by the National Natural Science Foundation of China(No: 10771088) and the Nature Science Foundation of Jiangsu (No:2007098) and Outstanding Personnel Program in Six Fields of Jiangsu Province (No: 6-A-029).

References

- [1] Ju Lin: On Solution of the Dullin-Gottwald-Holm Equation. *International Journal of Nonlinear Science*. 1(1), 43-48 (2006)
- [2] Matveev V.B., Salle M.A.: Darboux transformations and solitons. *Springer, Berlin* (1991)
- [3] Hirota R, Satsuma J: Soliton solutions for a coupled KdV equation. *Phys. Lett. A.* (85), 407-408 (1981)
- [4] M. L. Wang, Y. B. Zhou, Z. B. Li : Application of a homogeneous balance method to exact solution of nonlinear equations in mathematical physics. *Phys. Lett. A.*216, 67-75 (1996)

- [5] Malfliet W.: Solitary wave solutions of nonlinear wave equation. *Am. J. Phys.* 60(7), 650-654 (1992)
- [6] Dianchen Lu, Baojian Hong, Lixin Tian: Backlund Transformation and N-soliton-like Solutions to the Combined KdV-Burgers Equation with Variable Coefficients . *International Journal of Nonlinear Science.* 2(1), 3-10 (2006)
- [7] A. Khare, U. Sukhatme: Linear superposition in nonlinear equations. *Phys. Rev. Lett.* 88, 244101–244104 (2002)
- [8] Lixin Tian, Jiuli Yin: Stability of multi-compacton solutions and Backlund transformation in $K(m,n,l)$. *Chaos, Solitons & Fractals.* 23(1), 159–169 (2005)
- [9] Xiqiang Zhao, Dengbin Tang, Limin Wang: New soliton-like solutions for KdV equation with variable coefficient. *Physics Letters A.* 346(4), 288-291 (2005)
- [10] W. L. Chan, X. Zhang: *J. Phys. A: Math. Gen.* 27, 407-437 (1994)
- [11] Chou Tian: Symmetries and a hierarchy of the general KdV equation. *J. Phys. A: Math. Gen.* 20, 359-366 (1987)
- [12] Brugarino T., Pantano P.: The integration of Burgers and Korteweg-de Vries equations with nonuniformities. *Phys. Lett. A.* 80, 223-224 (1980)
- [13] Yuchun Wang, Lixia Wang, Wenbin Zhang: Application of the adomian decomposition method to fully nonlinear sine-Gordon equation. *International Journal of Nonlinear Science.* 2(1), 29-38 (2006)
- [14] Dianchen Lu, Baojian Hong: General riccati equations method and its applications to the general types of combined KdV-Burgers. *International Journal of Nonlinear Science.* 4(3), 223-230 (2007)