ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.5(2008) No.1,pp.39-42

Computing Klein-Gordon Equation on Turing machine

Dianchen Lu * , Donghai Jiang
Nonlinear Scientific Research Center, Faculty of Science, Jiangsu University
Zhenjiang, Jiangsu, 212013, P.R. China
(Received 3 March 2007, accepted 28 December 2007)

Abstract: In this paper we study the computability of the solution operation of the initial-value problem for the Klein-Gordon equation.Firstly we give some basic definitions about TTE. Then we transform the partial differential equation into the integral equation by Fourier transform, and prove that the solution operation of the integral equation is Turing- computable. Therefore the solutions of the Klein-Gordon equation can be computed on the Turing machine.

Key words: Turing machine; Klein-Gordon equation; sobolev space; limit space; initial value problem

1 Introduction

Some physicists believe that the future behavior of processes described by well- established theories can be computed with arbitrary precision, at least in principle.Accordingly,from the given initial conditions, we can perform the computation on digital computers, hence on Turing machines[1].In [2],Pour-El/Richards gave cause for speculations that it might be design "wave computers" beating the Turing machine. Meanwhile, Klaus has shown that wave propagation is computable[3]. However, are other Partial differential equations(cf.[4-7]) computable? Could we design theoretic computer on the basis of other physical equations?At present,there are many work about these problems. In [8], Klaus Weihrauch and Ning Zhong proved that the solution of KdV equation is Turing computable. In this paper, we study the computability of the solution operation of the Klein-Gordon equation.

2 Preliminaries

In this section let us introduce some basic definitions, for details on TTE we refer the readers to [9]. Let $\sum$ be a sufficiently large finite alphabet, $\sum^*$ the set of finite words over $\sum$ with the discrete topology, and $\sum^\omega$ the set of infinite words over $\sum$ with the Cantor topology. A notation (representation) of a set M is a surjective map $\delta :\subseteq \sum^* \to M(\delta :\subseteq \sum^\omega \to M)$. For any $x \in M$ and $p \in \sum^*(p \in \sum^\omega)$, p is called a $\delta-$ name(or a code) of x if $\delta(p) = x$. If δ and δ' are representations of M and M', respectively, then a function $\varphi :\subseteq \sum^\omega \to \sum^\omega$ is called a (δ, δ')-realization of $f :\subseteq M \to M'$, if and only if $f \circ \delta(p) = \delta' \circ \varphi(p)$ for $p \in dom(f \circ \delta)$. A function f is called (δ, δ')-continuous(-computable), if and only if it has a continuous(computable) (δ, δ')-realization. For two representations of M, $\delta_1 :\subseteq \sum^\omega \to M$ and $\delta_2 :\subseteq \sum^\omega \to M$, we say that δ_1 can be reduced to δ_2 if there is a function $\varphi :\subseteq \sum^\omega$ is called a (δ, δ')-realization of $f :\subseteq M \to M'$, if and only if $f \circ \delta(p) = \delta' \circ \varphi(p)$ for $p \in dom(f \circ \delta)$. A function f is called (δ, δ')-continuous(-computable), if and only if it has a continuous(computable) (δ, δ')-realization. For two representations of M, $\delta_1 :\subseteq \sum^\omega \to M$ and $\delta_2 :\subseteq \sum^\omega \to M$, we say that δ_1 can be reduced to δ_2 if there is a function $\varphi :\subseteq \sum^\omega \to \sum^\omega$ which translations δ_1-names to δ_2-names, that is, $\delta_1(p) = \delta_2 \circ \varphi(p)$ for all $p \in dom(\delta_1)$. If φ is continuous(computable), we write $\delta_1 \leq_t \delta_2 (\delta_1 \leq \delta_2)$. If, furthermore, $\delta_1 \leq_t \delta_2$

*Corresponding author. *E-mail address*: dclu@ujs.edu.cn

and $\delta_2 \leq_t \delta_1(\delta_1 \leq \delta_2$ and $\delta_2 \leq \delta_1)$, then we write $\delta_1 \equiv_t \delta_2(\delta_1 \equiv \delta_2)$. For the definition of computable string function(see also [8]).

Definition 2.1 (Limit Space) *A Limit space is a space $(X, \rightarrow)$ if it satisfies (L1),(L2) and (L3).*

$$(L1)(x, x, \cdots) \rightarrow x;$$
$$(L2) if (x_n)_n \rightarrow x, then (y_n)_n \rightarrow x for every subsequence (y_n)_n of (x_n)_n;$$
$$(L3)(x_n)_n \rightarrow x, if every subsequence of (x_n)_n has a subsequence converging to x.$$

Every topological space induces a limit space. The sequentially continuous function is defined naturally[9].

Definition 2.2 (Admissible Representation) *A representation $\delta :\subseteq \sum^\omega \rightarrow M$ of a limit space$(X, \rightarrow$ is called admissible, if δ is continuous and $f \leq_t \delta$ for every continuous function $f :\subseteq \sum^\omega \rightarrow M$.*

Lemma 2.3 *Let $(X, \ \rightarrow_X)$ and $(Y, \ \rightarrow_Y)$ be limit spaces with admissible representations δ and δ', respectively. Then $(C(X, Y), \ \rightarrow)$ such that $(f_n)_n \ \rightarrow f := (f_n(x_n))_n \ \rightarrow_Y f(x)$ if $(x_n)_n \ \rightarrow_X x$ is a limit space and $[\delta \ \rightarrow \delta']$ is an admissible representation of it.*

Lemma 2.4 *For admissible representations δ_i of limit spaces $(X_i, \ \delta_i)(i = 1, 2)$, a function $f :\subseteq X_1 \ \rightarrow X_2$ is continuous, if it is $(\delta, \ \delta')$- continuous.*

Lemma 2.5

$$(1) Evaluation (f, x \ \rightarrow f(x) is ([\delta \ \rightarrow \delta'], \ \delta, \ \delta') - continuous.$$
$$(2) Type Conversion Let (\delta_i, \ M_i)(0 \leq i \leq k) be representation spaces.$$
$$Let f :\subseteq X_1 \ \times \cdots X_k \rightarrow X_0 and define F(x_1, \cdots x_{k-1})(x_k) := f(x_1 \cdots x_k). Then, f is$$
$$(\delta_1, \cdots \delta_k, \ \delta_0) - computable iff F is (\delta_1, \cdots, \delta_{k-1}, [\delta_k \ \rightarrow \delta_0]) - computable.$$

Lemma 2.6 (Computable Metric Space[10]) *A computable metric space is a quadruple M=(M, d, D, υ_D) such that (M, d) is a metric space, D is a countable dense subset and $\upsilon_D :\subseteq (\sum \backslash \{\#\})^+ \rightarrow D$ is a notation of D such that $\{(u, v, w, x) | \upsilon_Q(w) < d(\upsilon_D(u), \ \upsilon_D(v)) < \upsilon_Q(x)\}$ is an r.e. set. The Cauchy representation $\delta :\subseteq \sum^\omega \rightarrow M$ is defined as follow: $\delta(p) = x :\Leftrightarrow p = \#u_0\#u_1\#\cdots$, such that $d(\upsilon_D t(u_i), \ x) \leq 2^{-i}, \ (i \in \mathrm{N})$. The Cauchy representation is admissible.*

Definition 2.7 ($L^p(R)$ **and** $H^s(R)$) *For any $0 \leq p < \infty$. the space $L^p(R)$ is the set of all measurable complex valued functions f such that $\int_\Re |f|^p dx < \infty$ with norm $\|f\|_{L^p} = \{\int_\Re |f|^p dx\}^{1/p}$. For any $s \in \Re$, the Sobolev space $H^s(R)$is the set of all functions$f \in L^2(R)$ such that $T_s(f) \in L^2(\Re)$, where $T_s(f)(\xi) := (1 + |\xi|^2)^{s/2} \cdot F(f)(\xi)$, ($F(f)$ is the Fourier transform of f) with norm $\|f\|_{H^s} = \|T_s(f)\|_{L^2}$.*

Lemma 2.8 *Let $(L^2(\mathrm{R}), d_{L^2}, \sigma, \upsilon_{L^2})$ be computable metric space, where σ is a set of all rational finite step functions $f = \sum_{i=0}^{k} c_i \Pi_{a_i b_i}$where $k \in \mathrm{N}$, c_i is a rational complex number, $a_i < b_i$ are rational numbers, and if $a_i < x < b_i \Pi_{a_i b_i} = 1$ else $\Pi_{a_i b_i} = 0$, and υ_{L^2} is a canonical notation of σ. The Cauchy representation δ_{L^2} is defined accordingly.*

Lemma 2.9

$$(1)(f, g) \rightarrow f + g is (\delta_{L^2}, \ \delta_{L^2}, \ \delta_{L^2}) - computable;$$
$$(2)(f, g, K) \rightarrow f \cdot g is ([\rho \ \rightarrow \rho^2]\delta_{L^2}, \rho, \delta_{L^2}) - computable, where f \in C(\Re, \mathrm{C}), g \in L^2(\Re) and \sup_{x \in \Re} |f(x)| < K;$$
$$(3)(a, b, f) \rightarrow \int_b^a f(\tau) d\tau is (\rho, \ \rho, \ [\rho \rightarrow \delta_{L^2}], \ \delta_{L^2}) - computable, where a, b \in \Re and f \in C(\Re, \ L^2(\Re)).$$

Lemma 2.10 *The representation δ_{H^2} of $H^s(\Re)$ is defined as follow: $\delta_{H^s}(q) := T_s^{-1} \circ \delta_{L^2}(q)$, $q \in dom(\delta_{H^s})$. Then δ_{H^s} is admissible and $\delta_{H^s} \leq \delta_{L^2}$.*

Lemma 2.11 $[\rho \rightarrow \delta_{H^s}] \equiv T_s^{-1} \circ [\rho \rightarrow \delta_{L^2} t]$.

3 Klein—Gordon equation

Theorem 3.1 *The solution operator* $S : C(\Re, H^s(\Re)) \times H^s(\Re) \times H^s(\Re) \to C(\Re,\ H^s(\Re))$, $(f, \phi, \varphi) \to u$ *is* $([\rho \to \delta_{H^s}], \delta_{H^s}, \delta_{H^s}, [\rho \to \delta_{H^s}])-$*computable, where* u *is the solution of the following initial-value problem for Klein—Gordon equation,*

$$u_{tt} - \Delta u + u = f(t)(x),\ t \in \Re,\ x \in \Re,\ f \in C(\Re,\ H^s(\Re))$$
$$u(0) = \phi(x), u_t(0) = \varphi(x), \phi \in H^s(\Re),\ \varphi \in H^s(\Re)$$

Proof: For any $\phi, \varphi \in H^s(\Re), f \in C(\Re, H^s(\Re))$ and $t \in \Re$, consider the integral equation:

$$F(u(t))(\xi) = \cos[(|\xi|^2+1)^{1/2}t] \cdot F\phi + \frac{\sin[(|\xi|^2+1)^{1/2}t]}{(|\xi|^2+1)^{1/2}} \cdot F\varphi + \int_0^t \frac{\sin[(|\xi|^2+1)^{1/2}(t-\tau)]}{(|\xi|^2+1)^{1/2}} \cdot F(f(x,\tau))d\tau, \quad (1)$$

where F is Fourier transform. Next multiplying (1) by $(1+|\xi|^2)^{s/2}$ yields the equation

$$T_s(ut(t))(\xi) = \cos[(|\xi|^2+1)^{1/2}t] \cdot T_s(\phi)(\xi) + \frac{\sin[(|\xi|^2+1)^{1/2}t]}{(|\xi|^2+1)^{1/2}} \cdot T_s(\varphi)(\xi) + \int_0^t \frac{\sin[(|\xi|^2+1)^{1/2}(t-\tau)]}{(|\xi|^2+1)^{1/2}} \cdot T_s(f(\tau))d\tau.$$

The following parts constitute this proof,

(a) It is easy to see that the function $(t, \xi) \to \cos[(|\xi|^2+1)^{1/2}t]$ is (ρ, ρ, ρ^2)-computable. By Lemma 2.5, the function $t \to \cos[(|\xi|^2+1)^{1/2}t]$ is $(\rho, [\rho \to \rho^2])$-computable.
Similarly, $t \to \frac{\sin[(|\xi|^2+1)^{1/2}t]}{(|\xi|^2+1)^{1/2}}$ is also $(\rho, [\rho \to \rho^2])$-computable.

(b) For any $t \in \Re$, $|\cos[(|\xi|^2+1)^{1/2}t]| \le 1$ and $|\frac{\sin[(|\xi|^2+1)^{1/2}t]}{(|\xi|^2+1)^{1/2}}| \le 1$. By lemma 2.9, $(t, \phi) \to \cos[(|\xi|^2+1)^{1/2}t] \cdot T_s(\phi)$ and $(t, \varphi) \to \frac{\sin[(|\xi|^2+1)^{1/2}t]}{(|\xi|^2+1)^{1/2}} \cdot T_s(\varphi)are(\rho, \delta_{H^s},\delta_{L^2})-$computable.

(c) It is easy to prove that $(t, \tau, f) \to \frac{\sin[(|\xi|^2+1)^{1/2}(t-\tau)]}{(|\xi|^2+1)^{1/2}} \cdot T_s(f(\tau))$ is $(\rho, \rho, [\rho \to \delta_{H^s}], \delta_{L^2})-$computable.

(d) By lemma 2.9 and (c), $(t, f) \to \int_0^t \frac{\sin[(|\xi|^2+1)^{1/2}(t-\tau)]}{(|\xi|^2+1)^{1/2}} \cdot T_s(f(\tau))d\tau$ is $(\rho, [\rho \to \delta_{H^s}], \delta_{L^2})-$computable.

(e) By lemma 2.9 and addition computable, the function $(f, \phi, \varphi, t) \to F(u(t))$ is $([\rho \to \delta_{H^s}], \delta_{H^s}, \delta_{H^s}, \rho, \delta_{L^2})-$ computable.

(f) By lemma 2.10 and type conversion, it is sufficient to prove Theorem 3.1.

4 Conclusion

In this paper, we prove that the solution operations of Klein-Gordon equation is Turing computable. This method can provide effective approach for the application of other linear equations, and furthermore establish basic position of Turing machines.

Acknowledgements

Research was supported by the National Nature Science Foundation of China(No:10420130638).

References

[1] Alan M.Turing: On computable numbers, with an application to the "Entscheid- ungsproblem". *Proceedings of the London Mathematical Society*. 42(2): 230-265 (1936)

[2] M.B.Pour-El, J.I.Richards: Computability in analysis and physics. *Perspectives in Mathematical Logic. Springer, Berlin* (1989)

[3] K.Weihrauch, N.Zhong: Is wave propagation computable or can wave computers beat the Turing machine. *Proc.London Math.Soc*. 85(2): 312-332 (2002)

[4] Dianchen Lu, Dejun Peng, Lixin Tian: On the well-posedness problem for the generalized Dullin-Gottwald-Holm equation. *International Journal of Nonlinear Science*. 1(3):178-186 (2006)

[5] Dianchen Lu, Baojian Hong, Lixin Tian: Bäcklund transformation and N soliton like solutions to the combined Kdv Burgers equations with variable coefficients. *International Journal of Nonlinear Science*. 2(1): 3-10 (2006)

[6] Suping Qian: Nonclassical Symmetry Reductions for Coupled Equations. *International Journal of Nonlinear Science*. 2(2): 97-103 (2006)

[7] Dianchen Lu, Guangjuan Yang: Compacton Solutions and Peakon Solutions for a Coupled Nonlinear Wave Equation. *International Journal of Nonlinear Science*.4(1):31-36 (2007).

[8] K.Weihrauch, N.Zhong: Computing the solution of the Korteweg-de Vries equation with arbitrary precision on Turing machines. *Theoret.Comput.Sci*.332(1-3): 337-366 (2005)

[9] K.Weihrauch: Computable Analysis.*Springer,Berlin*(2000)

[10] K.Weihrauch: Computability on computable metric spaces. *Theoret.Comput.Sci*.113(2): 191-210 (1993)