ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.5(2008) No.1,pp.25-32

A Generalized F-expansion Method and its Application to (2 + 1)-dimensional Breaking Solition Equations

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(Received 27 August 2007, accepted 10 December 2007)

Abstract: In this paper, a generalized F-expansion method is proposed to seek exact solutions of nonlinear evolution equations. With the aid of symbolic computation, we choose the (2 + 1)-dimensional breaking soliton equations to illustrate the validity and advantages of the proposed method. As a result, many new and more general non-travelling wave and coefficient function solutions are obtained including single and combined non-degenerate Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions, each of which contains an arbitrary function of two variables. Compared with the most existing F-expansion methods, the proposed method gives new and more general exact solutions. More importantly it provides a straightforward mathematical tool for solving a great many nonlinear evolution equations in mathematical physics.

Key words: F-expansion method; Jacobi elliptic function solutions; non-travelling wave and coefficient function solutions; soliton-like solutions; trigonometric function solutions

1 Introduction

Since the soliton phenomena were first observed by Scott Russell in 1834 [1] and the KdV equation was solved by the inverse scattering method by Gardner et al. in 1967 [2], finding exact solutions of nonlinear evolution equations (NLEEs) has become one of the most exciting and extremely active areas of research investigation. Many effective methods for obtaining exact solutions of NLEEs have been presented, such as Hirota's bilinear method [3], Bäcklund transformation [4], homogenous balance method [5], homotopy perturbation method [6], variational method [7], tanh-function method [8–10], Fan sub-equation method [11–13] and so on.

Recently, many exact solutions expressed by various Jacobi elliptic functions of many NLEEs in mathematical physics have been obtained by Jacobi elliptic function expansion method [14–16] and F-expansion method [17] which can be thought of as a generalization of Jacobi elliptic function expansion method. F-expansion method was later extended in different manners [18–22]. Very recently, we proposed a generalized F-expansion method [23–25] to obtain more general exact solutions which contain not only the results obtained by using the methods [17–22] but also a series of new and more general exact solutions.

In this paper, we further improve and develop our method [22–25]. In order to illustrate the effectiveness of the proposed method, we take the (2 + 1)-dimensional breaking soliton (BS) equations as an example. As a result, many new and more general exact non-travelling wave and coefficient function solutions are obtained including single and combined non-degenerate Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions, each of which contains an arbitrary function of two variables.

The rest of this paper is organized as follows: in Section 2, we give the description of the generalized F-expansion method; in Section 3, we use this method to obtain more general exact solutions of the (2 + 1)-dimensional BS equations; in Section 4, some conclusions are given.

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IJNS.2008.02.15/121

2 Description of the generalized F-expansion method

For a given NLEE with independent variables $X = (x, y, z, \ldots, t)$ and dependent variable u:

$$F(u, u_t, u_x, u_y, u_z, \ldots, u_{xt}, u_{yt}, u_{zt} \ldots, u_{tt}, u_{xx}, u_{yy}, u_{zz}, \ldots) = 0, \tag{1}$$

we seek its solutions in the new and more general form:

$$u = a_0 + \sum_{i=1}^{n} \left\{ a_i F^i(\xi) + b_i F^{-i}(\xi) + c_i F^{i-1}(\xi) F'(\xi) + d_i F^{-i}(\xi) F'(\xi) \right\}. \tag{2}$$

where $a_0 = a_0(X)$, $a_i = a_i(X)$, $b_i = b_i(X)$, $c_i = c_i(X)$, $d_i = d_i(X)$ $(i = 1, 2, \ldots, n)$ and $\xi = \xi(X)$ are undetermined functions, $F(\xi)$ and $F'(\xi)$ in (2) satisfy

$$F'^2(\xi) = PF^4(\xi) + QF^2(\xi) + R, \tag{3}$$

and hence holds for $F(\xi)$ and $F'(\xi)$

$$\begin{cases} F''(\xi) = 2PF^3(\xi) + QF(\xi), \\ F'''(\xi) = (6PF^2(\xi) + Q)F'(\xi), \\ F^{(4)}(\xi) = 24P^2F^5(\xi) + 20PQF^3(\xi) + (Q^2 + 12PR)F(\xi), \\ F^{(5)}(\xi) = (120P^2F^4(\xi) + 60PQF^2(\xi) + Q^2 + 12PR)F'(\xi), \\ \ldots \end{cases} \tag{4}$$

where P, Q and R are all parameters, the prime denotes $\mathrm{d}/\mathrm{d}\xi$. Given different values of P, Q and R, the different Jacobi elliptic function solutions $F(\xi)$ can be obtained from Eq. (3) (see Appendix A for its various Jacobi elliptic function solutions including some new ones). To determine u explicitly, we take the following four steps:

Step 1. Determine the integer n by balancing the highest order nonlinear term(s) and the highest order partial derivative term of u in Eq. (1).

Step 2. Substitute (2) along with (3) and (4) into Eq. (1) and collect all coefficients of $F'^i(\xi)F^j(\xi)$ $(i = 0, 1; j = 0, \pm 1, \pm 2, \ldots)$, then set each coefficient to zero to derive a set of over-determined partial differential equations for a_0, a_i, b_i, c_i, d_i $(i = 1, 2, \ldots, n)$ and ξ.

Step 3. Solve the system of over-determined partial differential equations obtained in Step 2 for a_0, a_i, b_i, c_i, d_i and ξ by use of Mathematica.

Step 4. From Appendix A select P, Q, R and $F(\xi)$, then substitute them along with a_0, a_i, b_i, c_i, d_i and ξ obtained in Step 3 into (2) to obtain Jacobi elliptic function solutions of Eq. (1) (see Appendix B for $F'(\xi)$), from which soliton-like solutions and trigonometric function solutions can be obtained when $m \to 1$ and $m \to 0$.

Remark 1. In order to determine the explicit solutions of the partial differential equations derived in Step 2, we may choose special forms of a_0, a_i, b_i, c_i, d_i and ξ (As we do in Section 3).

3 Application to the (2 + 1)-dimensional BS equations

Let us consider the (2 + 1)-dimensional BS equations:

$$u_t + bu_{xxy} + 4buv_x + 4bu_xv = 0, \quad u_y = v_x. \tag{5}$$

Eq. (5) describes the (2 + 1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. In the past years, many authors have studied Eq. (5). For instance, the Painlevé property was examined and localized coherent structures were presented in [26–28]. Recently, a class of periodic wave solutions were obtained by means of the mapping method in [29]. Some non-travelling wave solutions were obtained by the generalized expansion method of Riccatic equation in [30]. Multi-soliton solutions were obtained by further generalizing Hirota bilinear method in [31]. Very recently, using the singular manifold method, two classes of new exact solutions were obtained in [32].

According to Step 1, we get $n = 2$ for u and v. In order to search for explicit solutions of Eq. (5), we set $a_0 = a_0(y,t)$, $a_1 = a_1(y,t)$, $a_2 = a_2(y,t)$, $b_1 = b_1(y,t)$, $b_2 = b_2(y,t)$, $c_1 = c_1(y,t)$, $c_2 = c_2(y,t)$, $d_1 = d_1(y,t)$, $d_2 = d_2(y,t)$, $A_0 = A_0(y,t)$, $A_1 = A_1(y,t)$, $A_2 = A_2(y,t)$, $B_1 = B_1(y,t)$, $B_2 = B_2(y,t)$, $C_1 = C_1(y,t)$, $C_2 = C_2(y,t)$, $D_1 = D_1(y,t)$, $D_2 = D_2(y,t)$ and $\xi = kx + \eta$, here $\eta = \eta(y,t)$ and k is a nonzero constant. Thus we have the following formal solutions of Eq. (5):

$$u = a_0 + a_1F(kx+\eta) + a_2F^2(kx+\eta) + b_1F^{-1}(kx+\eta) + b_2F^{-2}(kx+\eta) + c_1F'(kx+\eta)$$
$$+c_2F'(kx+\eta)F(kx+\eta) + d_1F'(kx+\eta)F^{-1}(kx+\eta) + d_2F'(kx+\eta)F^{-2}(kx+\eta), \quad (6)$$
$$v = A_0 + A_1F(kx+\eta) + A_2F^2(kx+\eta) + B_1F^{-1}(kx+\eta) + B_2F^{-2}(kx+\eta) + C_1F'(kx+\eta)$$
$$+C_2F'(kx+\eta)F(kx+\eta) + D_1F'(kx+\eta)F^{-1}(kx+\eta) + D_2F'(kx+\eta)F^{-2}(kx+\eta). \quad (7)$$

With the aid of Mathematica, substituting (6) and (7) along with (3) and (4) into Eq. (5), the left-hand sides of Eq. (5) are converted into two polynomials of $F'^i(kx+\eta)F^j(kx+\eta)$ $(i = 0,1; j = 0, \pm1, \pm2, \cdots)$, then setting each coefficient to zero, we get a set of over-determined partial differential equations for a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, A_0, A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2 and η. Solving these over-determined partial differential equations by use of Mathematica, we get the following non-trivial results:

Case 1

$$a_0 = C, \quad a_2 = -\frac{3k^2P}{4}, \quad b_2 = -\frac{3k^2R}{4}, \quad c_1 = \pm\frac{3k^2\sqrt{P}}{4}, \quad d_2 = \pm\frac{3k^2\sqrt{R}}{4}, \quad (8)$$
$$A_0 = -\frac{\Psi_t(y,t) + b(\pm6k^2\sqrt{PR} + k^2Q + 4C)\Psi_y(y,t)}{4bk}, \quad A_2 = -\frac{3kP\Psi_y(y,t)}{4}, \quad (9)$$
$$B_2 = -\frac{3kR\Psi_y(y,t)}{4}, \quad C_1 = \pm\frac{3k\sqrt{P}\Psi_y(y,t)}{4}, \quad D_2 = \pm\frac{3k\sqrt{R}\Psi_y(y,t)}{4}, \quad (10)$$
$$a_1 = b_1 = c_2 = d_1 = A_1 = B_1 = C_2 = D_1 = 0, \quad \eta = \Psi(y,t),$$

where $\Psi(y,t)$ is arbitrary function of y and t, C is an arbitrary constant, k is a nonzero constant. The sign "$\pm$" in (8) means that all possible combinations of "$+$" and "$-$" can be used. The sign "$\pm$" in (9) and (10) depends on the sign "$\pm$" in (8), to be more precise, if the same sign is used in c_1 and d_2, then "$-$" must be used in A_0. If different signs are used in c_1 and d_2, then "$+$" must be used in A_0. Furthermore, the same sign must be used in c_1 and C_1. Also the same sign must be used in d_2 and D_2. Hereafter, the sign "$\pm$" always stands for this meaning in the similar circumstances.

Case 2

$$a_0 = C, \quad b_2 = -\frac{3k^2R}{4}, \quad d_2 = \pm\frac{3k^2\sqrt{R}}{4}, \quad B_2 = -\frac{3kR\Psi_y(y,t)}{4},$$
$$A_0 = -\frac{\Psi_t(y,t) + b(k^2Q + 4C)\Psi_y(y,t)}{4bk}, \quad D_2 = \pm\frac{3k\sqrt{R}\Psi_y(y,t)}{4},$$
$$a_1 = a_2 = b_1 = c_1 = c_2 = d_1 = A_1 = A_2 = B_1 = C_1 = C_2 = D_1 = 0, \quad \eta = \Psi(y,t),$$

where $\Psi(y,t)$ is arbitrary function of y and t, C is an arbitrary constant, k is a nonzero constant.

Case 3

$$a_0 = C, \quad a_2 = -\frac{3k^2P}{2}, \quad b_2 = -\frac{3k^2R}{2}, \quad A_2 = -\frac{3kP\Psi_y(y,t)}{2},$$
$$A_0 = -\frac{\Psi_t(y,t) + b(k^2Q + 4C)\Psi_y(y,t)}{4bk}, \quad B_2 = -\frac{3kR\Psi_y(y,t)}{2},$$
$$a_1 = b_1 = c_1 = c_2 = d_1 = d_2 = A_1 = B_1 = C_1 = C_2 = D_1 = D_2 = 0, \quad \eta = \Psi(y,t),$$

where $\Psi(y,t)$ is arbitrary function of y and t, C is an arbitrary constant, k is a nonzero constant.

Case 4

$$a_0 = C, \quad a_2 = -\frac{3k^2P}{2}, \quad A_2 = -\frac{3kP\Psi_y(y,t)}{2},$$

$$A_0 = -\frac{\Psi_t(y,t) + b(k^2Q + 4C)\Psi_y(y,t)}{4bk}, \quad \eta = \Psi(y,t),$$
$$a_1 = b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = A_1 = B_1 = B_2 = C_1 = C_2 = D_1 = D_2 = 0,$$

where $\Psi(y,t)$ is arbitrary function of y and t, C is an arbitrary constant, k is a nonzero constant.

Substituting Cases 1–4 into (6) and (7) respectively, we have four kinds of formal solutions of Eq. (5):

$$u = C - \frac{3k^2P}{4}F^2(\xi) - \frac{3k^2R}{4}F^{-2}(\xi) \pm \frac{3k^2\sqrt{P}}{4}F'(\xi) \pm \frac{3k^2\sqrt{R}}{4}F'(\xi)F^{-2}(\xi), \tag{11}$$

$$\begin{aligned} v = &-\frac{\Psi_t(y,t) + b(\pm 6k^2\sqrt{PR} + k^2Q + 4C)\Psi_y(y,t)}{4bk} - \frac{3kP\Psi_y(y,t)}{4}F^2(\xi) \\ &-\frac{3kR\Psi_y(y,t)}{4}F^{-2}(\xi) \pm \frac{3k\sqrt{P}\Psi_y(y,t)}{4}F'(\xi) \pm \frac{3k\sqrt{R}\Psi_y(y,t)}{4}F'(\xi)F^{-2}(\xi); \end{aligned} \tag{12}$$

$$u = C - \frac{3k^2R}{4}F^{-2}(\xi) \pm \frac{3k^2\sqrt{R}}{4}F'(\xi)F^{-2}(\xi), \tag{13}$$

$$v = -\frac{\Psi_t(y,t) + b(k^2Q + 4C)\Psi_y(y,t)}{4bk} \pm \frac{3k\sqrt{R}\Psi_y(y,t)}{4}F'(\xi)F^{-2}(\xi); \tag{14}$$

$$u = C - \frac{3k^2P}{2}F^2(\xi) - \frac{3k^2R}{2}F^{-2}(\xi), \tag{15}$$

$$v = -\frac{\Psi_t(y,t) + b(k^2Q + 4C)\Psi_y(y,t)}{4bk} - \frac{3kP\Psi_y(y,t)}{2}F^2(\xi) - \frac{3kR\Psi_y(y,t)}{2}F^{-2}(\xi); \tag{16}$$

$$u = C - \frac{3k^2P}{2}F^2(\xi), \tag{17}$$

$$v = -\frac{\Psi_t(y,t) + b(k^2Q + 4C)\Psi_y(y,t)}{4bk} - \frac{3kP\Psi_y(y,t)}{2}F^2(\xi), \tag{18}$$

where $\xi = kx + \Psi(y,t)$, C is an arbitrary constant, k is a non-zero constant.

With the aid of Appendices A, B and C, from the four kinds of formal solutions (11)–(18), we can obtain many exact solutions of Eq. (5).

For example, from Appendix A, selecting $F(\xi) = \mathrm{ns}\xi$, $P = 1$, $Q = -(1 + m^2)$, $R = m^2$, inserting them into (11) and (12) and using Appendix B, yields combined non-degenerate Jacobi elliptic function solutions of Eq. (5):

$$u = C - \frac{3k^2}{4}\mathrm{ns}^2\xi - \frac{3k^2m^2}{4}\mathrm{sn}^2\xi \mp \frac{3k^2}{4}\mathrm{cs}\xi\mathrm{ds}\xi \mp \frac{3k^2m}{4}\mathrm{cn}\xi\mathrm{dn}\xi,$$

$$\begin{aligned} v = &-\frac{\Psi_t(y,t) + b(\pm 6k^2m - k^2(1+m^2) + 4C)\Psi_y(y,t)}{4bk} - \frac{3k\Psi_y(y,t)}{4}\mathrm{ns}^2\xi - \frac{3km^2\Psi_y(y,t)}{4}\mathrm{sn}^2\xi \\ &\mp \frac{3k\Psi_y(y,t)}{4}\mathrm{cs}\xi\mathrm{ds}\xi \mp \frac{3km\Psi_y(y,t)}{4}\mathrm{cn}\xi\mathrm{dn}\xi \end{aligned}$$

where $\xi = kx + \Psi(y,t)$.

Selecting $F(\xi) = \mathrm{ns}\xi \pm \mathrm{cs}\xi$, $P = \frac{1}{4}$, $Q = \frac{1-2m^2}{2}$, $R = \frac{1}{4}$, inserting them into (11) and (12) and using Appendix B, yields combined non-degenerate Jacobi elliptic function solutions of Eq. (5):

$$u = C - \frac{3k^2}{16}(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2 - \frac{3k^2}{16(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2} \mp \frac{3k^2}{8}(\mathrm{cs}\xi\mathrm{ds}\xi \pm \mathrm{ns}\xi\mathrm{ds}\xi) \mp \frac{3k^2(\pm\mathrm{ds}\xi)}{8(\mathrm{ns}\xi \pm \mathrm{cs}\xi)}, \tag{19}$$

$$\begin{aligned} v = &-\frac{2\Psi_t(y,t) + b(\pm 3k^2 + k^2(1-2m^2) + 8C)\Psi_y(y,t)}{8bk} - \frac{3k\Psi_y(y,t)}{16}(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2 \\ &-\frac{3k\Psi_y(y,t)}{16(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2} \mp \frac{3k\Psi_y(y,t)}{8}(\mathrm{cs}\xi\mathrm{ds}\xi \pm \mathrm{ns}\xi\mathrm{ds}\xi) \mp \frac{3k\Psi_y(y,t)(\pm\mathrm{ds}\xi)}{8(\mathrm{ns}\xi \pm \mathrm{cs}\xi)}, \end{aligned} \tag{20}$$

where $\xi = kx + \Psi(y,t)$.

When $m \to 1$, from (31) and (32) we get soliton-like solutions of Eq. (5):

$$u = C - \frac{3k^2}{16}(\mathrm{coth}\xi \pm \mathrm{csch}\xi)^2 - \frac{3k^2}{16(\mathrm{coth}\xi \pm \mathrm{csch}\xi)^2} \mp \frac{3k^2}{8}(\mathrm{csch}^2\xi \pm \mathrm{coth}\xi\mathrm{csch}\xi) \mp \frac{3k^2(\pm\mathrm{csch}\xi)}{8(\mathrm{coth}\xi \pm \mathrm{csch}\xi)},$$

$$v = -\frac{2\Psi_t(y,t) + b(\pm 3k^2 - k^2 + 8C)\Psi_y(y,t)}{8bk} - \frac{3k\Psi_y(y,t)}{16}(\mathrm{coth}\xi \pm \mathrm{csch}\xi)^2 - \frac{3k\Psi_y(y,t)}{16(\mathrm{coth}\xi \pm \mathrm{csch}\xi)^2} \mp \frac{3k\Psi_y(y,t)}{8}(\mathrm{csch}^2\xi \pm \mathrm{coth}\xi\mathrm{csch}\xi) \mp \frac{3k\Psi_y(y,t)(\pm\mathrm{csch}\xi)}{8(\mathrm{coth}\xi \pm \mathrm{csch}\xi)},$$

where $\xi = kx + \Psi(y,t)$.

When $m \to 0$, from (19) and (20) we get trigonometric function solutions of Eq. (5):

$$u = C - \frac{3k^2}{16}(\mathrm{csc}\xi \pm \mathrm{cot}\xi)^2 - \frac{3k^2}{16(\mathrm{csc}\xi \pm \mathrm{cot}\xi)^2} \mp \frac{3k^2}{8}(\mathrm{cot}\xi\mathrm{csc}\xi \pm \mathrm{csc}^2\xi) \mp \frac{3k^2(\pm\mathrm{csc}\xi)}{8(\mathrm{csc}\xi \pm \mathrm{cot}\xi)},$$

$$v = -\frac{2\Psi_t(y,t) + b(\pm 3k^2 + k^2 + 8C)\Psi_y(y,t)}{8bk} - \frac{3k\Psi_y(y,t)}{16}(\mathrm{csc}\xi \pm \mathrm{cot}\xi)^2 - \frac{3k\Psi_y(y,t)}{16(\mathrm{csc}\xi \pm \mathrm{cot}\xi)^2} \mp \frac{3k\Psi_y(y,t)}{8}(\mathrm{cot}\xi\mathrm{csc}\xi \pm \mathrm{csc}^2\xi) \mp \frac{3k\Psi_y(y,t)(\pm\mathrm{csc}\xi)}{8(\mathrm{csc}\xi \pm \mathrm{cot}\xi)},$$

where $\xi = kx + \Psi(y,t)$.

From (11)–(18) and Appendices A, B and C, we can also obtain other Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions of Eq. (5), we omit them here for simplicity. All solutions obtained from (15)–(18) can not be obtained by methods in [17–20]. There are more solutions of Eq. (3) listed in Appendix A than those in [21–25], so other new formal exact solutions of Eq. (5) can be obtained. To our knowledge, these solutions have not been reported in former literature.

4 Conclusion

In this paper, we have proposed a generalized F-expansion method to construct more general exact solutions of NLEEs. With the help of Mathematica, the method provides a straightforward mathematical tool for obtaining more general exact solutions of NLEEs in mathematical physics. Applying this method to the (2 + 1)-dimensional BS equations, we have successfully obtained many new and more general non-degenerate Jacobi elliptic function solutions, solitary wave solutions and trigonometric function solutions, each of which contains an arbitrary function of two variables. The arbitrary function in the obtained solutions implies that these solutions have rich local structures. It may be important to explain some physical phenomena. Applications of the generalized F-expansion method proposed in this paper to other NLEEs are worth study, such as combined KdV-Burgers equation with variable coefficients [4], dispersive long wave equation [33], Burgers equation [34], and others.

Acknowledgements

This work was supported by the Natural Science Foundation of Educational Committee of Liaoning Province of China under Grant No. 2006022.

Appendix A

Relations between values of (P,Q,R) and corresponding $F(\xi)$ in Eq. (3) which are listed in Table 1 [35]:

Table 1:

P	Q	R	$F(\xi)$
m^2	$-(1+m^2)$	1	$\mathrm{sn}\xi,\quad \mathrm{cd}\xi=\frac{\mathrm{cn}\xi}{\mathrm{dn}\xi}$
$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{cn}\xi$
-1	$2-m^2$	m^2-1	$\mathrm{dn}\xi$
1	$-(1+m^2)$	m^2	$\mathrm{ns}\xi=(\mathrm{sn}\xi)^{-1},\quad \mathrm{dc}\xi=\frac{\mathrm{dn}\xi}{\mathrm{cn}\xi}$
$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{nc}\xi=(\mathrm{cn}\xi)^{-1}$
m^2-1	$2-m^2$	-1	$\mathrm{nd}\xi=(\mathrm{dn}\xi)^{-1}$
$1-m^2$	$2-m^2$	1	$\mathrm{sc}\xi=\frac{\mathrm{sn}\xi}{\mathrm{cn}\xi}$
$-m^2(1-m^2)$	$2m^2-1$	1	$\mathrm{sd}\xi=\frac{\mathrm{sn}\xi}{\mathrm{dn}\xi}$
1	$2-m^2$	$1-m^2$	$\mathrm{cs}\xi=\frac{\mathrm{cn}\xi}{\mathrm{sn}\xi}$
1	$2m^2-1$	$-m^2(1-m^2)$	$\mathrm{ds}\xi=\frac{\mathrm{dn}\xi}{\mathrm{sn}\xi}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\mathrm{ns}\xi\pm\mathrm{cs}\xi,\quad \frac{\mathrm{cn}\xi}{\sqrt{1-m^2}\mathrm{sn}\xi\pm\mathrm{dn}\xi}$
$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$\mathrm{nc}\xi\pm\mathrm{sc}\xi$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{ns}\xi\pm\mathrm{ds}\xi$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{sn}\xi\pm i\mathrm{cn}\xi,\quad \frac{\mathrm{dn}\xi}{\sqrt{1-m^2}\mathrm{sn}\xi\pm\mathrm{cn}\xi}$
$-\frac{1}{4}$	$\frac{1+m^2}{2}$	$-\frac{(1-m^2)^2}{4}$	$m\mathrm{cn}\xi\pm\mathrm{dn}\xi$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$m\mathrm{sn}\xi\pm i\mathrm{dn}\xi,\quad \frac{\mathrm{sn}\xi}{1\pm\mathrm{cn}\xi}$
$\frac{1}{4m^2}$	$\frac{1-2m^2}{2}$	$\frac{m^2}{4}$	$\frac{\mathrm{dn}\xi}{\sqrt{\frac{1-m^2}{m^2}}\pm\mathrm{cn}\xi}$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$\frac{\mathrm{sn}\xi}{1\pm\mathrm{dn}\xi}$
$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{m^2-1}{4}$	$\frac{\mathrm{dn}\xi}{1\pm m\mathrm{sn}\xi}$
$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$\frac{\mathrm{cn}\xi}{1\pm\mathrm{sn}\xi}$
$\frac{(1-m^2)^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1}{4}$	$\frac{\mathrm{sn}\xi}{\mathrm{cn}\xi\pm\mathrm{dn}\xi}$
$\frac{m^4}{4}$	$\frac{2-m^2}{2}$	$\frac{1}{4}$	$\frac{\mathrm{cn}\xi}{\sqrt{1-m^2}\pm\mathrm{dn}\xi}$

Appendix B

Derivatives of Jacobi elliptic functions

$$\mathrm{sn}'\xi=\mathrm{cn}\xi\mathrm{dn}\xi,\quad \mathrm{cd}'\xi=-(1-m^2)\mathrm{sd}\xi\mathrm{nd}\xi,\quad \mathrm{cn}'\xi=-\mathrm{sn}\xi\mathrm{dn}\xi,\quad \mathrm{dn}'\xi=-m^2\mathrm{sn}\xi\mathrm{cn}\xi,$$

$$\mathrm{ns}'\xi=-\mathrm{cs}\xi\mathrm{ds}\xi,\quad \mathrm{dc}'\xi=(1-m^2)\mathrm{nc}\xi\mathrm{sc}\xi,\quad \mathrm{nc}'\xi=\mathrm{sc}\xi\mathrm{dc}\xi,\quad \mathrm{nd}'\xi=m^2\mathrm{cd}\xi\mathrm{sd}\xi,$$

$$\mathrm{sc}'\xi=\mathrm{dc}\xi\mathrm{nc}\xi,\quad \mathrm{cs}'\xi=-\mathrm{ns}\xi\mathrm{ds}\xi,\quad \mathrm{ds}'\xi=-\mathrm{cs}\xi\mathrm{ns}\xi,\quad \mathrm{sd}'\xi=\mathrm{nd}\xi\mathrm{cd}\xi.$$

Appendix C

Jacobi elliptic functions degenerate into hyperbolic functions when $m\to 1$:

$$\mathrm{sn}\xi\to\tanh\xi,\quad \mathrm{cn}\xi\to\mathrm{sech}\xi,\quad \mathrm{dn}\xi\to\mathrm{sech}\xi,\quad \mathrm{sc}\xi\to\sinh\xi,\quad \mathrm{sd}\xi\to\sinh\xi,\quad \mathrm{cd}\xi\to 1,$$

$$\mathrm{ns}\xi\to\coth\xi,\quad \mathrm{nc}\xi\to\cosh\xi,\quad \mathrm{nd}\xi\to\cosh\xi,\quad \mathrm{cs}\xi\to\mathrm{csch}\xi,\quad \mathrm{ds}\xi\to\mathrm{csch}\xi,\quad \mathrm{dc}\xi\to 1.$$

Jacobi elliptic functions degenerate into trigonometric functions when $m\to 0$:

$$\mathrm{sn}\xi\to\sin\xi,\quad \mathrm{cn}\xi\to\cos\xi,\quad \mathrm{dn}\xi\to 1,\quad \mathrm{sc}\xi\to\tan\xi,\quad \mathrm{sd}\xi\to\sin\xi,\quad \mathrm{cd}\xi\to\cos\xi,$$

$$\mathrm{ns}\xi\to\csc\xi,\quad \mathrm{nc}\xi\to\sec\xi,\quad \mathrm{nd}\xi\to 1,\quad \mathrm{cs}\xi\to\cot\xi,\quad \mathrm{ds}\xi\to\csc\xi,\quad \mathrm{dc}\xi\to\sec\xi.$$

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