

The Global Attractor for the Viscous Weakly Damped Forced Korteweg-de Vries Equations in $H^1(R)$

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Abstract: The existence of the global attractor for the viscous weakly damped forced Korteweg-de Vries Equations in the phase space $H^1(R)$ is studied in this paper. Bourgain spaces and the energy-type equation method are used here to obtain the well-posedness for the viscous weakly damped forced KdV equations. The existence of the bounded absorbing sets for the viscous weakly damped forced KdV equations is obtained by the standard procedure. Finally, the existence of the attractor for the viscous weakly damped forced KdV equations is proved.

Keywords: Bourgain space; energy equation; viscous term; global attractor

1 Introduction

The paper is mainly concerned with the existence of the global attractor of the viscous weakly damped forced Korteweg-de Vries equations([1–4, 6–8, 13–16, 18, 20]):

$$\begin{cases} u_t + uu_x + u_{xxx} + \gamma u + \beta u_x = f \\ u|_{t=0} = u_0 \in H^1(R) \end{cases} \quad (1)$$

in the phase space $H^1(R)$, For $(x, t) \in R \times R$, f is time-independent and belongs to $H^2(R)$. γ, β are constants, $\gamma, \beta > 0$.

The weakly damped KdV equation which was derived by Ott and Sudan as a model for ionsound waves damped by ion-neutral collisions([19]), has been considered in different contexts, such as the global attractor for the Korteweg-de Vries Equations on the real line. By using the energy-type equation and the weighted space, P.Laurencot got the existence of the global attractor of the weakly damped KdV equation in $H^2(R)$ ([10]). Similarly the work was done by R.Rosa and I.Moise who only used the energy-type equation([9]). Ricardo Rosa investigated the existence of the global attractor of the weakly damped forced KdV equation in $H^1(R)$ ([12]). Oliver Groubet and Ricardo Rosa investigated the asymptotic and the global attractor of the weakly damped forced KdV equation on the real line([5]). From the mathematical point of view, the term with the factor γ accounts for a weakly dissipation with no regularization, and the term with the factor η accounts for viscous term. Lixin Tian and Wenbin Zhang investigated the the existence of the global attractor of the weakly damped forced KdV equation with the viscous term in $L^2(R)$ ([17]). As $H^1(R)$ has further regularity than $L^2(R)$ and with the continuous injection $H^1(R) \subseteq L^2(R)$, we get the existence of the global attractor of the weakly damped forced KdV equation with the viscous term in $H^1(R)$ for the first time.

The outline of this paper is as follows. In section 1, the researches of the KdV equation are introduced. In section 2, the definitions and main theorems are introduced. In section 3, we obtain the local well-posedness of the Korteweg-de Vries equation in the mild sense. In section 4, the global existence of the solutions is established. In section 5, we obtain the existence of the attractor for the viscous weakly damped forced Korteweg-de Vries Equations.

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2 Function Spaces and Main Theorems

We consider the spaces $L^2(R)$ and $H^1(R)$ with its normal norms respectively, as well as $L^\infty(R)$, with its norm denoted by $\|\cdot\|_{L^\infty(R)} = \text{ess sup } |\cdot|$. We write $((\cdot, \cdot))_{L^2(R)}$ and $((\cdot, \cdot))_{H^1(R)}$ for the standard inner product. We set $A = (1 - \partial_x^2)$ as a positive self-adjoint operator in $L^2(R)$ and denote $H^{-1}(R)$ the dual of $H^1(R)$. The operator A can be extended to an isomorphism from $H^1(R)$ onto $H^{-1}(R)$ and the norm in $H^{-1}(R)$ can be written as:

$$\|u\|_{H^1(R)} = \|A^{-1}u\|_{H^1(R)} = \|A^{\frac{1}{2}}u\|_{L^2(R)}$$

We also use other useful spaces $H_0^1(-r, r)$, of functions in $H^1(R)$ which vanish outside $(-r, r)$, where $r > 0$ and the dual $H_0^{-1}(-r, r)$. For all $r > 0$,

$$H_0^1(-r, r) \subseteq H^1(R) \subseteq L^2(R) \subseteq H^{-1}(R) \subseteq H_0^{-1}(-r, r)$$

with continuous injections, and $H_0^1(-r, r) \subseteq L^2(R) \subseteq H_0^{-1}(-r, r)$ with compact injections.

Other useful relations are:

$$((Au, v))_{L^2(R)} = ((u, v))_{H^1(R)}, \quad \frac{1}{2} \|u\|_{L^2(R)} \leq \|A^{-1}u\|_{H^1(R)}^2 \leq \|u\|_{L^2(R)}$$

For each $T > 0$ and each measurable function $u : R \times [-T, T] \rightarrow R$, now we consider the space similar to that introduced by C. E. Kenig, G. Ponce and L. Vega [7].

We set: $X_T = \{u : R \times [-T, T] \rightarrow R; \wedge(T; u) < \infty\}$

and

$$\wedge(T; u) = \max \{ \lambda_1(T; u), \lambda_2(T; u), \lambda_3(T; u), \lambda_4(T; u), \lambda_5(T; u) \}$$

$$\lambda_1(T; u) = \text{ess. sup}_{t \in [-T; T]} \|u(\cdot, t)\|_{H^1(R)}$$

$$\lambda_2(T; u) = \left(\text{ess. sup}_{t \in [-T; T]} \int_{-T}^T |u_{xx}(\cdot, t)|^2 dt \right)^{\frac{1}{2}}$$

$$\lambda_3(T; u) = \left(\text{ess. sup}_{t \in [-T; T]} \int_{-T}^T |u_{xxx}(\cdot, t)|^2 dt \right)^{\frac{1}{2}}$$

$$\lambda_4(T; u) = \left(\int_{-T}^T \|u_x(\cdot, t)\|_{L^\infty(R)}^6 dt \right)^{\frac{1}{6}}$$

$$\lambda_5(T; u) = \frac{1}{1+T} \left(\int_R \text{ess. sup}_{t \in [-T; T]} |u(x, t)|^2 dx \right)^{\frac{1}{2}}$$

The space X_T is a Banach space endowed with the norm $\|\cdot\|_{X_T} = \wedge(T, \cdot)$. Now, we consider the linear part of Eq.(1):

$$\begin{cases} u_t + u_{xxx} + \gamma u + \beta u_x = 0 \\ u|_{t=0} = u_0 \in H^1(R) \end{cases} \quad (2)$$

For each $\gamma, \beta > 0$. We denote $\{W_\gamma(t)\}_{t \in R}$ the group associated with (2) and define $W_\gamma(t) = e^{-rt}W_0(t)$. We multiply (2) by u , then integrate over R , and then we find the estimate $\|W_\gamma(t)u_0\|_{L^2(R)} = e^{\gamma t} \|u_0\|_{L^2(R)}$. By the linearity, we have $\|W_\gamma(t)u_0\|_{H^1(R)} = e^{\gamma t} \|u_0\|_{H^1(R)}$. If the initial condition u_0 belongs to $H^1(R)$, the solution is continuous as a function values in $H^1(R)$, so we have $u_0 \in H_1(R) \Rightarrow t \mapsto W_\gamma(t)u_0 \in C_b([-T, T]; H_1(R))$, for all $T > 0$. The case $\gamma = 0, \beta = 0$ is connected to the KdV equation, and the fundamental estimates for well-posedness of the KdV in $H^1(R)$ were obtained by C. E. Kenig, G. Ponce and L. Vega ([7]) and R.Rosa ([12]). We borrow the estimates from them as follows:

Lemma 1 ([7]) Let $\gamma, \beta > 0$ and $T > 0$. There is a numerical constant C_1 , which satisfies:

$$\left(\int_{-T}^T \|W_\gamma(t) u_0\|_{L^\infty(R)}^6 dx \right)^{\frac{1}{6}} \leq C_1 e^{|\gamma|T} \|u_0\|_{L^2(R)}$$

for all u_0 in $L^2(R)$,

$$\left(\sup_{x \in R} \int_{-T}^T |\partial_x W_\gamma(t) u_0|^2 dx \right)^{\frac{1}{2}} \leq \frac{\sqrt{3}}{3} e^{|\gamma|T} \|u_0\|_{L^2(R)}$$

for all u_0 in $L^2(R)$,

$$\left(\int_{-\infty}^{-\infty} \sup_{t \in [-T, T]} \|W_\gamma(t) u_0\|_{L^\infty(R)}^2 dx \right)^{\frac{1}{2}} \leq C_1 (1 + T) e^{|\gamma|T} \|u_0\|_{H^1(R)}$$

for all u_0 in $H^1(R)$, and all $T > 0$.

Lemma 2 ([12]) Let $T > 0$ and $u, v \in X_T$. There is

$$\|(uv)_x\|_{L^1(-T, T; H^1(R))} \leq 4(1 + \sqrt{2}) (1 + T) T^{\frac{1}{2}} \|u\|_{X_T} \|v\|_{X_T}$$

From Lemma 1, it is straightforward to prove the following results:

Lemma 3 Letting $\gamma, \beta > 0, T > 0, u_0(\cdot, t) \in H^1(R)$ and $\omega(\cdot, t) = W_\gamma(t) u_0$, then, there exists $u \in X_T$, with $\|\omega\|_{X_T} \leq C_1 e^{|\gamma|t} \|u_0\|_{H^1(R)}$, where C_1 is the constant from lemma 1, which is independent of γ, β, T and f .

We proceed essentially as in ([7]) and we derive estimates for the no homogeneous equation with (2):

$$\begin{cases} u_t + u_{xxx} + \gamma u + \beta u_x = f \\ u|_{t=0} = u_0 \in H^1(R) \end{cases}$$

where f is time-dependent and assumed to belong to $L^1(-T, T; H^2(R))$.

Lemma 4 Let $\gamma, \beta > 0, T > 0, f \in L^1(-T, T; H^2(R))$ and

$$u(\cdot, t) = \int_0^t W_\gamma(t - \tau) f(\cdot, \tau) d\tau \tag{3}$$

then $u \in X_T$, with $\|u\|_{X_T} \leq C_1 e^{|\gamma|T} \|f\|_{L^1(-T, T; H^j(R))}, j = 1, 2$, where C_1 is the constant from Lemma 1, which is independent of γ, β, T and f .

Proof. We will only illustrate the cases $i = 3$. Let $\chi_t = \chi_t(T)$ be the characteristic function of the closed interval with end points 0 and t , other positive or negative. Hence we can write t given in (3) as:

$$u(\cdot, t) = \int_{-T}^T \chi_t(\tau) W_\gamma(\tau) f(\cdot, \tau) d\tau$$

Then for $i = 3$,

$$\begin{aligned} \lambda_3 &\leq \int_{-T}^T \lambda_3(T; \chi_t(\tau)) W_\gamma(t - \tau) f(\cdot, \tau) d\tau \\ &\leq \int_{-T}^T \left(\text{ess. sup}_{t \in [-T, T]} \int_{-T}^T |\partial_x^2 (W_\gamma(s) f_x(\cdot, \tau))|^2 ds \right)^{\frac{1}{2}} d\tau \\ &\leq \frac{\sqrt{3}}{3} \int_{-T}^T e^{|\gamma|T} \|f_{xx}(\cdot, \tau)\|_{L^2(R)} d\tau \\ &\leq \frac{\sqrt{3}}{3} e^{|\gamma|T} \|f\|_{L^1(-T, T; H^2(R))} \end{aligned}$$

■

We borrow the following estimates ([12]):

Lemma 5 Let $\gamma, \beta > 0, T > 0, u, v \in X_T$ and $\omega(\cdot, t) = \int_0^t W_\gamma(t - \tau)(uv)_x d\tau$. If $t \in [-T, T]$, then

$$\|\omega\|_{X_T} \leq C_2(1+T)T^{\frac{1}{2}}e^{|\gamma|T} \|u\|_{X_T} \|v\|_{X_T}$$

where $C_2 = 4(1 + \sqrt{2})C_1$, C_1 is the constant as above, which is independent of γ, β, T, u and v .

In this paper, we will obtain the following main theorems:

Theorem 1 Let $\gamma, \beta > 0, f \in L^1_{loc}(R; H^2(R))$ and $u_0 \in H^1(R)$. There exists a unique continuous solution u in X_{T_R} of Eq.(1).

Theorem 2 Let $\gamma, \beta > 0, f \in H^2(R)$ and $u_0 \in H^1(R)$. There exists a solution $u \in C(R; H^1(R))$ of Eq.(1), which is the unique solution which belongs to X_T for all $T > 0$; moreover, the solution $t \mapsto u(t)$ satisfies the energy-type equation:

$$\frac{d}{dt} I_j(u(t)) + 2\gamma I_j(u(t)) = K_j(u(t)) \quad (4)$$

for all $t \in R$ and for $j = 0, 1$, where I_j, K_j are given as follows. Further more, the map which associates the data (r, f, u_0) to the corresponding unique u is continuous from $R \times H^2(R) \times H^1(R)$ into $X_T \cap C([-T, T]; H^1(R))$ for all $T > 0$. In particular,

$$\|u\|_{X_T} \leq C(\gamma, \|f\|_{H^2(R)}, \|u_0\|_{H^1(R)}, T)$$

for some constant C .

Theorem 3 Let $\gamma, \beta > 0, f \in H^2(R)$. The solution operator $\{S(t)\}_{t \in R}$ in $H^1(R)$ associated with the Eq.(1) possesses a global attractor.

3 Local well-posedness

In the proof of theorem 1, we use the approaches in ([7]) to obtain the local well-posedness of Eq.(1) in the mild sense. Consider $\gamma, \beta > 0, f \in L^1_{loc}(R; H^2(R))$ and $u_0 \in H^1(R)$, as the unique fixed point of the map $\sum(u) : X_T \mapsto X_T$, defined:

$$\sum(u(t)) = W_\gamma(t)u_0 + \frac{1}{2} \int_0^t W_\gamma(t-s) \left[2f - 2\beta u_x(s) - (u(s)^2)_x \right] ds$$

For $t \in [-T, T]$, from Lemmas as above, it follows:

$$\begin{aligned} \|\sum(u)\|_{X_T} &\leq C_1 e^{|\gamma|T} \|u_0\|_{H^1(R)} + C_1 e^{|\gamma|T} \|f\|_{L^1(-T, T; H^2(R))} \\ &+ \frac{C_2}{2} T^{\frac{1}{2}} (1+T) e^{|\gamma|T} \|u\|_{X_T}^2 + \beta C_1 e^{|\gamma|T} \|u\|_{X_T} \end{aligned}$$

Similarly, by writing $u^2 - v^2 = (u - v)(u + v)$, we find:

$$\left\| \sum(u) - \sum(v) \right\|_{X_T} \leq \left(\frac{C_2}{2} T^{\frac{1}{2}} (1+T) e^{|\gamma|T} \|u + v\|_{X_T} + \beta C_1 e^{|\gamma|T} \right) \|u - v\|_{X_T}$$

We let $R^* = 2C_1 e^{|\gamma|} (\|u_0\|_{H^1(R)} + \|f\|_{L^1(-T, T; H^2(R))})$. For $0 < T \leq 1, \|u\|_{X_T}, \|v\|_{X_T} \leq R^*$, we have:

$$\left\| \sum(u) \right\|_{X_T} \leq \frac{R^*}{2} + \left(\frac{C_2}{2} T^{\frac{1}{2}} e^{|\gamma|} R^* + \beta C_1 e^{|\gamma|} \right) R^*$$

and

$$\left\| \sum(u) - \sum(v) \right\|_{X_T} \leq \left(C_2 T^{\frac{1}{2}} e^{|\gamma|} R^* + \beta C_1 e^{|\gamma|} \right) \|u - v\|_{X_T}$$

Taking $T_R, 0 < T_R \leq 1$ which is sufficiently small, we have:

$$\frac{R^*}{2} + \left(\frac{C_2}{2} T^{\frac{1}{2}} e^{|\gamma|} R^* + \beta C_1 e^{|\gamma|} \right) R^* \leq R$$

and $C_2 T^{\frac{1}{2}} e^{|\gamma|} R^* + \beta C_1 e^{|\gamma|} \leq \frac{1}{2}$, with the choice of T_R , following that:

$$\left\| \sum(u) \right\|_{X_T} \leq R, \left\| \sum(u) - \sum(v) \right\|_{X_T} \leq \frac{1}{2} \|u - v\|_{X_{T_R}}$$

for every $u, v \in X_{T_R}$, with $\|u\|_{X_{T_R}}, \|v\|_{X_{T_R}} \leq R$. So, \sum is a strict contraction when restricted to the ball in X_{T_R} of radius R and centered at the origin. Following the Banach Fixed Point Theorems, there is a unique u in this ball, which is the fixed point of \sum as well as a solution of Eq.(1) in the mild sense. By the uniform contraction, one can check that u is continuous and unique. Finally, we find that the solution u is continuous as a function with values in $H^1(R)$, i.e.

$$t \mapsto u(t) \in C_b([-T_R, T_R]; H^1(R))$$

4 Global Solutions and Energy-Type Equation

In this section, we will establish the global existence of the solutions and obtain theorem 2. This is achieved by the help of two of the invariants of the KdV equation, namely:

$$I_0(u) = \int_R u^2(x) dx, I_1(u) = \int_R \left(u_x^2(x) - \frac{1}{3} u^3(x) \right) dx$$

Multiplying equation Eq.(1) by $2\tilde{u}$, and integrating on R ,

$$\left(\tilde{u}_t + \tilde{u}\tilde{u}_x + \tilde{u}_{xxx} + \gamma\tilde{u} + \beta\tilde{u}_x - \tilde{f}, 2\tilde{u} \right) = \frac{d}{dt} \|\tilde{u}\|^2 + 2\gamma \|\tilde{u}\|^2 - 2 \int_R \tilde{f}(x) \tilde{u}(x) dx$$

We see that \tilde{u} satisfies the energy-type equation:

$$\frac{d}{dt} \|\tilde{u}\|^2 + 2\gamma \|\tilde{u}\|^2 = 2 \left(\tilde{f}(x), \tilde{u}(x) \right)_{L^2(R)} \tag{5}$$

Multiplying equation Eq.(1) by $-2\tilde{u}_{xx} - \tilde{u}^2$, and integrating on R , then we get:

$$\frac{d}{dt} \int_R \left(\tilde{u}_x^2 - \frac{1}{3} \tilde{u}^3 \right) dx + 2\gamma \int_R \left(\tilde{u}_x^2 - \frac{1}{3} \tilde{u}^3 \right) dx = \int_R \left(2\tilde{f}_x \tilde{u}_x + \frac{\gamma}{3} \tilde{u}^3 - \tilde{u}^2 \tilde{f} \right) dx \tag{6}$$

Now, we use the invariants of the KdV equation and set:

$$\tilde{K}_0(\tilde{u}) = 2 \int_R \tilde{f}(x) \tilde{u}(x) dx, \tilde{K}_1(\tilde{u}) = \int_R \left(2\tilde{f}_x \tilde{u}_x + \frac{\gamma}{3} \tilde{u}^3 - \tilde{u}^2 \tilde{f} \right) dx$$

Eqs.(5) ~ (6) can be written as:

$$\frac{d}{dt} I_j(\tilde{u}(t)) + 2\gamma I_j(\tilde{u}(t)) = \tilde{K}_j(\tilde{u}(t)) \tag{7}$$

We integrate (7) and find:

$$I_j(\tilde{u}(\tau)) + 2\gamma \int_0^\tau I_j(\tilde{u}(\tau)) = I_j(\tilde{u}_0) + \int_0^\tau \tilde{K}_j(\tilde{u}(\tau)) d\tau \tag{8}$$

for $t \in [-T, T]$ and $j = 0, 1$.

Taking the limit in (8) and using the continuity of the solution that we get in section 2, we have that the solution \tilde{u} with initial condition $\tilde{u}(0) = \tilde{u}_0$ and forcing term f converge in X_T , for some appropriate $T > 0$, to the solution $u \in X_T$, with the initial condition $u(0) = u_0$ and forcing term f . Using

$$I_j(u(t)) = \lim I_j(\tilde{u}(t)), K_j(u(t)) = \lim K_j(\tilde{u}(t))$$

for all $t \in [-T, T]$, where

$$K_0(u) = \int_R f(x)u(x)dx, K_1(u) = \int_R \left(2f_x u_x + \frac{\gamma}{3}u^3 - u^2 f\right)dx$$

We find $I_j(u(\tau)) + 2\gamma \int_0^t I_j(u(\tau))d\tau = I_j(u_0) + \int_0^t K_j(u(\tau))d\tau$. for all $t \in [-T, T]$ and $j = 1, 2$. From the energy-type equation (7), one can extend the solution u indefinitely and obtain a global solution $u = u(t), t \in R$, with $u \in X_T \cap C_b([-T, T]; H^1(R))$, for all $T > 0$. One can also check that for each $T > 0$ and each initial condition $u_0 \in H^1(R)$, there exists a constant $C(\gamma, \|f\|_{H^2(R)}, \|u_0\|_{H^1(R)}, T)$, such that $\|u\|_{X_T} \leq C$. Then, theorem 2 is obtained.

Owing to Theorem 2, we define a group associated with Eq.(1):

Definition 1 For $\gamma, \beta > 0, f \in H^2(R)$, we denote $\{S(t)\}_{t \in R}$, the group in $u_0 \in H^1(R)$ defined by $u_0 = u(t)$, where $u = u(t), t \in R$ is the unique solution of Eq.(1), which belongs to X_T for all $T > 0$.

5 Bounded Absorbing Sets and Global Attractor

In this section, we will obtain the existence of the bounded absorbing sets and the attractor of the solution operator $\{S(t)\}_{t \in R}$. We follow the standard procedure to obtain an absorbing ball in $H^1(R)$. By using Young's inequality, Cauchy-Schwartz inequality and (4), we find that:

$$\limsup_{t \rightarrow \infty} \|u(t)\|_{L^2(R)} \leq \rho_0 = \frac{1}{\gamma} \|f\|_{L^2(R)} \quad (9)$$

Proving absorbing ball in $H^1(R)$, we need to estimate $u_x(t)$. From Young's inequality and (2), we have:

$$\int_R |u(x)|^3 dx \leq \|u\|_{\infty} \|u\|_{L^2(R)}^2 \leq \|u\|_{X_T} \|u\|_{L^2(R)}^2 \leq \frac{C}{\gamma} \|f\|_{L^2(R)}$$

Therefore, from the energy-type equation (4), for $j = 1$,

$$\frac{d}{dt} I_1(u(t)) + 2\gamma I_1(u(t)) \leq 2C \|f_x\|_{L^2(R)} + \frac{C}{3} \|f\|_{L^2(R)} + \frac{1}{\gamma} \|f\|_{L^\infty(R)} \|f\|_{L^2(R)}$$

Applying the Grownwall Lemma, we find:

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|I_1(u)\|_{L^2(R)} &\leq \exp^{2\gamma} C^* \left(C, \|f\|_{L^2(R)}\right) + \exp^{2\gamma} C \frac{1}{\gamma} \|f_x\|_{L^2(R)} \\ &+ \exp^{2\gamma} \frac{C}{6} \frac{1}{\gamma} \|f\|_{L^2(R)} + \exp^{2\gamma} \frac{1}{2\gamma^2} \|f\|_{L^\infty(R)} \|f\|_{L^2(R)} \end{aligned}$$

Finally, by using the estimates above, we obtain:

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|u_x(t)\|_{L^2(R)}^2 &\leq \exp^{2\gamma} C^* \left(C, \|f\|_{L^2(R)}\right) + \exp^{2\gamma} C \frac{1}{\gamma} \|f_x\|_{L^2(R)} \\ &+ \frac{\exp^{2\gamma} C + 2C}{6\gamma} \|f\|_{L^2(R)} + \exp^{2\gamma} \frac{1}{2\gamma^2} \|f\|_{L^\infty(R)} \|f\|_{L^2(R)} = \rho_0^* \end{aligned}$$

so we have:

$$\limsup_{t \rightarrow \infty} \|u_x(t)\|_{L^2(R)}^2 \leq \rho_0^* \quad (10)$$

Then we have proven the result:

Lemma 6 Let $\gamma, \beta > 0, f \in H^2(R)$. The solution operator associated with Eq.(1) possesses a bounded absorbing set in $H^1(R)$ with the radius of absorbing ball given according to (9) and (10).

We assume $\gamma, \beta > 0, f \in H^2(R)$. $\{S(t)\}_{t \in R}$ is the solution operator associated with Eq.(1). We borrow the result from Ricardo Rosa in ([12]).

Lemma 7 ([12]) The solution operator $\{S(t)\}_{t \in R}$ is weakly continuous in $H^1(R)$. It means if u_{0n} converges weakly in $H^1(R)$ to some u_0 , as $n \rightarrow \infty$, then $S(t) u_{0n}$ converges to $S(t) u_0$ weakly in $H^1(R)$, for all $t \in R$.

With the previous Lemmas in mind, we can proceed as in I. Moise; R. Rosa and X. Wang ([11]). We let u_{0n} be bounded in $H^1(R)$, with $S(t'_n) u_{0n} \rightarrow u$, weakly in $H^1(R)$, for some subsequences $n' \rightarrow \infty$ and some $u \in H^1(R)$, with $\|u(t)\|_{L^2(R)} \leq \rho_0, \|u_x(t)\|_{L^2(R)} \leq \rho_1$. From above lemmas, it follows that for every $t \in R$, $S(t'_n - t) u_{0n} \rightarrow S(-t) u$ weakly in $H^1(R)$, as $n' \rightarrow \infty$ with: $\|S(-t) u\|_{L^2(R)} \leq \rho_0, \|S(-t) u_x\|_{L^2(R)} \leq \rho_1$. For $j = 0$, we use the energy-type equation to find:

$$\|S(t'_n) u'_{0n}\|_{L^2(R)}^2 = \|S(t'_n - t) u'_{0n}\|_{L^2(R)}^2 e^{-2T} + 2 \int_0^T e^{-2\gamma(T-\tau)} ((f, S(\tau) S(t'_n - T) u'_{0n}))_{L^2(R)} d\tau$$

Similarly, for the solution $S(\cdot) u$, we find:

$$\|u\|_{L^2(R)}^2 = \|S(-T) u\|_{L^2(R)}^2 e^{-2T} + 2 \int_0^T e^{-2\gamma(T-\tau)} ((f, S(\tau) S(-T) u))_{L^2(R)} d\tau$$

By above equations, taking the limit as $n' \rightarrow \infty$, we obtain:

$$\limsup_{n' \rightarrow \infty} \|S(t'_n) u_{0n'}\|_{L^2(R)}^2 \leq \|u\|_{L^2(R)}^2 + 2\rho_0^2 e^{-2\gamma t}$$

Letting $t \rightarrow \infty$, we find that $\limsup_{n' \rightarrow \infty} \|S(t'_n) u_{0n'}\|_{L^2(R)}^2 \leq \|u\|_{L^2(R)}^2$. Similarly, we also find for $j = 1$, there exists $\limsup_{n' \rightarrow \infty} \|(S(t'_n) u_{0n'})_x\|_{L^2(R)}^2 \leq \|u_x\|_{L^2(R)}^2$. Since $H^1(R)$ is a Hilbert space, there exists the strong convergence: $S(t'_n) u'_{0n} \rightarrow u$ in $H^1(R)$. Hence, we obtain theorem 3 as above.

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