Stabilization of Higher Periodic Orbits of Discrete-time Chaotic Systems

Guoliang Cai*, Weihuai Zhou, Zhenmei Tan
(Faculty of science, Jiangsu University, Zhenjiang, Jiangsu, 212013, P. R. China)
(Received 3 April 2007, accepted 29 June 2007, will be set by the editor)

Abstract. Two controller design methods for stabilizing higher periodic orbits of discrete-time chaotic systems, which are based on the invariant manifold theory and the sliding mode control concept, are presented. The first controller is a linear one designed by using locally linearized knowledge. The second one is constructed by states transformation and feedback linearization. Both of them are time-varying while the system is explicitly expressed based on the local knowledge about the points on the prescribed orbit. The effectiveness of the proposed methods are tested by numerical examples of the Hénon map for stabilizing orbit of period 7.

Keywords: higher periodic orbit; invariant manifold; sliding mode concept; chaotic systems; stabilization

1 Introduction

Stabilization of chaotic systems has been received much attention in the past two decades[1-4]. Several control schemes have been successfully established, which consist of attempting to stabilize a chaotic system to either the fixed points or the higher periodic orbits.

The well-known OGY-approach[5] was firstly proposed to stabilize unstable periodic orbits embedded in a chaotic attractor to a fixed stable point by a linear controller, which was designed only requiring local knowledge about the system on the fixed point. Recently, The local linear controller has been applied for the stabilization of higher periodic orbits of chaotic systems in [6-8]. For an orbit of the period \( p \geq 2 \) of the chaotic system, where symbol \( p \) denotes the period, a successive linearization around the points on the orbit gives rise to a time-varying system. In [6,7], the controller was restricted to be time-invariant, but which did not often exist. However, a time-varying controller was designed in [8], moreover, an extra degree of freedom to optimize the controller parameters was supported. Due to their design, taking only the linear parts of the system into account, the system errors (i.e. modeling error and external varying disturbances) will affect controlled system performance.

As a widely used robust control technique, sliding mode control has proven to be very effective to control systems with uncertainties by designing the sliding manifold. In the sliding mode, the system motion is independent of certain system parameter variations and external disturbances. Using sliding mode control concept, X. Yu, G. Chen et al[9] extended OGY method to deal with higher order chaotic systems, which not only removed the reliance of the controller on the system Jacobian eigenvalues and eigenvectors, but also was robust against the disturbances.

In this paper, based on the invariant manifold theory presented by [9], we present a new local linear controller for stabilizing the higher periodic orbits of discrete-time chaotic systems, which not only preserves the spirit of the method in [8] to assign arbitrary poles regardless of the number of positive Lyapunov exponents, but also can deal with the external varying disturbances.

* Corresponding author. Tel.: +86-511-8780164; Fax: +86-511-8780164. E-mail address: glcai@ujs.edu.cn

Copyright © World Academic Press, World Academic Union
IJNS.2007.10.15/101
Although the local linear controller is easily implemented, the complicated linearization computation and to determine the region of the effectivity of the controllers are drawback. Yet there exists modeling errors. In order to overcome these problems, another efficient controller different from the local linear one is proposed in section 3. Instead of linear approximation of the system dynamics, a states transformation concept, which is insensitive to the external disturbances.

This paper is organized as follows. The new local linear controller is presented and used to stabilize the Hénon system orbit of period 7 in section 2, and the controller is designed based on feedback linearization with sliding mode control concept in section 3, which is used to the Hénon system, as well. In section 4, we summarize that two control methods for their advantages and disadvantages.

2 The local linear sliding mode controller

Consider a discrete-time nonlinear chaotic system in the general form,

$$x(k + 1) = f(x(k), u(k))$$

(1)

where the states vector $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a nonlinear function, and $u(k)$ is a scalar input. In general, a periodic orbit $O = \{\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_p\}$ of system (1) can be determined by the following sequence of points:

$$\begin{align*}
\tilde{x}_2 &= f(\tilde{x}_1) \\
\tilde{x}_3 &= f(\tilde{x}_2) = f^2(\tilde{x}_1) \\
&\cdots \\
\tilde{x}_p &= f(\tilde{x}_{p-1}) = f^{p-1}(\tilde{x}_1), \\
\tilde{x}_{p+1} &= f^p(\tilde{x}_1) = \tilde{x}_1
\end{align*}$$

(2)

where the $p$ vectors $\tilde{x}_i$ are distinct, the period $p$ of the orbit is the length of the orbit.

Local linear stabilization of all $p$ points on the periodic orbit brings on a time-variant representation of the linearized, controlled system:

$$\Delta x(k + 1) = A_k \Delta x(k) + B_k u(k) + d(k)$$

(3)

where $\Delta x(k) = x(k) - \tilde{x}_k$, $d(\cdot) \in \mathbb{R}^n$ is the vector of disturbances, consisting of external disturbances and modeling errors.

$$A_k = \left. \frac{\partial}{\partial x} f(x) \right|_{\tilde{x}_k} = \begin{bmatrix} a_{11,k} & a_{12,k} & \cdots & a_{1n,k} \\ a_{21,k} & a_{22,k} & \cdots & a_{2n,k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1,k} & a_{n2,k} & \cdots & a_{nn,k} \end{bmatrix}$$

(4)

$$B_k = \begin{bmatrix} b_{1,k} & b_{2,k} & \cdots & b_{n,k} \end{bmatrix}^T$$

(5)

and the elements $a_{ij,k}$ and $b_{ik}$ are time-varying. The index $k$ stands for time, while $i$ and $j$ indicate the position in the system-matrix and input vector respectively, and $u(k)$ is the control input. For each of the points on the orbit, we can choose the different control vector $B_k$.

According to Eq.(2), due to the periodicity of the desired trajectory, the following relationship holds:

$$A_{k+p} = A_k = A_1, \quad A_{k+2p-1} = A_{k+p-1} = A_p$$

(6)

Assumption 1. The pair $(A_k, B_k)$ is completely controllable.

Assumption 2. The matching condition holds for this system (3), which implies that it exists $\varsigma(k) \in \mathbb{R}$, such that $d(k) = B_k \varsigma(k)$.
Due to the freedom for the design of the time-varying controller, generally, we can select appropriate vector $B_k$ to meet the two assumptions above, commonly. For the sake of simplify, we assume the same relationship holds:

$$B_{k+p} = B_k = B_1, \ldots, B_{k+2p-1} = B_{k+p-1} = B_p$$

(7)

Therefore, Eq. (3) can be rewritten in the discrete-time form,

$$\Delta x(k+1) = A_k \Delta x(k) + B_k u(k) + B_k \varsigma(k)$$

(8)

where $\varsigma(k)$ is the generalized disturbance vector.

In [8], the linear state feedback $\Delta q_k = k^T \Delta x(k)$ is used to stabilize system dynamics, when $p$ proper gain vectors are chosen for $p$ points of the periodic orbit. When $\varsigma(k)$ exists and varies, the parametric control input $\Delta q(k)$ is difficult to stabilize the system to given orbit. Here, we make use of the sliding mode control concept to stabilize the system dynamics, which not only preserves the spirit of the method in [8] to assign arbitrary poles, but also is robust against to the disturbances.

With assumption 1, an equivalent transformation $T_k$ is proposed by [8] to transform discrete-time periodic system to the controller canonical form.

$$\Delta x_c(k) = T_k \Delta x(k)$$

(9)

$$T_k = \begin{bmatrix} t_{1,1}^T & t_{1,2}^T & t_{1,3}^T & \cdots & t_{1,n}^T \\ t_{2,1}^T & t_{2,2}^T & t_{2,3}^T & \cdots & t_{2,n}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n,1}^T & t_{n,2}^T & t_{n,3}^T & \cdots & t_{n,n}^T \end{bmatrix}^T = \begin{bmatrix} t_{1,1}^T & t_{1,1}^T A_k & t_{1,2}^T A_k & \cdots & t_{1,n-1}^T A_k \\ t_{2,1}^T A_{p+1} & t_{2,2}^T A_{p+1} & \cdots & \prod_{j=1}^{n-1} A_{p+j-n} B_{p+i-n} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$$

(10)

The transformed system is in the controller canonical form:

$$\Delta x_c(k+1) = A_{C,k} \Delta x_c(k) + B_{C,k} u(k) + B_{C,k} \varsigma(k)$$

(12)

with

$$A_{C,k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_{c1,k} & a_{c2,k} & a_{c3,k} & \cdots & a_{cn,k} \end{bmatrix}$$

(13)

and

$$B_{C,k} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T$$

(14)

The transformed system matrix and the transformed input-vector are given in Eqs. (15)-(16).

$$A_{C,k} = T_{k+1} A_k T_k^{-1}$$

(15)

$$B_{C,k} = T_{k+1} B_k$$

(16)

First, we construct an invariant manifold for chaotic system, on which the system orbit is prescribed independently of the system Jacobians. It is represented by

$$s(k) = C \Delta x_c(k)$$

(17)

where $C$ is a control vector. We choose $C = [ C_1 \ 1 ]$, $C_1^T \in \mathbb{R}^{n-1}$. When the system states are on the invariant manifold, the control input and the controlled system can be obtained.

---

*IJNS email for contribution: editor@nonlinearscience.org.uk*
u(k) = -CAc,\Delta x_c(k) - \varsigma(k) \tag{18}

\Delta x_c(k + 1) = (AC,k - BC,kCA,k)\Delta x_c(k) \tag{19}

Compared to Eq. (2), when the system trajectory is on the desired periodic orbit, the linearized, controlled system holds as follows:

\Delta x_c(k + p) = \left( \prod_{i=1}^{p} (AC,i - BC,iCA,i) \right)\Delta x_c(k) \tag{20}

According to the design process simplified by [8], we order

\begin{align*}
AC,i - BC,iCA,i &= F, \forall i = 1, 2, \ldots, p \tag{21}
\end{align*}

where $F$ is some time-invariant, i.e. constant matrix, and it satisfies,

|\text{eig}(F)| < 1 \tag{22}

where “eig” stands for the eigenvalues of the $F$ matrix.

Since $\text{eig}(FP) = (\text{eig}(F))^p$, it is sufficient to guarantee that

$$|\text{eig} \left( \prod_{i=1}^{p} (AC,i - BC,iCA,i) \right)| = |\text{eig}(FP)| < 1.$$  

Therefore, the Eq. (20) can be replaced as:

$$\Delta x_c(k + i) = F^i\Delta x_c(k) \tag{23}$$

Since each $T_i$ is bounded and nonsingular (otherwise $AC,k$ would not exist), when Eq. (9) is recursively applied to itself leading to:

$$\Delta x(k + 1) = T_i^{-1}F_iT_p\Delta x(k) \tag{24}$$

Finally, after one period, i.e. $i = p$, the stability of the closed-loop system in the original domain is obvious since

$$\Delta x(k + p) = T_p^{-1}F_pT_p\Delta x(k) = \tilde{F}_p\Delta x(k) \tag{25}$$

i.e. $\text{eig}(\tilde{F}_p) = (\text{eig}(F))^p$ due to the similarity transformation.

Choose $C_1 = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} \end{bmatrix}$, when the system orbit lies on the invariant manifold, i.e. $s(k) = 0$, the matrix $F$ can be designed as:

$$F = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 1 \\
0 & -c_1 & -c_2 & \cdots & -c_{n-1}
\end{bmatrix}_{n \times n} \tag{26}$$

where $c_i \in R$.

The eigenvalues $\lambda_i$ of $F$ are performed by

$$z^n + \sum_{i=1}^{n-1} c_i z^i = \prod_{i=1}^{n-1} (z - \lambda_i) \tag{27}$$

According to Eq.(22), it satisfies $|\lambda_i| < 1$, $i=1,2,\ldots,n-1$. When $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda_n = 0$, then the controller (18) will be a dead-beat controller.

Therefore, the method based on invariant manifold theory preserves the freedom of the design of the controller, meanwhile, it is effectively against disturbances.

IINS homepage: http://www.nonlinearscience.org.uk/
The controller in the original domain is defined in Eq. (28).

\[
    u(k) = -CT_{k+1}A_k \Delta x(k) - \varsigma(k)
\]  

(28)

Usually, the uncertainty term \( \varsigma(k) \) is not known at the \( k \)th sampling time, which means that the invariant manifold can not be exactly maintained. To ensure to close the invariant manifold enough, Su et al [10] approximated \( \varsigma(k) \) by \( \varsigma(k-1) \) with the assumption that \( \varsigma(k) \) is a smooth function. The control input can be calculated by

\[
    u(k) = -CT_{k+1}A_k \Delta x(k) - \varsigma(k-1)
\]  

(29)

**Example for Hénon system by the local linear sliding mode controller**

The Hénon system is described by

\[
    \begin{align*}
        x_1(k + 1) &= 1 - px_1^2(k) + x_2(k) + u(k) + d(k) \\
        x_2(k + 1) &= qx_1(k)
    \end{align*}
\]

(30)

where \( p \) and \( q \) are real parameters, \( d(k) \) is the external disturbance. This system exhibits chaotic behavior in a large neighborhood of the parameter values \( p=1.4 \) and \( q=0.3 \), when \( u(k) = 0 \) and \( d(k) = 0 \).

The around \( \tilde{x}_i \) linearized system is

\[
    \Delta x(k + 1) = \begin{bmatrix} -2.8x_{1,k} & 1 \\ 0 & 0 \end{bmatrix} \Delta x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \varsigma(k)
\]

(31)

with \( \Delta x(k) = x(k) - \tilde{x}_i \).

We simulate the proposed algorithm to the periodic orbit of the Hénon map with period 7, which is given by [9] as:

\[
    O=((0.8191, 0.1477), (0.2153, 0.2457), (1.1808, 0.0646), (-0.8876, 0.3542), (0.2514, -0.2663), (0.6453, 0.0754), (0.4925, 0.1936)).
\]

The transformation matrix is given in [9]:

\[
    T = \begin{bmatrix} 0 & 10/3 \\ 1 & 0 \end{bmatrix}
\]

(32)

We design the control input \( u(k) \) as Eq. (28). \( C = [ \begin{array}{c} 0 \\ 1 \end{array} ] \) is choose to optimize the controller as a dead-beat one, when \( F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \).

Assuming the initial state is already positioned within the contracting region of the higher periodic orbit.

The region can be obtained by [11], which is bounded by \( |\Delta x(k)| < 0.68 \).

To consider the effects of measurement noise, a varying distributed noise term \( \varsigma \) with the intensity \( \varsigma = \cos(2\pi k) \) is added to the state variable \( x_1 \) of the Hénon map. Fig.1 shows that the system state \( x_1 \) is stabilized on the desired orbit, which is the same as in [8] although the system is disturbed by \( \varsigma \). When the disturbances term \( \varsigma(k) \) is not known at \( k \)th sampling time, but \( \varsigma(k) \) is a smooth and time varying matched disturbance vector, which is 0.1 bigger than \( \varsigma(k-1) \) in our simulation. Clearly, the controlled system states are stabilized on an orbit which departs from the desired one slightly in Fig.2.

3 The Feedback linearized sliding mode controller

Consider the following nonlinear system,

\[
    \begin{align*}
        x(k + 1) &= f(x(k), u(k)) \quad k = 0, 1, 2, \ldots \\
        y(k) &= h(x(k))
    \end{align*}
\]

(33)

where \( k = 0, 1, \ldots \) is the discrete-time index, \( x(k) \in \mathbb{R}^n \) is the vector of state variables, \( u(k) \in \mathbb{R} \) is the input variable and \( y(k) \in \mathbb{R} \) is the output variable. It is assumed that \( f(x, u) \) is a real analytic vector function on \( \mathbb{R}^n \times \mathbb{R} \), and the output map \( h(x) \) is a real analytic scalar function on \( \mathbb{R}^n \).
rewritten as follows:

\[(33)\] therefore, it can be used as a local coordinate transformation. The new coordinate system \((33)\) may be

\[\begin{align*}
h^0(x) &= h(x) \\
h^k(x, u) &= h^{k-1} \circ f(x, u)
\end{align*}\]

with \(k \geq 1\). The relative degree \(r\) of system \((33)\) in the neighborhood of \(x_0\) is defined as the smallest integer

\[\frac{\partial}{\partial u} h^r(x, u) |_{x_0} \neq 0\]

on \((R^n \times R)\). The relative degree \(r\) determines the time delay of the input signal \(u\) before it can influence

\[\text{Figure 1: The system state } x_1 \text{ is stabilized on the desired orbit when it is disturbed by } \cos(2\pi k).\]

\[\text{Figure 2: The system states } x_1 \text{ is stabilized when it is disturbed by unknown term bounded by } \varsigma(k) - \varsigma(k-1) = 0.1.\]

**Definition**[12] For system \((33)\), let us denote by \(\circ\) the usual composition of functions and recursively define the following functions:

\[h^0(x) = h(x)\]

\[h^k(x, u) = h^{k-1} \circ f(x, u)\]

\[\text{(34)}\]

where \(z_i = \phi_i(x) = h^{i-1}(x, u), (i = 1, \ldots, n - r)\) has a nonsingular Jacobian matrix at the point \(x_0\) of system

\[\text{(33)}\], therefore, it can be used as a local coordinate transformation. The new coordinate system \((33)\) may be rewritten as follows:

\[z_1(k+1) = z_2(k)\]

\[\ldots\]

\[z_{r-1}(k+1) = z_r(k)\]

\[z_r(k+1) = \varphi(z(k), u(k))\]

\[\pi(k+1) = \eta(z(k), \pi(k), u(k))\]

\[y(k) = z_1(k)\]

\[\text{(37)}\]

with \(z = (z_1, z_2, \ldots, z_r), \pi = (\pi_1, \pi_2, \ldots, \pi_{n-r})\). the function \(\pi(k)\) shows the internal dynamics of the

\[\text{system.}\]

If \(r = n\), the system can be exact-transformed by \(\Phi(x)\), where \(z = (z_1, z_2, \ldots, z_n)\).

Assume that the mapping \(f(x, u)\) can be decomposed as: \(\alpha(x(k)) + \beta(x(k))u(k)\), then \(z_r(k+1)\) can be rewritten as

\[z_r(k+1) = \varphi(z(k), u(k)) = a(z(k)) + b(z(k))u(k)\]

\[\text{(38)}\]

where \(a(z(k)) = h^{r-1} \circ \alpha(\Phi^{-1}(z)), b(z(k)) = h^{r-1} \circ \beta(\Phi^{-1}(z))\).
When the output is constrained to a certain value, i.e. \( h(x) = y_0 \), according to (37), we can easily deduce the values of \( z_i(k) \), for all \( k \in N^+(i = 1, \ldots, r) \). Correspondingly, we denote the corresponding value of \( z_i(k) \) as \( y_0(i = 1, 2, \ldots, r) \), \( \xi = [ y_0 \quad y_0^2 \quad \ldots \quad y_0^T] \).

We consider a nonlinear chaotic system which is given a desired periodic orbit \( O = \{ \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_p \} \). The objective is to stabilize the system states to desired points \( \tilde{x}_i \) in the periodic form. In general, the controlled chaotic system can be expressed as system (33). Then, we can transform the system into Eq. (37) in the neighborhoods of the desired points of the periodic orbit. According to analysis above, the stabilization can be converted as an output tracking. When the output is relative to the desired points of the periodic orbit, the stabilization of higher periodic orbit can be transform to output tracking. In other words, \( y(k) \) is equal to a prescribed value \( \xi_i \), the system trajectory is stabilized on a desired point \( \tilde{x}_i \), for the length of the orbit is \( p \), the output will equal to \( p \) prescribed values \( \{ \xi_1, \ldots, \xi_p \} \) in turn. Therefore, the key is to choose a proper output function \( y(k) = z_1(k) \).

First, we consider the chaotic system denoted as system (37) on a desire point of the periodic orbit through the coordinate transformation. We choose the invariant manifold as,

\[
s(k) = C[z(k) - \xi_i]
\]  

(39)

where \( C = [C_1, \ldots, C_r] \in R^{1 \times (r-1)} \).

When \( s(k + 1) = 0 \), substitute the front \( r \) Eqs. of (36) into (39),

\[
u(k) = -\frac{1}{b(z(k))} \left[ \begin{array}{c} 0 \\ C_1 \end{array} \right] z(k) + a(z(k)) - C_1 \xi_i
\]

(40)

where \( C_1 = \begin{bmatrix} c_1 & c_2 & \cdots & c_{r-1} \end{bmatrix} \), \( c_i(i = 1, \ldots, r - 1) \) are chosen so that the following polynomial of \( z \),

\[
  H(z) = z^{r-1} + c_1 z^{r-2} + \cdots + c_{r-1}
\]

(41)

is Hurwitz, i.e. the roots of the equation \( H(z) = 0 \) lie inside the unit circle in the \( z \)-plane.

In order to ensure the whole system controlled by input (40) asymptotically stabilize as \( r < n \), an additional condition that the internal dynamics is exponentially asymptotically stabilized at the point \( \tilde{x}_i \) is to maintain internal stability. The evolution of the internal dynamics is described by:

\[
  \pi(k + 1) = \eta(\xi, \pi(k), u(k))
\]

(42)

where \( u(k) \) is calculated by Eq. (40).

The controller in the original domain is obtained as

\[
u(k) = -\frac{1}{b(\Phi^{-1}(x(k)))} \left[ \begin{array}{c} 0 \\ C_1 \end{array} \right] \phi^{-1}(x(k)) + a(\Phi^{-1}(x(k))) - C_1 \xi_i
\]

(43)

We notice that this class of control methods, based on invariant manifold, are robust against certain external disturbance(i.e., the matching condition holds [13]). Due to variable term, \( \beta(x(k)) \) can be determined freely, so we can choose a proper control input to satisfy the matching condition. When the external disturbance cannot be measured at the \( k \) th sampling time, ensure that the orbit stay as close as possible to the invariant manifold, one can choose a control consisting of a time delayed term of the external disturbance under the assumption that the disturbance is slowly varying.

**Example for Hénon system by the feedback linearized sliding mode controller**

Consider the Hénon system again. Define \( y(k) = x_1(k) \) as the system’s output so that the controlled system becomes

\[
x_1(k + 1) = 1 - px_1^2(k) + x_2(k)
\]

\[
x_2(k + 1) = qx_1(k) + u(k)
\]

\[
y(k) = x_1(k)
\]

(44)

Then, the state equations and output equation of the system (44),can be represented as follows,
\[
\alpha(x(k)) = \begin{bmatrix} 1 - px_1^2(k) + x_2(k) \\ qx_1(k) \end{bmatrix}
\]
\[
\beta(x) = 1
\]
\[
h(x(k)) = x_1(k)
\]

For system (36), we obtain \( \frac{\partial}{\partial u} h \circ f = 0, \frac{\partial}{\partial u} h \circ f^2 \neq 0 \), Thereby the relative degree of the Hénon system is equal to 2, which is equal to the system states dimension. Correspondingly, the coordinate transformation for the system is given as follows,

\[
z_1(k) = x_1(k) \\
z_2(k) = 1 - px_1^2(k) + x_2(k)
\]

From Eq. (37), we obtain the normal form,

\[
\begin{bmatrix} z_1(k + 1) \\ z_2(k + 1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xi(k)
\]

Assume \( \tilde{x}_i = (\tilde{x}_{i,1}, \tilde{x}_{i,2}) \) is one of the points on the desired orbit, when the output \( y(k) \) is equal to \( \tilde{x}_{i,1} \), then \( y_i^1 = \tilde{x}_{i,1}, y_i^2 = 1 - px_1^2 + \tilde{x}_{i,2}, \xi_i = [y_i^1, y_i^2] \).

Here, we design \( s(k) \) as Eq. (39), and the control input \( u(k) \) as Eq. (43).

\[
u(k) = C(\xi_i - a(z(k))) - C \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}
\]

A varying distributed noise term \( d(k) \) with the intensity \( d(k) = \cos(2\pi k) \) is added to the state variable \( x_2 \) of the Hénon map, which satisfies the matching condition. The Fig.3 shows the system states \( x_1 \) is stabilized on the scheduled orbit which is the same as Fig.1. When the disturbance term is unknown at \( k \) th sampling time, but \( d(k) \) is a smooth and time varying matched disturbance vector. We assume that \( d(k) \) is bigger than \( d(k - 1) \) in our simulation, which is bounded by 0.1. Clearly, the controlled system states are stabilized on an orbit which departs from the desired one a little in Fig.4.

![Figure 3](image1.png)  
![Figure 4](image2.png)

Figure 3: The system state \( x_1 \) is stabilized on the desired orbit when system state \( x_2 \) is disturbed by \( \cos(2\pi k) \).  
Figure 4: The system state \( x_1 \) is stabilized when system state \( x_2 \) is disturbed by unknown term bounded by \( \varsigma(k) - \varsigma(k - 1) = 0.1 \).

## 4 Conclusion

The two different controllers designed above are time-variant, while the different control inputs are supported for the different points on the desired orbit. The linear one can be implemented easily, but complicated computation is its drawback. The other is a nonlinear one, which is designed simply and rigorously, but requires more global knowledge of the system. Meanwhile, a proper output should be chosen. Both

IJNS homepage: http://www.nonlinearscience.org.uk/
of them are designed based on invariant manifold theory, and the numerical simulations for Hénon system demonstrate the closed-loop system’s performance and robustness. The two methods proposed for stabilizing higher period orbit all could extend to high dimension chaotic systems.

Acknowledgments

The work is supported by the National Nature Science Foundation of China (Grant No. 70571030 ) and the Advanced Talents’ Foundation of Jiangsu University (No. 07JBG054).

References


