Global Synchronization for Time-delay of HCSA System

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(Received 31 October 2006, accepted 9 January 2007)

Abstract: Considering a time-delay in the receiver as compared with the transmitter, a practical issue of global chaos synchronization is proposed to the HCSA system which is based on the Lyapunov stabilization theory and matrix measure. By choosing proper coupling parameters, the error system is globally asymptotical stable, such that the state of the slave system at time \( t + \tau \) is asymptotically synchronizing with the master at time \( t \). The Mathematical software is used to simulate. Both theoretical analysis and simulation result show the effectiveness and anti-interference ability of this method.

Key words: HCSA system, linear feedback functional method, channel time-delay, global chaos synchronization

1 Introduction

Chaos synchronization is one of the most interesting problems in nonlinear sciences [1]-[7]. It can be applied to many fields, such as secure communication, human heartbeat regulation and ecological systems. Paper [8]-[9] study the chaos synchronization, where the unavoidable signal propagation delays in the typical master-slave synchronization. In the practical environment with signal propagation delays, it is not reasonable to require the slave system to synchronize the master system at exactly the same time. It is just like the telephone communication systems, where one hears the voice at time \( t + \tau \) on the receiver side, which was said from the transmitter side some time ago, at time \( t \). For this reason, in paper [8], chaotic synchronization is re-defined in such a way that the state of the slave system at time \( t + \tau \) is asymptotically synchronizing with the master at time \( t \), namely,

\[
\lim_{t \to \infty} \left\| X(t) - \tilde{X}(t + \tau) \right\| = 0
\]

where \( X \) and \( \tilde{X} \) are the state of the master and slave systems, respectively. Consequently, the control signal uses the output error between the drive system at time \( t \) and the slave system at time \( t + \tau \), i.e., \( X(t) - \tilde{X}(t + \tau) \).

The HCSA system is a complex dynamical system, which exhibits a variety of dynamical states. Paper [10] describes the HCSA system. In this paper, we address a scheme on chaos synchronization of HCSA system where the unavoidable signal propagation delays. Using Lyapunov stabilization theory we study the synchronization of HCSA system, give the numerical results and disturb to the third variable of the system \( z(t) \). Finally, conclusion remarks are given.

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IJNS.2007.06.15/081
2 Time-delay chaotic synchronization

Consider a chaotic continuous system described by

\[
\dot{X}(t) = AX(t) + g(X(t)) \tag{1}
\]

where \( X(t) \in \mathbb{R}^n \) is the state vector, \( A \in \mathbb{R}^{n \times n} \) is constant matrix, and \( g(X) \) is a continuous nonlinear function. Assume that

\[
g(X(t_1)) - g(\tilde{X}(t_2)) = M_{X,\tilde{X}}(X(t_1) - \tilde{X}(t_2)) \tag{2}
\]

for a bounded matrix \( M_{X,\tilde{X}} \) with elements depending on \( X(t_1) \) and \( \tilde{X}(t_2) \). At this point, it is important to note that most chaotic systems can be described by (1) and (2) in paper [8]. From the typical unidirectional linear error feedback coupling approach, and taking the time-delay into account, the slave system based on the chaotic system (1) is described by (see Fig.1)

\[
\dot{\tilde{X}}(t + \tau) = A\tilde{X}(t + \tau) + g(\tilde{X}(t + \tau)) + L(X(t) - \tilde{X}(t + \tau)) \tag{3}
\]

where \( \tau \) is a finite time-delay which is an unknown constant, and \( L \in \mathbb{R}^{n \times m} \) is the coupling matrix to be designed to achieve synchronization.

![Figure 1: Synchronization with time-delay](IJNS homepage: http://www.nonlinearscience.org.uk/)
From [10] we learn: when \( a = 1.27 \), the Lyapunov exponents are \((0.7899, 0, -4.5999)\) then system (5) is chaotic. (see Fig.2). The following slave system is constructed by using (3) for (5)

\[
\begin{align*}
\hat{x}(t + \tau) &= -a\hat{x}(t + \tau) - 4\hat{y}(t + \tau) - 4\hat{z}(t + \tau) - \hat{y}^2(t + \tau) + l_1(x(t) - \hat{x}(t + \tau)) \\
\hat{y}(t + \tau) &= -a\hat{y}(t + \tau) - 4\hat{x}(t + \tau) - 4\hat{y}(t + \tau) - \hat{z}^2(t + \tau) + l_2(y(t) - \hat{y}(t + \tau)) \\
\hat{z}(t + \tau) &= -a\hat{z}(t + \tau) - 4\hat{x}(t + \tau) - 4\hat{y}(t + \tau) - \hat{z}^2(t + \tau) + l_3(z(t) - \hat{z}(t + \tau))
\end{align*}
\]

(6)

Subtracting (6) from (5), we obtain

\[
\begin{align*}
\dot{e}_x(t) &= -ae_x(t) - 4e_y(t) - 4e_z(t) - \left[ y^2(t) - \hat{y}^2(t + \tau) \right] - l_1e_x(t) \\
\dot{e}_y(t) &= -ae_y(t) - 4e_x(t) - 4e_z(t) - \left[ z^2(t) - \hat{z}^2(t + \tau) \right] - l_2e_y(t) \\
\dot{e}_z(t) &= -ae_z(t) - 4e_x(t) - 4e_y(t) - \left[ x^2(t) - \hat{x}^2(t + \tau) \right] - l_3e_z(t)
\end{align*}
\]

where

\[
\begin{align*}
e_x(t) &= x(t) - \hat{x}(t + \tau), \\
e_y(t) &= y(t) - \hat{y}(t + \tau), \\
e_z(t) &= z(t) - \hat{z}(t + \tau),
\end{align*}
\]

which can be rewritten as

\[
\dot{e}(t) = Ae(t) + g(X(t)) - g \left( \hat{X}(t + \tau) \right) - Le(t)
\]

(7)

where

\[
A = \begin{bmatrix}
-a & -4 & -4 \\
-4 & -a & -4 \\
-4 & -4 & -a
\end{bmatrix},
\]

\[
L = \begin{bmatrix}
l_1 & 0 & 0 \\
0 & l_2 & 0 \\
0 & 0 & l_3
\end{bmatrix},
\]

\[
e(t) = \begin{bmatrix}
x(t) - \hat{x}(t + \tau) \\
y(t) - \hat{y}(t + \tau) \\
z(t) - \hat{z}(t + \tau)
\end{bmatrix},
\]

\[
g(X) = \begin{bmatrix}
y^2 \\
z^2 \\
x^2
\end{bmatrix}
\]
Then
\[
    g(X(t)) - g(\dot{X}(t + \tau)) = \begin{bmatrix}
        y^2(t) - \tilde{y}^2(t + \tau) \\
        z^2(t) - \tilde{z}^2(t + \tau) \\
        x^2(t) - \tilde{x}^2(t + \tau)
    \end{bmatrix}
\]
\[
= \begin{bmatrix}
    0 & y(t) + \tilde{y}(t + \tau) & 0 \\
    0 & z(t) + \tilde{z}(t + \tau) & 0 \\
    x(t) + \tilde{x}(t + \tau) & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    x(t) - \tilde{x}(t + \tau) \\
    y(t) - \tilde{y}(t + \tau) \\
    z(t) - \tilde{z}(t + \tau)
\end{bmatrix}
\]
\[
= M_{X,\tilde{X}} e(t)
\]
where
\[
M_{X,\tilde{X}} = \begin{bmatrix}
    0 & y(t) + \tilde{y}(t + \tau) & 0 \\
    0 & z(t) + \tilde{z}(t + \tau) & 0 \\
    x(t) + \tilde{x}(t + \tau) & 0 & 0
\end{bmatrix}
\]

**Theorem 1** If there exists a positive definite symmetric constant matrix such that
\[
(A - L + M_{X,\tilde{X}})^T P + P (A - L + M_{X,\tilde{X}}) \leq \mu I < 0
\]
uniformly for all in the phase space, where is a negative constant and is the identity matrix, then the error system (7) is globally exponentially stable about zero, implying that the two systems (5) and (6) are globally asymptotically synchronized.

**Proof.** Choose a Lyapunov function of the form
\[
V = e(t)^T P e(t)
\]
where is a positive definite symmetric constant matrix. Its derivative is
\[
\dot{V} = \dot{e}(t) P e(t) + e(t)^T P \dot{e}(t)
\]
\[
= [(A - L + M_{X,\tilde{X}}) e(t)]^T P e(t) + e(t)^T P [(A - L + M_{X,\tilde{X}}) e(t)]
\]
\[
= e(t)^T [(A - L + M_{X,\tilde{X}})^T P + P (A - L + M_{X,\tilde{X}})] e(t) \leq \mu \| e(t) \|^2 < 0
\]
where \(\| \|\) denotes the Euclidean norm.

Based on the Lyapunov stability theory, system (7) is globally exponentially stable about zero. Hence the two systems (5) and (6) are globally asymptotically synchronized. □

Theorem 1 proves the effectiveness of the method, next work is to determine .

Let
\[
Q = (A - L + M_{X,\tilde{X}})^T P + P (A - L + M_{X,\tilde{X}}), \lambda_i, i = 1, 2, 3 \text{ are the eigenvalues of } Q. \text{ If inequality } (9) \text{ holds, then}
\]
\[
\lambda_i \leq \mu < 0, \quad i = 1, 2, 3
\]

So we choose , so as to satisfy (10).

**Theorem 2** Choose \(P = \text{diag}(p_1, p_2, p_3), \text{ and let}
\]
\[
P \left( A + M_{X,\tilde{X}} \right)^T + \left( A + M_{X,\tilde{X}} \right)^T P = [\bar{a}_{ij}]
\]

If a suitable \(L\) is chosen such that
\[
l_i \geq \frac{1}{2p_i} (\bar{a}_{ii} + R_i - \mu), \quad i = 1, 2, 3 \quad \text{and} \quad R_i = \sum_{j=1, j \neq i}^{3} |\bar{a}_{ij}|
\]

then the two coupled chaotic systems (7) and (8) are globally synchronized.

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Proof:

\[ Q = \left( A - L + M_{X,\tilde{X}} \right)^T P + P \left( A - L + M_{X,\tilde{X}} \right) \]
\[ = P \left( A + M_{X,\tilde{X}} \right) + \left( A + M_{X,\tilde{X}} \right)^T P - LP - PL \]
\[ = P \left( A + M_{X,\tilde{X}} \right) + \left( A + M_{X,\tilde{X}} \right)^T P - 2LP \]

From Gerschgorin’s theorem in matrix theory we know, every eigenvalue is in its Gerschgorin, i.e. \( \lambda_i - \bar{a}_{ii} - 2l_i p_i \leq R_i \), while \( \mu \geq \lambda_i \), it means \( \mu \) is out of all Gerschgoeins, so \( \mu - \bar{a}_{ii} - 2l_i p_i \geq R_i \), then we get following inequalities,

\[ l_i \geq \frac{1}{2p_i} (\bar{a}_{ii} + R_i - \mu) \]

In particular, we choose \( P = I \), we obtain

\[ \left( A + M_{X,\tilde{X}} \right) + \left( A + M_{X,\tilde{X}} \right)^T \]
\[ = \begin{bmatrix}
-a & -4 + y + \tilde{y} & -4 \\
-4 & -a & -4 + z + \tilde{z} \\
-4 + x + \tilde{x} & -4 & -a
\end{bmatrix} \begin{bmatrix}
-a & -4 & -4 + x + \tilde{x} \\
-4 + y + \tilde{y} & -a & -4 \\
-4 & -4 + z + \tilde{z} & -a
\end{bmatrix} \]
\[ = \begin{bmatrix}
-2a & -8 + y + \tilde{y} & -8 + x + \tilde{x} \\
-2a & -8 + y + \tilde{y} & -8 + x + \tilde{x} \\
-8 + x + \tilde{x} & -8 + z + \tilde{z} & -2a
\end{bmatrix} \]

We can choose

\[ l_1 \geq 1/2 (-2a + |-8 + y + \tilde{y}| + |-8 + x + \tilde{x}| - u) \]
\[ l_2 \geq 1/2 (-2a + |-8 + y + \tilde{y}| + |-8 + z + \tilde{z}| - u) \]
\[ l_3 \geq 1/2 (-2a + |-8 + x + \tilde{x}| + |-8 + z + \tilde{z}| - u) \]

From Fig.1, we can get \(-15 < x < 5, -15 < y < 5, -15 < z < 5\), so we choose \( x = -5, y = 0, z = 0, \tilde{x} = -4, \tilde{y} = 1, \tilde{z} = 0.5 \) and choose \( \mu = -1 \), we can obtain \( l_1 = 13, l_2 = 21, l_3 = 18 \), the inequality (3) holds. According to Theorem 1, the two chaotic systems (5) and (6) are globally asymptotically synchronized (see Fig. 3-5).

![Figure 3: Time evolution of valuable \( e_x(t) \)](image)

![Figure 4: Time evolution of valuable \( e_y(t) \)](image)

**Theorem 3** The two chaotic systems (1) and (3) are globally asymptotically synchronized, if one condition is satisfied at least as follows:

1. \[ \max_j \{ \bar{a}_{jj} + \sup_{i=1,i\neq j} |\bar{a}_{ij}| \} < 0 \]
2. \[ \max_i \{ \bar{a}_{ii} + \sup_{j=1,j\neq i} |\bar{a}_{ij}| \} < 0 \]

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Figure 5: Time evolution of valuable $e_z(t)$

Figure 6: Time evolution of valuable $e_y(t)$

where $A - L + M_{\tilde{X}, \tilde{X}} = (\tilde{a}_{ij})_{3 \times 3}$.

In this paper, we have

$$A - L + M_{\tilde{X}, \tilde{X}} = \begin{bmatrix} -a - l_1 & -4 + y + \tilde{y} & -4 \\ -4 & -a - l_2 & -4 + z + \tilde{z} \\ -4 + x + \tilde{x} & -4 & -a - l_3 \end{bmatrix}$$

We choose $l_1$, $l_2$, $l_3$ to satisfy

$$\max \{-a - l_1 + 4 + | -4 + x + \tilde{x}|, -a - l_2 + 4 + | -4 + y + \tilde{y}|, -a - l_3 + 4 + | -4 + z + \tilde{z}|\} < 0$$

or

$$\max \{-a - l_1 + 4 + | -4 + y + \tilde{y}|, -a - l_2 + 4 + | -4 + z + \tilde{z}|, -a - l_3 + 4 + | -4 + x + \tilde{x}|\} < 0$$

Then, We can determine the bound of $l_1$, $l_2$, $l_3$ and realize the chaos synchronization by choosing the proper parameters (see Fig.6-8).

Figure 7: Time evolution of valuable $e_y(t)$

Figure 8: Time evolution of valuable $e_z(t)$

In practical appliances, systems get interfere inevitably from environment. Disturb to the error system, assuming that $z = (1.0 + 0.002 \sin t) z$. And the Mathematic software is used to simulate, the simulation result show the effectiveness and anti-interference ability of this method (see Fig.9-11).
4 Conclusion

The chaos synchronization problem, with time-delay in the channel, has been studied from the typical approach of unidirectional linear error feedback using the received time-delay signal directly without requiring any knowledge of the constant unknown time-delay. This paper address a practical issue in chaos synchronization of HCSA system. From the attractor of the system, we can get the bound of the parameters. Then we can choose the proper coupling parameters, which can make the systems globally synchronize. How to choose the better coupling parameter is the further research.

References


