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On the Principal Eigenvalues to Some Boundary Value Problems with Indefinite Weight

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Abstract: This paper deals with principal eigenvalues of the following class of boundary value problems

$$\begin{cases} -\Delta u = \lambda a(x)u, & x \in \Omega, \\ \frac{\partial u}{\partial n} + f(x)u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded region in R^N with smooth boundary $\partial\Omega$, $a(x)$ and $f(x)$ are indefinite weight functions which assumed to be continuous in $\bar{\Omega}$ and $\partial\Omega$, respectively, and at least one of them is not identically zero. We give a variational approach to find principal eigenvalues of this problem, and especially we find a necessary condition on $f(x)$ to have principal eigenvalues. Our method extends those of Afrouzi and Brown (Proc. Amer. Math. Soc. 146 (1998)) in the sense that boundary condition in this paper is a continuous function of x.

Key words: Principal eigenvalue; Indefinite weight function
AMS Subject Classification: 35J15, 35J25

1 Introduction

This paper deals with the existence of principal eigenvalues (i.e., eigenvalues corresponding to positive eigenfunctions) for the boundary value problem

$$\begin{cases} -\Delta u = \lambda a(x)u, & x \in \Omega, \\ \frac{\partial u}{\partial n} + f(x)u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subseteq R^N$ is a connected bounded domain with smooth boundary $\partial\Omega$, the outward unit normal to which is denoted by n. The functions $a(x)$ and $f(x)$ are indefinite weight functions which assumed to be continuous in $\bar{\Omega}$ and $\partial\Omega$, respectively, and at least one of them is not identically zero, and $a(x)$ may change sign on Ω. Here we say a function $a(x)$ changes sign if the measure of the sets $\{x \in \Omega; a(x) > 0\}$ and $\{x \in \Omega; a(x) < 0\}$ are both positive.
We consider only the cases that the function $f(x) : \partial\Omega \to R$ satisfies: $f(x) \geq 0$ on $\partial\Omega$ or $f(x) < 0$ on $\partial\Omega$, and find a necessary condition to have principal eigenvalues for the problem (1) in each case.
The problem such as (1) in the case $f(x) = \alpha \in R$ have been studied in recent years because of associated nonlinear problem arising in the study of population genetics (see [2]). The study of the ordinary differential equation case, however, goes back to Picone and Bôcher (see[1]). Attention has been confined mainly to the

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cases of Dirichlet ($f(x) = \infty$) and Neumann ($f(x) \equiv 0$) boundary conditions.
In the case of Dirichlet boundary condition, we have there exists a double sequence of eigenvalues for (1)

$$\cdots \lambda_2^- < \lambda_1^- < 0 < \lambda_1^+ < \lambda_2^+ \cdots ,$$

$\lambda_1^+(\lambda_1^-)$ being the unique positive (negative) principal eigenvalue. It is also well known that the case where $0 < f(x) < \infty$ is similar to the Dirichlet case. In the case of Neumann boundary conditions, 0 is clearly a principal eigenvalue and there is a positive (negative) principal eigenvalue if and only if $\int_\Omega a(x)dx < 0(> 0)$; in the case where $\int_\Omega a(x)dx = 0$ there are no positive and no negative principal eigenvalues.
In this paper we shall investigate how the principal eigenvalues of (1) depend on $f(x)$, obtaining new results for the case where $f(x) < 0$. This case seems to have been considered far less often than the case $f(x) \geq 0$, probably because it is more natural that the flux across the boundary should be outward if there is a positive concentration at the boundary, and also because $f(x) \geq 0$ is an easier condition to use when applying the maximum principle to discuss positive solutions. By studying the case $f(x) < 0$ we obtain a necessary condition to have principal eigenvalues of the problem (1).
Our analysis is based on a method used by Hess and Kato([3]). In the next section the existence and multiplicity of the principal eigenvalues of (1) depend on $f(x)$ are proved.

2 Existence results

In this section we will setup an appropriate functional analysis framework for our problem. We work in the Sobolev space $X = W^{1,2}(\Omega)$ with an ordinary norm

$$||u||_X = ||\nabla u||_{L^2(\Omega)} = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}.$$

We consider for fixed λ, the eigenvalue problem

$$\begin{cases} -\Delta u - \lambda a(x)u = \mu u, & x \in \Omega, \\ \frac{\partial u}{\partial n} + f(x)u = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

We denote the lowest eigenvalue of (2) by $\mu(\lambda)$. Let

$$S_\lambda = \{J_\lambda(\phi) = \int_\Omega |\nabla \phi|^2 dx - \lambda \int_\Omega a(x)\phi^2 dx + \int_{\partial\Omega} f(x)\phi^2 d\sigma; \phi \in X, ||u||_{L^2(\Omega)} = 1\}.$$

Suppose that $f(x) \geq 0$. It is easy to see that for fixed λ, S_λ is bounded below and $\mu(\lambda) = \inf S_\lambda$, by using variational arguments and that an eigenfunction corresponding to $\mu(\lambda)$ does not change sign on Ω. Thus, clearly, λ is a principal eigenvalue of the problem (1) if and only if $\mu(\lambda) = 0$.
When $f(x) < 0$, there exists $m < 0$ such that $m < f(x) < 0$, by using the continuity of $f(x)$ on $\partial\Omega$. The boundedness below of S_λ is a consequence of the following lemma.

Lemma 1 *For every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that*

$$\int_{\partial\Omega} \phi^2 d\sigma \leq \varepsilon \int_\Omega |\nabla \phi|^2 dx + C(\varepsilon) \int_\Omega \phi^2 dx$$

for all $\phi \in X$.

Proof. Suppose that the result does not hold, i.e., there exist $\varepsilon_0 > 0$ and a sequence $\{\phi_n\} \subset X$ such that $||\phi_n||_X = 1$ and

$$\int_{\partial\Omega} \phi_n^2 d\sigma \geq \varepsilon_0 + n \int_\Omega \phi_n^2 dx. \tag{3}$$

Suppose first that $\{||\phi_n||_{L^2(\Omega)}\}$ is unbounded. Then we assume without loss of generality that $||\phi_n||_{L^2(\Omega)} \to \infty$. Hence using Sobolev embedding theorem, we obtain $||\phi_n||_X \to \infty$, which is impossible.
Suppose now that $\{||\phi_n||_{L^2(\Omega)}\}$ is bounded. Then $\{\phi_n\}$ is bounded in X. So we may assume without loss of generality that $\phi_n \rightharpoonup \phi_0$ in X for some $\phi_0 \in X$. Since X may be compactly embedded in $L^2(\Omega)$ and $L^2(\partial\Omega)$, it follows that $\phi_n \to \phi_0$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$. Thus $\{\phi_n\}$ is bounded in $L^2(\partial\Omega)$. It follows from (3) that $\phi_n \to 0$ in $L^2(\Omega)$ and so in $L^2(\partial\Omega)$, but this is impossible because of (3). □

Corollary 1 *Suppose that $f(x) < 0$ on $\partial\Omega$, then S_λ is bonded below independently of $\phi \in X$.*

Proof. Choose $\varepsilon > 0$ such that $\varepsilon < -\frac{1}{m}$. By using above lemma, there exists $c(\varepsilon) > 0$ such that

$$\int_{\partial\Omega} \phi^2 d\sigma \leq \varepsilon \int_\Omega |\nabla\phi|^2 dx + C(\varepsilon) \int_\Omega \phi^2 dx$$

for all $\phi \in X$. Let $\phi \in X$ with $||\phi_n||_{L^2(\Omega)} = 1$ be arbitrary. Then

$$\begin{aligned} J_\lambda(\phi) &\geq \int_\Omega |\nabla\phi|^2 dx - \lambda \int_\Omega a(x)\phi^2 dx + m \int_{\partial\Omega} \phi^2 d\sigma \\ &\geq \int_\Omega |\nabla\phi|^2 dx - \lambda \int_\Omega a(x)\phi^2 dx + m(\varepsilon \int_\Omega |\nabla\phi|^2 dx + C(\varepsilon) \int_\Omega \phi^2 dx) \\ &\geq (1 + m\varepsilon) \int_\Omega |\nabla\phi|^2 dx - \lambda \sup_{x\in\Omega} |a(x)| + mC(\varepsilon) \\ &\geq -\lambda \sup_{x\in\Omega} |a(x)| + mC(\varepsilon), \end{aligned}$$

i.e., the set S_λ is bounded below. □

Using above corollary we have $\mu(\lambda) = \inf S_\lambda$ and that an eigenfunction corresponding to $\mu(\lambda)$ does not changes sign on Ω. Thus it is again the case that λ is a principal eigenvalue of (1) if and only if $\mu(\lambda) = 0$.

For fixed $\phi \in X$; $\lambda \to J_\lambda(\phi)$ is affine and so a concave function. It follows that $\lambda \to \inf J_\lambda(\phi) = \mu(\lambda)$ is a concave function. Also it is easy to see that $\mu(\lambda) \to -\infty$ as $\lambda \to \pm\infty$. Thus $\mu(\lambda)$ is an increasing function until it attains its maximum, and is a decreasing function thereafter.
As can be seen from variational characterization of $\mu(\lambda)$, $\mu(0) > 0$ for $0 < f(x) < \infty$, and so $\mu(\lambda)$ has exactly two zeroes for $0 < f(x) < \infty$. Thus in this case (1) has exactly two principal eigenvalues; one positive and one negative.

In the case $f(x) \leq 0$ we have that $\mu(\lambda) \leq 0$, and the situation is less clear.

Lemma 2 *Suppose that u_0 is a positive eigenfunction of (2) corresponding to the principal eigenvalue $\mu(\lambda)$. Then*

$$\frac{d\mu}{d\lambda}(\lambda) = -\frac{\int_\Omega a(x)u_0^2 dx}{\int_\Omega u_0^2 dx}.$$

Proof. Regarding u_0 and μ as functions of λ, we have

$$\begin{cases} -\Delta u_0 - \lambda a(x)u_0 = \mu u_0, & x \in \Omega, \\ \frac{\partial u_0}{\partial n} + f(x)u_0 = 0, & x \in \partial\Omega. \end{cases}$$

Then $u_0' = \frac{du_0}{d\lambda}$ satisfies

$$\begin{cases} -\Delta u_0' - \lambda a(x)u_0' - \mu u_0' = a(x)u_0 + \frac{d\mu}{d\lambda}u_0, & x \in \Omega, \\ \frac{\partial u_0'}{\partial n} + f(x)u_0' = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

Multiplying (4) by u_0 and integrating over Ω gives

$$\int_\Omega a(x)u_0^2 dx + \frac{d\mu}{d\lambda} \int_\Omega u_0^2 dx = 0$$

and so the result follows. □

The above lemma shows that where $\mu(\lambda)$ is an increasing (decreasing) function we have that $\int_\Omega a(x)u_0^2 dx < 0 (> 0)$, and at critical point we must have $\int_\Omega a(x)u_0^2 dx = 0$. The next lemma shows that $\mu(\lambda)$ has a unique critical point.

Lemma 3 *Suppose that* u_0 *is an eigenfunction of (2) corresponding to the principal eigenvalue* $\mu(\lambda_0)$*, such that* $\int_\Omega a(x)u_0^2dx = 0$*. Then* $\mu(\lambda_0) > \mu(\lambda)$ *for* $\lambda \neq \lambda_0$*.*

Proof. Regarding μ as function of λ, we have

$$\begin{cases} -\Delta u_0 - \lambda a(x)u_0 = \mu(\lambda_0)u_0, & x \in \Omega, \\ \frac{\partial u_0}{\partial n} + f(x)u_0 = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

Multiplying (5) by u_0 and integrating over Ω gives

$$\int_\Omega |\nabla u_0|^2 dx + \int_{\partial\Omega} f(x)u_0^2 d\sigma = \mu(\lambda_0)\int_\Omega u_0^2 dx.$$

Let $v_0 = \frac{u_0}{||u_o||_X}$. Then

$$\mu(\lambda_0) = \int_\Omega |\nabla v_0|^2 dx + \int_{\partial\Omega} f(x)v_0^2 d\sigma,$$

and

$$\begin{aligned} \mu(\lambda) &\leq \int_\Omega |\nabla v_0|^2 dx + \int_{\partial\Omega} f(x)v_0^2 d\sigma - \lambda\int_\Omega a(x)v_0^2 dx \\ &= \int_\Omega |\nabla v_0|^2 dx + \int_{\partial\Omega} f(x)v_0^2 d\sigma = \mu(\lambda_0). \end{aligned}$$

We now show that $\mu(\lambda) < \mu(\lambda_0)$, whenever $\lambda \neq \lambda_0$.
Suppose otherwise. Then $\mu = \mu(\lambda) = \mu(\lambda_0)$ satisfies

$$-\Delta u_0 - \lambda_0 a(x)u_0 = \mu u_0 \text{ in } \Omega; \quad \frac{\partial u_0}{\partial n} + f(x)u_0 = 0 \text{ on } \partial\Omega,$$

and

$$-\Delta u_0 - \lambda a(x)u_0 = \mu u_0 \text{ in } \Omega; \quad \frac{\partial u_0}{\partial n} + f(x)u_0 = 0 \text{ on } \partial\Omega,$$

while, $\lambda \neq \lambda_0$, and this is a contradiction. □

The above result shows that the unique global maximum of $\mu(\lambda)$ occurs when $\lambda = \lambda_0$. Hence the graph of $\lambda \to \mu(\lambda)$ may have 2, 1 and 0 intersections with the μ-axis, and so (1) may have 2, 1 and 0 principal eigenvalues.
As we shall see in the next theorem when $f(x) > 0$ on $\partial\Omega$, (1) has 2 principal eigenvalues, one positive and one negative.
When $f(x) \equiv 0$ on $\partial\Omega$, i.e., we have Neumann boundary condition, $\mu(0) = 0$ and the corresponding eigenfunction is a constant. Hence $\frac{d\mu}{d\lambda}(0) > 0(= 0)(< 0)$ as $\int_\Omega a(x)dx < 0(= 0)(> 0)$. Thus when $f(x) \equiv 0$ on $\partial\Omega$, $\mu = 0$ is a principal eigenvalue in all cases; if $\int_\Omega a(x)dx < 0$, there is an additional positive principal eigenvalue; and, if $\int_\Omega a(x)dx > 0$, there is an additional negative principal eigenvalue and, if $\int_\Omega a(x)dx = 0$, $\mu = 0$ is the only principal eigenvalue.
We now find a necessary condition on $f(x)$, when $f(x) < 0$ on $\partial\Omega$, that under it there exist principal eigenvalues to problem (1). We first assume that $\int_\Omega a(x)dx < 0$.

Theorem 1 *There exists* $c(\lambda_0, u_0) < 0$ *such that the problem (1) has an eigenfunction* u_0 *corresponding to the principal eigenvalue* λ_0 *if* $f(x)$ *satisfies*

$$c(\lambda_0, u_0) < \int_{\partial\Omega} f(x)d\sigma < 0.$$

Proof. Suppose $f(x) < 0$ on $\partial\Omega$ and u_0 is a positive eigenvalue of (1) corresponding to a positive principal eigenvalue λ_0. By using the maximum principle, it follows that $u_0(x) > 0$ for all $x \in \Omega \cup \partial\Omega = \overline{\Omega}$. Since $f(x) < 0$, we have

$$\begin{aligned} 0 = \mu_1(\lambda_0) &= \inf\{\int_\Omega |\nabla\phi|^2 dx - \lambda\int_\Omega a(x)\phi^2 dx + \int_{\partial\Omega} f(x)\phi^2 d\sigma\} \\ &\leq \inf\{\int_\Omega |\nabla\phi|^2 dx - \lambda\int_\Omega a(x)\phi^2 dx\} = \mu_2(\lambda_0), \end{aligned}$$

where μ_1 is the eigenvalue of the problem (2) for $f(x) \leq 0$ and μ_2 is the eigenvalue of the problem (2) for $f(x) \equiv 0$, i.e., the Neumann problem. Hence $\lambda_0 < \mu_0$ (the positive eigenvalue of the Neumann problem). Dividing (1) by u_0 and integrating over Ω, we have

$$\int_\Omega \frac{-\Delta u_0}{u_0} dx = \lambda_0 \int_\Omega a(x)dx,$$

and so

$$-\int_\Omega \frac{|\nabla u_0|^2}{u_0^2} dx - \int_{\partial\Omega} \frac{\partial u_0}{\partial n} u_0^{-1} d\sigma = \lambda_0 \int_\Omega a(x)dx,$$

i.e.,

$$\int_{\partial\Omega} f(x)d\sigma = \int_\Omega \frac{|\nabla u_0|^2}{u_0^2} dx + \lambda_0 \int_\Omega a(x)dx.$$

Since $\lambda_0 < \mu_0$, $f(x)$ can not be too negative, and the proof is complete. □

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