On the Principal Eigenvalues to Some Boundary Value Problems with Indefinite Weight

G.A.Afrouzi *, S.Khademloo
Department of Mathematics, Faculty of Basic Sciences, Mazandaran University, Babolsar, Iran
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Abstract: This paper deals with principal eigenvalues of the following class of boundary value problems

\[ \begin{align*}
-\Delta u &= \lambda a(x)u, \quad x \in \Omega, \\
\frac{\partial u}{\partial n} + f(x)u &= 0, \quad x \in \partial\Omega,
\end{align*} \]

where \( \Omega \) is a bounded region in \( \mathbb{R}^N \) with smooth boundary \( \partial\Omega \), \( a(x) \) and \( f(x) \) are indefinite weight functions which assumed to be continuous in \( \bar{\Omega} \) and \( \partial\Omega \), respectively, and at least one of them is not identically zero. We give a variational approach to find principal eigenvalues of this problem, and especially we find a necessary condition on \( f(x) \) to have principal eigenvalues. Our method extends those of Afrouzi and Brown (Proc. Amer. Math. Soc. 146 (1998)) in the sense that boundary condition in this paper is a continuous function of \( x \).

Key words: Principal eigenvalue; Indefinite weight function

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1 Introduction

This paper deals with the existence of principal eigenvalues (i.e., eigenvalues corresponding to positive eigenfunctions) for the boundary value problem

\[ \begin{align*}
-\Delta u &= \lambda a(x)u, \quad x \in \Omega, \\
\frac{\partial u}{\partial n} + f(x)u &= 0, \quad x \in \partial\Omega,
\end{align*} \]  

(1)

where \( \Omega \subseteq \mathbb{R}^N \) is a connected bounded domain with smooth boundary \( \partial\Omega \), the outward unit normal to which is denoted by \( n \). The functions \( a(x) \) and \( f(x) \) are indefinite weight functions which assumed to be continuous in \( \Omega \) and \( \partial\Omega \), respectively, and at least one of them is not identically zero, and \( a(x) \) may change sign on \( \Omega \). Here we say a function \( a(x) \) changes sign if the measure of the sets \( \{ x \in \Omega; a(x) > 0 \} \) and \( \{ x \in \Omega; a(x) < 0 \} \) are both positive.

We consider only the cases that the function \( f(x) : \partial\Omega \to \mathbb{R} \) satisfies: \( f(x) \geq 0 \) on \( \partial\Omega \) or \( f(x) < 0 \) on \( \partial\Omega \), and find a necessary condition to have principal eigenvalues for the problem (1) in each case.

The problem such as (1) in the case \( f(x) = \alpha \in \mathbb{R} \) have been studied in recent years because of associated nonlinear problem arising in the study of population genetics (see [2]). The study of the ordinary differential equation case, however, goes back to Picone and Bôcher (see[1]). Attention has been confined mainly to the

*Corresponding author. E-mail address: afrouzi@umz.ac.ir
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cases of Dirichlet \((f(x) = \infty)\) and Neumann \((f(x) \equiv 0)\) boundary conditions.

In the case of Dirichlet boundary condition, we have there exists a double sequence of eigenvalues for \((1)\)

\[ \cdots \lambda_2^- < \lambda_1^- < 0 < \lambda_1^+ < \lambda_2^+ \cdots, \]

\(\lambda_1^+ (\lambda_1^-)\) being the unique positive (negative) principal eigenvalue. It is also well known that the case where \(0 < f(x) < \infty\) is similar to the Dirichlet case. In the case of Neumann boundary conditions, \(0\) is clearly a principal eigenvalue and there is a positive (negative) principal eigenvalue if and only if \(\int_{\Omega} a(x) dx < 0\); in the case where \(\int_{\Omega} a(x) dx = 0\) there are no positive and no negative principal eigenvalues.

In this paper we shall investigate how the principal eigenvalues of \((1)\) depend on \(f(x)\), obtaining new results for the case where \(f(x) < 0\). This case seems to have been considered far less often than the case \(f(x) \geq 0\), probably because it is more natural that the flux across the boundary should be outward if there is a positive concentration at the boundary, and also because \(f(x) \geq 0\) is an easier condition to use when applying the maximum principle to discuss positive solutions. By studying the case \(f(x) < 0\) we obtain a necessary condition to have principal eigenvalues of the problem \((1)\).

Our analysis is based on a method used by Hess and Kato[3]). In the next section the existence and multiplicity of the principal eigenvalues of \((1)\) depend on \(f(x)\) are proved.

### 2 Existence results

In this section we will setup an appropriate functional analysis framework for our problem. We work in the Sobolev space \(X = W^{1,2} (\Omega)\) with an ordinary norm

\[ ||u||_X = ||\nabla u||_{L^2(\Omega)} = \left( \int_{\Omega} |\nabla u|^2 dx \right)^{1/2}. \]

We consider for fixed \(\lambda\), the eigenvalue problem

\[ \begin{cases} 
- \Delta u - \lambda a(x) u = \mu u, & x \in \Omega, \\
\frac{\partial u}{\partial n} + f(x) u = 0, & x \in \partial \Omega. 
\end{cases} \tag{2} \]

We denote the lowest eigenvalue of \((2)\) by \(\mu(\lambda)\). Let

\[ S_\lambda = \{ J_\lambda(\phi) = \int_{\Omega} |\nabla \phi|^2 dx - \lambda \int_{\Omega} a(x) \phi^2 dx + \int_{\partial \Omega} f(x) \phi^2 d\sigma; \phi \in X, ||\phi||_{L^2(\Omega)} = 1 \}. \]

Suppose that \(f(x) \geq 0\). It is easy to see that for fixed \(\lambda\), \(S_\lambda\) is bounded below and \(\mu(\lambda) = \inf S_\lambda\), by using variational arguments and that an eigenfunction corresponding to \(\mu(\lambda)\) does not change sign on \(\Omega\). Thus, clearly, \(\lambda\) is a principal eigenvalue of the problem \((1)\) if and only if \(\mu(\lambda) = 0\).

When \(f(x) < 0\), there exists \(m < 0\) such that \(m < f(x) < 0\), by using the continuity of \(f(x)\) on \(\partial \Omega\). The boundedness below of \(S_\lambda\) is a consequence of the following lemma.

**Lemma 1** For every \(\varepsilon > 0\) there exists a constant \(C(\varepsilon) > 0\) such that

\[ \int_{\partial \Omega} \phi^2 d\sigma \leq \varepsilon \int_{\Omega} |\nabla \phi|^2 dx + C(\varepsilon) \int_{\Omega} \phi^2 dx \]

for all \(\phi \in X\).

**Proof.** Suppose that the result does not hold, i.e., there exist \(\varepsilon_0 > 0\) and a sequence \(\{ \phi_n \} \subset X\) such that \(||\phi_n||_X = 1\) and

\[ \int_{\partial \Omega} \phi_n^2 d\sigma \geq \varepsilon_0 + n \int_{\Omega} \phi_n^2 dx. \tag{3} \]

Suppose first that \(\{ ||\phi_n||_{L^2(\Omega)} \}\) is unbounded. Then we assume without loss of generality that \(||\phi_n||_{L^2(\Omega)} \to \infty\). Hence using Sobolev embedding theorem, we obtain \(||\phi_n||_X \to \infty\), which is impossible.

Suppose now that \(\{ ||\phi_n||_{L^2(\Omega)} \}\) is bounded. Then \(\{ \phi_n \}\) is bounded in \(X\). So we may assume without loss of generality that \(\phi_n \to \phi_0\) in \(X\) for some \(\phi_0 \in X\). Since \(X\) may be compactly embedded in \(L^2(\Omega)\) and \(L^2(\partial \Omega)\), it follows that \(\phi_n \to \phi_0\) in \(L^2(\Omega)\) and \(L^2(\partial \Omega)\). Thus \(\{ \phi_n \}\) is bounded in \(L^2(\partial \Omega)\). It follows from (3) that \(\phi_n \to 0\) in \(L^2(\Omega)\) and so in \(L^2(\partial \Omega)\), but this is impossible because of (3). \(\Box\)

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Corollary 1 Suppose that \( f(x) < 0 \) on \( \partial \Omega \), then \( S_\lambda \) is bonded below independently of \( \phi \in X \).

Proof. Choose \( \varepsilon > 0 \) such that \( \varepsilon < -\frac{1}{m} \). By using above lemma, there exists \( c(\varepsilon) > 0 \) such that
\[
\int_{\partial \Omega} \phi^2 d\sigma \leq \varepsilon \int_{\Omega} |\nabla \phi|^2 dx + C(\varepsilon) \int_{\Omega} \phi^2 dx
\]
for all \( \phi \in X \). Let \( \phi \in X \) with \( ||\phi_n||_{L^2(\Omega)} = 1 \) be arbitrary. Then
\[
J_\lambda(\phi) \geq \int_{\Omega} |\nabla \phi|^2 dx - \lambda \int_{\Omega} a(x) \phi^2 dx + m \int_{\partial \Omega} \phi^2 d\sigma
\]
\[
\geq \int_{\Omega} |\nabla \phi|^2 dx - \lambda \int_{\Omega} a(x) \phi^2 dx + m(\varepsilon) \int_{\Omega} |\nabla \phi|^2 dx + C(\varepsilon) \int_{\Omega} \phi^2 dx
\]
\[
\geq (1 + \varepsilon m) \int_{\Omega} |\nabla \phi|^2 dx - \lambda \sup_{x \in \Omega} |a(x)| + mC(\varepsilon)
\]
i.e., the set \( S_\lambda \) is bounded below. \( \Box \)

Using above corollary we have \( \mu(\lambda) = \inf S_\lambda \) and that an eigenfunction corresponding to \( \mu(\lambda) \) does not changes sign on \( \Omega \). Thus it is again the case that \( \lambda \) is a principal eigenvalue of (1) if and only if \( \mu(\lambda) = 0 \).

For fixed \( \phi \in X \); \( \lambda \rightarrow J_\lambda(\phi) \) is affine and so a concave function. It follows that \( \lambda \rightarrow \inf J_\lambda(\phi) = \mu(\lambda) \) is a concave function. Also it is easy to see that \( \mu(\lambda) \rightarrow -\infty \) as \( \lambda \rightarrow \pm \infty \). Thus \( \mu(\lambda) \) is an increasing function until it attains its maximum, and is a decreasing function thereafter.

As can be seen from variational characterization of \( \mu(\lambda) \), \( \mu(0) > 0 \) for \( f(x) < \infty \), and so \( \mu(\lambda) \) has exactly two zeroes for \( 0 < f(x) < \infty \). Thus in this case (1) has exactly two principal eigenvalues; one positive and one negative.

In the case \( f(x) \leq 0 \) we have that \( \mu(\lambda) \leq 0 \), and the situation is less clear.

Lemma 2 Suppose that \( u_0 \) is a positive eigenfunction of (2) corresponding to the principal eigenvalue \( \mu(\lambda) \). Then
\[
\frac{d\mu}{d\lambda}(\lambda) = -\int_{\Omega} a(x) u_0^2 dx \int_{\Omega} u_0^2 dx.
\]

Proof. Regarding \( u_0 \) and \( \mu \) as functions of \( \lambda \), we have
\[
\left\{ \begin{array}{ll}
-\Delta u_0 - \lambda a(x) u_0 = \mu u_0, & x \in \Omega, \\
\frac{\partial u_0}{\partial n} + f(x) u_0 = 0, & x \in \partial \Omega.
\end{array} \right.
\]

Then \( u'_0 = \frac{du_0}{d\lambda} \) satisfies
\[
\left\{ \begin{array}{ll}
-\Delta u'_0 - \lambda a(x) u'_0 - \mu u'_0 = a(x) u_0 + \frac{d\mu}{d\lambda} u_0, & x \in \Omega, \\
\frac{\partial u'_0}{\partial n} + f(x) u'_0 = 0, & x \in \partial \Omega.
\end{array} \right.
\]

Multiplying (4) by \( u_0 \) and integrating over \( \Omega \) gives
\[
\int_{\Omega} a(x) u_0^2 dx + \frac{d\mu}{d\lambda} \int_{\Omega} u_0^2 dx = 0
\]
and so the result follows. \( \Box \)

The above lemma shows that where \( \mu(\lambda) \) is an increasing (decreasing) function we have that \( \int_{\Omega} a(x) u_0^2 dx < 0(> 0) \), and at critical point we must have \( \int_{\Omega} a(x) u_0^2 dx = 0 \). The next lemma shows that \( \mu(\lambda) \) has a unique critical point.

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Lemma 3 Suppose that $u_0$ is an eigenfunction of (2) corresponding to the principal eigenvalue $\mu(\lambda_0)$, such that $\int_\Omega a(x)u_0^2dx = 0$. Then $\mu(\lambda_0) > \mu(\lambda)$ for $\lambda \neq \lambda_0$.

Proof. Regarding $\mu$ as function of $\lambda$, we have

$$\begin{cases}
-\Delta u_0 - \lambda a(x)u_0 = \mu(\lambda_0)u_0, \quad x \in \Omega, \\
\frac{\partial u_0}{\partial n} + f(x)u_0 = 0, \quad x \in \partial \Omega.
\end{cases} \tag{2}$$

Multiplying (5) by $u_0$ and integrating over $\Omega$ gives

$$\int_\Omega |\nabla u_0|^2 dx + \int_{\partial \Omega} f(x)u_0^2 d\sigma = \mu(\lambda_0) \int_\Omega u_0^2 dx.$$ 

Let $v_0 = \frac{u_0}{||u_0||_X}$. Then

$$\mu(\lambda_0) = \int_\Omega |\nabla v_0|^2 dx + \int_{\partial \Omega} f(x)v_0^2 d\sigma,$$

and

$$\mu(\lambda) \leq \int_\Omega |\nabla v_0|^2 dx + \int_{\partial \Omega} f(x)v_0^2 d\sigma - \lambda \int_\Omega a(x)v_0^2 dx$$

$$= \int_\Omega |\nabla v_0|^2 dx + \int_{\partial \Omega} f(x)v_0^2 d\sigma = \mu(\lambda_0).$$

We now show that $\mu(\lambda) < \mu(\lambda_0)$, whenever $\lambda \neq \lambda_0$. Suppose otherwise. Then $\mu = \mu(\lambda) = \mu(\lambda_0)$ satisfies

$$-\Delta u_0 - \lambda_0 a(x)u_0 = \mu u_0 \text{ in } \Omega; \quad \frac{\partial u_0}{\partial n} + f(x)u_0 = 0 \text{ on } \partial \Omega,$$

and

$$-\Delta u_0 - \lambda a(x)u_0 = \mu u_0 \text{ in } \Omega; \quad \frac{\partial u_0}{\partial n} + f(x)u_0 = 0 \text{ on } \partial \Omega,$$

while, $\lambda \neq \lambda_0$, and this is a contradiction. 

The above result shows that the unique global maximum of $\mu(\lambda)$ occurs when $\lambda = \lambda_0$. Hence the graph of $\lambda \rightarrow \mu(\lambda)$ may have 2, 1 and 0 intersections with the $\mu$-axis, and so (1) may have 2, 1 and 0 principal eigenvalues.

As we shall see in the next theorem when $f(x) > 0$ on $\partial \Omega$, (1) has 2 principal eigenvalues, one positive and one negative.

When $f(x) \equiv 0$ on $\partial \Omega$, i.e., we have Neumann boundary condition, $\mu(0) = 0$ and the corresponding eigenfunction is a constant. Hence $\frac{\partial u}{\partial n}(0) > 0 (= 0)(< 0)$ as $\int_\Omega a(x)dx < 0 (= 0)(> 0)$. Thus when $f(x) \equiv 0$ on $\partial \Omega$, $\mu = 0$ is a principal eigenvalue in all cases; if $\int_\Omega a(x)dx < 0$, there is an additional positive principal eigenvalue; and, if $\int_\Omega a(x)dx > 0$, there is an additional negative principal eigenvalue and, if $\int_\Omega a(x)dx = 0$, $\mu = 0$ is the only principal eigenvalue.

We now find a necessary condition on $f(x)$, when $f(x) < 0$ on $\partial \Omega$, that under it there exist principal eigenvalues to problem (1). We first assume that $\int_\Omega a(x)dx < 0$.

Theorem 1 There exists $c(\lambda_0, u_0) < 0$ such that the problem (1) has an eigenfunction $u_0$ corresponding to the principal eigenvalue $\lambda_0$ if $f(x)$ satisfies

$$c(\lambda_0, u_0) < \int_{\partial \Omega} f(x)d\sigma < 0.$$ 

Proof. Suppose $f(x) < 0$ on $\partial \Omega$ and $u_0$ is a positive eigenvalue of (1) corresponding to a positive principal eigenvalue $\lambda_0$. By using the maximum principle, it follows that $u_0(x) > 0$ for all $x \in \Omega \cup \partial \Omega = \bar{\Omega}$. Since $f(x) < 0$, we have

$$0 = \mu_1(\lambda_0) = \inf\{\int_\Omega |\nabla \phi|^2 dx - \lambda \int_\Omega a(x)\phi^2 dx + \int_{\partial \Omega} f(x)\phi^2 d\sigma\}$$

$$\leq \inf\{\int_\Omega |\nabla \phi|^2 dx - \lambda \int_\Omega a(x)\phi^2 dx\} = \mu_2(\lambda_0),$$

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where $\mu_1$ is the eigenvalue of the problem (2) for $f(x) \leq 0$ and $\mu_2$ is the eigenvalue of the problem (2) for $f(x) \equiv 0$, i.e., the Neumann problem. Hence $\lambda_0 < \mu_0$ (the positive eigenvalue of the Neumann problem). Dividing (1) by $u_0$ and integrating over $\Omega$, we have

$$\int_{\Omega} \frac{-\Delta u_0}{u_0} dx = \lambda_0 \int_{\Omega} a(x) dx,$$

and so

$$-\int_{\Omega} \frac{|\nabla u_0|^2}{u_0^2} dx - \int_{\partial \Omega} \frac{\partial u_0}{\partial n} u_0^{-1} d\sigma = \lambda_0 \int_{\Omega} a(x) dx,$$

i.e.,

$$\int_{\partial \Omega} f(x) d\sigma = \int_{\Omega} \frac{|\nabla u_0|^2}{u_0^2} dx + \lambda_0 \int_{\Omega} a(x) dx.$$

Since $\lambda_0 < \mu_0$, $f(x)$ can not be too negative, and the proof is complete. □

**References**

