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Blow-up Phenomena for the Degasperis-Procesi Equation with Strong Dispersive Term

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Abstract: In this paper, we study the blow-up phenomena for the Degasperis-Procesi equation with strong dispersive term. Applying some inequalities in Sobolev space and comparison theorem, we first establish two blow-up results with certain initial profiles and then investigate the blow-up rate.

Key words: Degasperis-Procesi equation; blow-up; blow-up rate

1 Introduction

Recently, Degasperis and Procesi [1] studied the following family of third order dispersive PDE conservation laws,

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x, \tag{1.1}$$

where α, c_0, c_1, c_2 and c_3 are real constants. They found that there are at least four equations that satisfy the asymptotic integrability condition within this family: the KdV eqution, the Camassa-Holm equation, the Dullin-Gottwald-Holm equation and the Degasperis-Procesi equation.

For $\alpha = c_2 = c_3 = 0$ in Eq.(1.1), it becomes the well-known Korteweg-deVeris equation, which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. The KdV equation is completely integrable and its solitary waves are solitions [2], [3]. The Cauchy problem of the KdV equation has been the subject of a number of studies, and a satisfactory local or global (in time) existence theory is now in hand [4]. It is shown that as long as $u_0 \in H^s(R), s \geq 1$, the KdV equation is globally well-posed. It is observed that the KdV equation does not have wave breaking, that is, wave remains bounded, but its slope becomes unbounded in finite time.

If $c_1 = -\frac{3}{2}c_3/\alpha^2, c_2 = c_3/2$, Eq.(1.1) becomes the Camassa-Holm equation, modeling the unidirectional propagation of shallow water waves over a flat bottom. The Cammassa-Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [5]. The Cauchy problem of the Camassa-Holm equation has been studied extensively. It has been shown that the Camassa-Holm equation is locally well-posed [6], [7] with the initial data $u_0 \in H^s(R), s > \frac{3}{2}$. More interestingly, it has global strong solutions and also blow-up solutions in finite time [8], [9] with a different class of initial profiles in the Sobolev spaces H^s, $s > \frac{3}{2}$. The advantage of the Camassa-Holm equation in comparison with the KdV equation is clear: the Camassa-Holm equation has peaked solutions, but no shock waves. In [10] Danping Ding and Lixin Tian researched solution of dissipative Camassa-Holm equation on total space.

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Dullin, Gottwald, Holm [11] discussed the following 1+1 quadratically nonlinear equation in this class for a unidirectional water wave with fluid velocity $u(x,t)$,

$$m_t + c_0 u_x + u m_x + 2 m u_x = -\gamma u_{xxx}, \quad x \in R, \quad t \in R. \tag{1.2}$$

where $m = u - u_{xx}$ is a momentum variable, the constants α^2 and $\frac{\gamma}{c_0}$ are squares of length scales, and $c_0 = \sqrt{gh}$ is the linear wave speed for undisturbed water at rest at spatial infinity. Eq.(1.2) was derived by using asymptotic expansions directly in the Hamiltonian for Euler's equations in the shallow water regime and thereby shown to be bi-Hamiltonian and has a Lax formulation in [11], [12]. Eq.(1.2) combines the linear dispersion of the Korteweg-de Vries(KdV) with the nonlinear/nonlocal dispersion of the Camassa-Holm (CH) equation, yet preserves integrability via the inverse scatting transform (IST) method. Lixin Tian, Guilong Gui,Yue Liu [13] studied the Well-posedness problem and the scattering problem for Eq.(1.2). Lin Ju [14] studied the solution of Eq.(1.2). Dianchen Lu, Dejun Peng, Lixin Tian [15] studied the Well-posedness problem for the Generalized Eq.(1.2).

With $c_1 = -2c_3/\alpha^2, c_2 = c_3$ in Eq.(1.1), we find the Degasperis-Procesi equation of the form

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + u u_{xxx}, \quad t > 0\ , \quad x \in R. \tag{1.3}$$

Degasperis,Holm and Hone [16] proved the exact integrability of Eq.(1.3) by constructing a Lax pair. They also showed that Eq.(1.3) has bi-Hamiltonian structure and an infinite sequence of conserved quantities. It also admits exact peakon solutions which are analogous to the Camassa-Holm peakons.

The Degasperis-Procesi equation can be regarded as a model for nonlinear shallow water dynamics and its asmptotic accuracy is the same as for the Camassa-Holm shallow water equation. Dullin, Gottwald and Holm[17] showed that the Degasperis-Procesi equation can be obtained from the shallow water elevation equation by an appropriate Kodama transformation. Lundmark and Szmigielski [18] presented an inverse scattering approach for computing n-peakon solutions to Eq.(1.3).

Since the Degasperis-Procesi Eq.(1.3) was derived, many papers have been devoted to its study. Yin proved local well-posedness to Eq.(1.3) with initial data $u_0 \in H^s(R)$, $(s > \frac{3}{2})$ on the line [19] and on the circle [20]. In these two papers the precise blow-up scenario and a blow-up result were derived. The global existence of strong solutions and global weak solutions to Eq.(1.3) are also investigated in [21],[22].

Eq.(1.3) with a strong dispersive term, we get

$$u_t - u_{txx} + 4uu_x + \gamma(u - u_{xx})_x = 3u_x u_{xx} + u u_{xxx}, \qquad t > 0, \quad x \in R. \tag{1.4}$$

In this paper, we mainly study the Blow-up phenomena for Eq.(1.4).

Notation. In the following part, we denote by " $*$ " the spatial convolution on R. For $1 \le p \le \infty$, the norm in Lebesgue space L^p will be writtern $\|.\|_{L^p}$.

2 Preliminaries

With $y = u - u_{xx}$, Eq.(1.4) takes the form:

$$\begin{cases} y_t + u y_x + 3u_x y + \gamma y_x = 0, & t > 0, x \in R, \\ y(0,x) = u_0(x) - u_{0,xx}(x), & x \in R. \end{cases} \tag{2.1}$$

Note that if $p(x) = \frac{1}{2}e^{-|x|}$, then $(1 - \partial_x^2)^{-1} f = \mathrm{p} * f$ for all $f \in L^2(R)$. Using this identity, we can rewrite Eq.(1.4) as a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} u_t + u u_x + \gamma u_x = -\partial_x p * (\frac{3}{2}u^2), & t > 0, x \in R, \\ u(0,x) = u_0(x), & x \in R. \end{cases} \tag{2.2}$$

Lemma 2.1 *[23] Given $u_0 \in H^s(R), s > \frac{3}{2}$, there exists a maximal $T = T(u_0) > 0$ and a unique solution u to Eq.(1.4), and $u = u(., u_0) \in C([0,T); H^s(R)) \cap C^1([0,T); H^{s-1}(R))$. Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \to u(., u_0) : H^s(R) \to C([0,T); H^s(R) \cap C^1([0,T); H^{s-1}(R))$ is continuous and the maximal time of existence $T > 0$ can be chosen to be independent of s .*

Following the similar argument in [20], we obtain the following Lemma.

Lemma 2.2 *Given $u_0 \in H^s(R), s > \frac{3}{2}$, blow up of the solution $u = u(., u_0)$ in finite time $T < +\infty$ occurs if and only if $\liminf_{t\uparrow T}\{\inf_{x\in R}[u_x(t,x)]\} = -\infty$.*

Lemma 2.3 *[23] If $u_0 \in H^s(R), s > \frac{3}{2}$, then as long as the solution $u(t,x)$ given by Lemma2.1 exists, we have $\|u(t)\|_{L^2}^2 \le 4\|u_0\|_{L^2}^2$.*

Lemma 2.4 *[23] Assume $u_0 \in H^s(R), s > \frac{3}{2}$, $y_0 = (1-\partial_x^2)u_0 \ge 0$. Let $T > 0$ be the maximal existence time of the solution u to Eq.(1.4) guaranteed by Lemma2.1. Then we have:*
$\|u(t,x)\|_{L^\infty} \le 3\|u_0(x)\|_{L^2}^2 t + \|u_0(x)\|_{L^\infty}, \forall t \in [0,T]$.

Assume $u(t,x), t \in [0,t)$ is the solution of Eq.(2.2). Consider the following differential equation

$$\begin{cases} \xi_t(t,x) = u(t,\xi(t,x)) + \gamma, & t \ge 0,\, x \in R, \\ \xi(0,x) = x, & x \in R. \end{cases} \tag{2.3}$$

Lemma 2.5 *[23] Let $u_0 \in H^s(R),\ s \ge 3$, and let $T > 0$ be the maximal existence time of the corresponding solution u to Eq.(2.2). Then the Eq.(2.3) has a unique solution $\xi \in C^1([0,T) \times R, R)$. Moreover, the map $\xi(t,.)$ is an increasing diffeomorphism of R with $\xi_x(t,x) = \exp\left(\int_0^t u_x(z,\xi(z,x))\,dz\right) > 0$.*

Lemma 2.6 *[23] Let $u_0 \in H^s(R), s \ge 3$, and let $T > 0$ be the maximal existence time of the corresponding solution u to Eq.(2.2). Setting $y = u - u_{xx}$, we have $y(t,\xi(t,x))\xi_x^3(t,x) = y_0(x), \forall (t,x) \in [0,T) \times R$.*

Lemma 2.7 *[24] Let $T > 0$ and $u \in C^1([0,T); H^2(R))$. Then for every $t \in [0,T)$, there exists at least one point $\delta(t) \in R$ with $I(t) = \inf_{x\in R}(u_x(t,x)) = u_x(t,\delta(t))$. The function I is almost everywhere differentiable on $(0,T)$, with $\frac{d}{dt}I(t) = u_{tx}(t,\delta(t)), a.e.(0,T)$.*

3 Blow-up phenomena

Theorem 3.1 *Let $\varepsilon > 0, u_0 \in H^s(R),\ s > \frac{3}{2}$. Assume there is $x_0 \in R$ such that*

$$u_0'(x_0) < -\frac{\sqrt{6}(1+\varepsilon)}{4}(\|u_0\|_{L^\infty} + (2\sqrt{6}\|u_0\|_{L^2}^2 \ln(1+\frac{2}{\varepsilon}) + \|u_0\|_{L^\infty}^2)^{\frac{1}{2}}).$$

Then the corresponding solution to Eq.(2.2) blows up in finite time. Moreover, the maximal time of existence is estimated above by

$$\frac{(2\sqrt{6}\|u_0\|_{L^2}^2 \ln(1+\frac{2}{\varepsilon}) + \|u_0\|_{L^\infty}^2)^{\frac{1}{2}} - \|u_0\|_{L^\infty}}{6\|u_0\|_{L^2}^2}.$$

Proof. We only need to show that the above theorem holds for $s = 3$. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(2.2) with the initial data $u_0 \in H^3(R)$, Differentiating Eq.(2.2) with respect to x, in view of $\partial_x^2 p * f = p * f - f$, we have

$$u_{tx} = -u_x^2 - uu_{xx} - \gamma u_{xx} + \frac{3}{2}u^2 - p * (\frac{3}{2}u^2). \tag{3.1}$$

Note that

$$\begin{aligned} \frac{du_x(t,\xi(t,x))}{dt} &= u_{xt}(t,\xi(t,x)) + u_{xx}(t,\xi(t,x))\frac{d\xi(t,x)}{dt} \\ &= u_{xt}(t,\xi(t,x)) + (u(t,\xi(t,x)) + \gamma)u_{xx}(t,\xi(t,x)). \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), in view of $p * (\frac{3}{2}u^2)(t,\xi(t,x)) \ge 0$, we deduce that

$$\begin{aligned}\frac{du_x(t,\xi(t,x))}{dt} &= -u_x^2(t,\xi(t,x)) + \frac{3}{2}u^2(t,\xi(t,x)) - p*(\frac{3}{2}u^2)(t,\xi(t,x)) \\ &\le -u_x^2(t,\xi(t,x)) + \frac{3}{2}u^2(t,\xi(t,x)) \\ &\le -u_x^2(t,\xi(t,x)) + \frac{3}{2}(3\left\|u_0\right\|_{L^2}^2 t + \left\|u_0\right\|_{L^\infty})^2.\end{aligned}$$

Set $m(t) = u_x(t,\xi(t,x_0))$ and fix $\varepsilon > 0$. Take

$$T_1 = \frac{(2\sqrt{6}\left\|u_0\right\|_{L^2}^2 \ln(1+\frac{2}{\varepsilon}) + \left\|u_0\right\|_{L^\infty}^2)^{\frac{1}{2}} - \left\|u_0\right\|_{L^\infty}}{6\left\|u_0\right\|_{L^2}^2},$$

and K$(T_1) = \frac{\sqrt{6}}{2}(3\left\|u_0\right\|_{L^2}^2 T_1 + \left\|u_0\right\|_{L^\infty})$. Then following the same argument in [22], we finish the proof of the theorem. $\square$

Theorem 3.2 *Let $u_0 \in H^s(R)$, $s > \frac{3}{2}$. Assume there exists $x_0 \in R$ such that*

$$\begin{cases} y_0(x) = u_0(x) - u_{0,xx}(x) \ge 0, & x \le x_0, \\ y_0(x) = u_0(x) - u_{0,xx}(x) \le 0, & x \ge x_0. \end{cases}$$

and y_0 changes sign. Then, the corresponding solution to Eq.(2.2) blows up in finite time.

Proof. We only need to show that the above theorem holds for $s = 3$. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(2.2) with the initial data $u_0 \in H^3(R)$. Note that the function $\xi(t,x)$ is an increasing diffeomorphism of R with $\xi_x(t,x) > 0$ with respect to time t. We infer from the assumption of the theorem and Lemma2.6 that for $t \in [0,T)$, we have

$$\begin{cases} y(t,x) \ge 0, & x \le \xi(t,x_0), \\ y(t,x) \le 0, & x \ge \xi(t,x_0). \end{cases} \tag{3.3}$$

and $y(t,\xi(t,x_0)) = 0$, $t \in [0,T)$. Define

$$M(t,x) = e^{-x}\int_{-\infty}^{x} e^{\eta} y(t,\eta)d\eta,\ t \in [0,T). \tag{3.4}$$

$$N(t,x) = e^{x}\int_{x}^{-\infty} e^{-\eta} y(t,\eta)d\eta,\ t \in [0,T). \tag{3.5}$$

By (3.4) and (3.5), in view of (3.3), we have

$$M(t,\xi(t,x_0))N(t,\xi(t,x_0)) = \int_{-\infty}^{\xi(t,x_0)} e^{\eta}y(t,\eta)d\eta \int_{\xi(t,x_0)}^{\infty} e^{-\eta}y(t,\eta)d\eta < 0,\ t \in [0,T)$$

Note that

$$u(t,x) = p*y = \int_R \frac{1}{2}e^{-|x-\eta|}y(t,\eta)d\eta = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\eta}y(t,\eta)d\eta + \frac{e^{x}}{2}\int_{x}^{\infty} e^{-\eta}y(t,\eta)d\eta,$$

and $u_x(t,x) = -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\eta}y(t,\eta) + \frac{e^{x}}{2}\int_{x}^{\infty} e^{-\eta}y(t,\eta)d\eta$, We deduce that

$$u(t,x) - u_x(t,x) = e^{-x}\int_{-\infty}^{x} e^{\eta}y(t,\eta)d\eta = M(t,x),$$

$$u(t,x) + u_x(t,x) = e^{x}\int_{x}^{\infty} e^{-\eta}y(t,\eta)d\eta = N(t,x).$$

and

$$M(t,\xi(t,x_0)) = u(t,\xi(t,x_0)) - u_x(t,\xi(t,x_0)), t \in [0,T).$$

$$N(t,\xi(t,x_0)) = u(t,\xi(t,x_0)) + u_x(t,\xi(t,x_0)), t \in [0,T).$$

Then we have

$$u^2(t,\xi(t,x_0)) - u_x^2(t,\xi(t,x_0)) = M(t,\xi(t,x_0))N(t,\xi(t,x_0)) < 0.$$

Since $y(t,\xi(t,x_0)) = 0$, on $t \in [0,T)$, We can obtain by taking derivative with respect to t on $[0,T)$,

$$\frac{dM(t,\xi(t,x_0))}{dt} = -\xi_t(t,x_0)M(t,\xi(t,x_0)) + e^{-\xi(t,x_0)} \int_{-\infty}^{\xi(t,x_0)} e^{\eta} y_t(t,\eta)d\eta. \tag{3.6}$$

Using Eq.(2.1) and integrating by parts, in view of $y = u - u_{xx}$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\xi(t,x_0)} e^{\eta} y_t(t,\eta)d\eta = -\int_{-\infty}^{\xi(t,x_0)} e^{\eta}((u(t,\eta)+\gamma)y(t,\eta))_{\eta}d\eta - 2\int_{-\infty}^{\xi(t,x_0)} e^{\eta}y(t,\eta)u_{\eta}(t,\eta)d\eta \\
&= \int_{-\infty}^{\xi(t,x_0)} e^{\eta}(u(t,\eta)+\gamma)y(t,\eta)d\eta - 2\int_{-\infty}^{\xi(t,x_0)} e^{\eta}u(t,\eta)u_{\eta}(t,\eta)d\eta + 2\int_{-\infty}^{\xi(t,x_0)} e^{\eta}u_{\eta\eta}(t,\eta)u_{\eta}(t,\eta)d\eta \\
&= \int_{-\infty}^{\xi(t,x_0)} e^{\eta}u(t,\eta)y(t,\eta)d\eta + \gamma e^{\xi(t,x_0)}M(t,\xi(t,x_0)) - 2\int_{-\infty}^{\xi(t,x_0)} e^{\eta}u(t,\eta)u_{\eta}(t,\eta)d\eta \\
&+ 2\int_{-\infty}^{\xi(t,x_0)} e^{\eta}u_{\eta\eta}(t,\eta)u_{\eta}(t,\eta)d\eta \\
&= \int_{-\infty}^{\xi(t,x_0)} e^{\eta}u^2(t,\eta)d\eta - \int_{-\infty}^{\xi(t,x_0)} e^{\eta}u(t,\eta)u_{\eta\eta}(t,\eta)d\eta + \gamma e^{\xi(t,x_0)}M(t,\xi(t,x_0)) \\
&- 2\int_{-\infty}^{\xi(t,x_0)} e^{\eta}u(t,\eta)u_{\eta}(t,\eta)d\eta + 2\int_{-\infty}^{\xi(t,x_0)} e^{\eta}u_{\eta\eta}(t,\eta)u_{\eta}(t,\eta)d\eta \\
&= \int_{-\infty}^{\xi(t,x_0)} e^{\eta}u^2(t,\eta)d\eta - \int_{-\infty}^{\xi(t,x_0)} e^{\eta}u(t,\eta)u_{\eta}(t,\eta)d\eta - e^{\xi(t,x_0)}u(t,\xi(t,x_0))u_x(t,\xi(t,x_0)) \\
&+ e^{\xi(t,x_0)}u_x^2(t,\xi(t,x_0)) + \gamma e^{\xi(t,x_0)}M(t,\xi(t,x_0)) \\
&= \frac{3}{2}\int_{-\infty}^{\xi(t,x_0)} e^{\eta}u^2(t,\eta)d\eta - \frac{1}{2}e^{\xi(t,x_0)}u^2(t,\xi(t,x_0)) - e^{\xi(t,x_0)}u(t,\xi(t,x_0))u_x(t,\xi(t,x_0)) \\
&+ e^{\xi(t,x_0)}u_x^2(t,\xi(t,x_0)) + \gamma e^{\xi(t,x_0)}M(t,\xi(t,x_0))
\end{aligned} \tag{3.7}$$

Note that $p*(\frac{1}{2}u_x^2+u^2)(t,x) \geq \frac{1}{2}u^2(t,x), \forall (t,x) \in [0,T)\times R$, and $\xi_t(t,x) = u(t,\xi(t,x))+\gamma$, Substituting (3.7) into the last term in the expression (3.6) yields

$$\begin{aligned}
&\frac{dM(t,\xi(t,x_0))}{dt} = -(u(t,\xi(t,x_0))+\gamma)M(t,\xi(t,x_0)) + \frac{3}{2}e^{-\xi(t,x_0)}\int_{-\infty}^{\xi(t,x_0)} e^{\eta}u^2(t,\eta)d\eta \\
&- \frac{1}{2}u^2(t,\xi(t,x_0)) - u(t,\xi(t,x_0))u_x(t,\xi(t,x_0)) + u_x^2(t,\xi(t,x_0)) + \gamma M(t,\xi(t,x_0)) \\
&= -\frac{3}{2}u^2(t,\xi(t,x_0)) + \frac{3}{2}e^{-\xi(t,x_0)}\int_{-\infty}^{\xi(t,x_0)} e^{\eta}u^2(t,\eta)d\eta + u_x^2(t,\xi(t,x_0)) \\
&= -\frac{3}{2}u^2(t,\xi(t,x_0)) + e^{-\xi(t,x_0)}\int_{-\infty}^{\xi(t,x_0)} e^{\eta}(\frac{1}{2}u_{\eta}^2(t,\eta)+u^2(t,\eta))d\eta \\
&+ u_x^2(t,\xi(t,x_0)) + \frac{1}{2}e^{-\xi(t,x_0)}\int_{-\infty}^{\xi(t,x_0)} e^{\eta}(u^2(t,\eta)-u_{\eta}^2(t,\eta))d\eta \\
&\geq -u^2(t,\xi(t,x_0)) + u_x^2(t,\xi(t,x_0)) + \frac{1}{2}e^{-\xi(t,x_0)}\int_{-\infty}^{\xi(t,x_0)} e^{\eta}(u^2(t,\eta)-u_{\eta}^2(t,\eta))d\eta \\
&= -M(t,\xi(t,x_0))N(t,\xi(t,x_0)) + \frac{1}{2}e^{-\xi(t,x_0)}\int_{-\infty}^{\xi(t,x_0)} e^{\eta}(u^2(t,\eta)-u_{\eta}^2(t,\eta))d\eta
\end{aligned}$$

Then following the similar argument in [22], we complete the proof of this theorem. □

4 Blow-up rate

Theorem 4.1 *If $T < \infty$ is the blow-up time of the solution u of Eq.(2.2) with initial data $u_0 \in H^s(R)$, $s > \frac{3}{2}$. Then we have* $\lim\limits_{t\to T}(\inf\limits_{x\in R}\{u_x(t,x)\}(T-t)) = -1.$

Proof. Again we may assume $s = 3$ to prove the above theorem. By the assumption $T < \infty$ and Lemma 2.2, we know that $\liminf\limits_{t\to T}(I(t)) = -\infty$, where $I(t) = \inf\limits_{x\in R}(u_x(t,x))$, for $t \in [0,T)$. Lemma 2.7 implies $I(t)$ is locally Lipschitz with $I(t) = u_x(t,\delta(t)), t \in [0,T)$. Note that $u_{xx}(t,\delta(t)) = 0$ for $t \in (0,T)$. Then from (3.1), we deduce that

$$\frac{dI(t)}{dt} = -I^2(t) + \frac{3}{2}u^2(t,\delta) - p * (\frac{3}{2}u^2)(t,\delta). \tag{4.1}$$

According to Young's inequality, in view of Lemma 2.3, we have

$$\left\|p * u^2(t,.)\right\|_{L^\infty} \le \|p\|_{L^\infty}\left\|u^2(t,.)\right\|_{L^1} \le \frac{1}{2}\|u\|_{L^2}^2 \le 2\|u_0\|_{L^2}^2.$$

Now Lemma 2.4 and the above inequality imply that for all $t \in [0,T]$,

$$\begin{aligned}\left|\frac{3}{2}u^2(t,\xi) - \frac{3}{2}p*(u^2(t,\delta))\right| &\le \frac{3}{2}(\|u(t,.)\|_{L^\infty}^2 + \left\|p*(u^2)(t,\delta)\right\|_{L^\infty}) \\ &\le \frac{3}{2}((3\|u_0(x)\|_{L^2}^2 t + \|u_0(x)\|_{L^\infty})^2 + 2\|u_0\|_{L^2}^2) \\ &\le \frac{3}{2}((3\|u_0(x)\|_{L^2}^2 T + \|u_0(x)\|_{L^\infty})^2 + 2\|u_0\|_{L^2}^2).\end{aligned} \tag{4.2}$$

Set

$$U(T) = \frac{3}{2}((3\|u_0(x)\|_{L^2}^2 T + \|u_0(x)\|_{L^\infty})^2 + 2\|u_0\|_{L^2}^2). \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we deduce that

$$-U(T) \le \frac{dI(t)}{dt} + I^2(t) \le U(T), a.e.\ t \in [0,T). \tag{4.4}$$

Let $\varepsilon \in (0,1)$, since $\liminf\limits_{t\uparrow T} I(t) = -\infty$ by Lemma 2.2, there is some $t_0 \in (0,T)$ with $I(t_0) < 0$ and $I^2(t_0) > \frac{N}{\varepsilon}$. Let us first prove that

$$I^2(t) > \frac{U}{\varepsilon}, t \in (t_0,T) \tag{4.5}$$

Since $I(t)$ is locally Lipschitz, there is some $\sigma > 0$, such that $I^2(t) > \frac{U}{\varepsilon}$, $t \in (t_0, t_0+\sigma)$, Pick $\sigma > 0$ maximal with this property.
If $t_0 + \sigma < T$, we would have $I^2(t_0+\sigma) = \frac{U}{\varepsilon}$, while

$$\frac{dI(t)}{dt} \le -I^2 + U < -I^2 + \varepsilon I^2 = (\varepsilon - 1)I^2 < 0,\ a.e.\ (t_0, t_0+\sigma).$$

Being locally Lipschitz, the function $I(t)$ is absolutely continuous and therefore we would obtain by integrating the previous relation on $[t_0, t_0+\sigma]$, that $I(t_0+\sigma) \le I(t_0) < 0$, which on its turn would yield $I^2(t_0+\sigma) \ge I^2(t_0) > \frac{U}{\varepsilon}$, which is contradicted with $I^2(t_0+\sigma) = \frac{U}{\varepsilon}$. Thus (4.5) holds for $\forall t \in [t_0,T)$.

A combination of (4.4) and (4.5) enables us to infer

$$1 - \varepsilon \le \frac{-\frac{dI}{dt}}{I^2} \le 1 + \varepsilon. \quad a.e.(0,T). \tag{4.6}$$

Since $I(t)$ is locally Lipschitz on $[0,T)$ and (4.5) holds, it is easy to check that $\frac{1}{I(t)}$ is Lipschitz on (t_0,T), in view of $\frac{d}{dt}(\frac{1}{I}) = \frac{-\frac{dI}{dt}}{I^2}$, $a.e\ (t_0,T)$. For $t \in (t_0,T)$, integrate (4.6) on (t,T) to obtain

$$(1-\varepsilon)(T-t) \le -\frac{1}{I(t)} \le (1+\varepsilon)(T-t), \quad t \in (t_0,T).$$

That is, $\frac{1}{1+\varepsilon} \le -I(t)(T-t) \le \frac{1}{1-\varepsilon}$, $t \in (t_0,T)$. Since $\varepsilon \in (0,1)$ is arbitrary, the statement of Theorem 4.1 follows. $\square$

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