The Range of an Affine Fractal Interpolation Function

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Abstract: This paper discusses the range of a class of affine interpolation functions in terms of vertical scaling factors.

Key words: fractal; affine interpolation function; vertical scaling factor

1 Introduction

The research of fractal geometry and chaotic phenomena([7]-[11]) are very interesting. In particular, the affine fractal interpolation function provides a new approach to fit experimental data. It has been applied to model discrete sequence and signal ([3],[6]). In order to determine effectively vertical scaling factors when modeling experimental data, Dalla and Drakopoulos [2] gave some conditions that a vertical scaling factor must obey.

Let \( \{(x_i, y_i)\}_{i=0}^{N} \) be a given data set, where \( x_0 < x_1 < \cdots < x_N \). For \( i = 1, 2, \cdots, N \), affine map \( \omega_i \) is defined as

\[
\omega_i \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{cc} a_i & 0 \\ c_i & s_i \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) + \left( \begin{array}{c} d_i \\ e_i \end{array} \right),
\]

where real numbers \( a_i, c_i, s_i, d_i \) and \( e_i \) are chosen such that \( |s_i| < 1 \) and

\[
\omega_i \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) = \left( \begin{array}{c} x_{i-1} \\ y_{i-1} \end{array} \right), \quad \omega_i \left( \begin{array}{c} x_N \\ y_N \end{array} \right) = \left( \begin{array}{c} x_i \\ y_i \end{array} \right).
\]

Then there exists a unique attractor \( E = \bigcup_{i=1}^{N} \omega_i(E) \) ([11]). We call such \( \{s_i\}_{i=1}^{N} \) vertical scaling factors. Furthermore, \( E \) is the graph of a continuous function \( f : [x_0, x_N] \rightarrow \mathbb{R} \) satisfying \( f(x_i) = y_i \) (\( i = 0, 1, \cdots, N \)). Such a function is said to be an affine fractal interpolation function or AFIF for short. By the definition of AFIF, an AFIF is determined by \( \{(x_i, y_i)\}_{i=0}^{N} \) and vertical scaling factors \( \{s_i\}_{i=1}^{N} \).

In particular, we let \( f_{(h,s_1,s_2,s_3)}(x) \) denote the AFIF which is determined by vertical scaling factors \( \{s_i\}_{i=1}^{3} \) and interpolation points \( \{(x_i, y_i)\}_{i=0}^{3} = \{(0,0), (\frac{1}{4}, h), (\frac{3}{4}, h), (1,0)\} \). In this paper, we consider the range of \( f_{(h,s_1,s_2,s_3)}(x) \), and obtain the following main Theorem.

Theorem 1 Let \( h > 0 \). Then \( 0 \leq f_{(h,s_1,s_2,s_3)}(x) \leq h \) for any \( x \in [0,1] \) if and only if the vertical scaling factors \( (s_1, s_2, s_3) \in \Gamma \) where \( \Gamma = [-\frac{1}{3}, \frac{1}{3}] \times [-1,0] \times [-\frac{1}{3}, \frac{1}{3}] \setminus \{(x, y, z) : x + z > \frac{1}{3}\} \).

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2 Some Lemmas

Let $L_i(x) = a_i x + d_i$ and $F_i(x, y) = c_i x + s_i y + e_i$ for $i = 1, 2, 3$. Let $f(x) = f_{(h,s_1,s_2,s_3)}(x)$. It follows from the property of AFIF \[1,5\] that $f(L_i(x)) = F_i(x, f(x))$, where $i = 1, 2, 3$ and $x \in [0, 1]$, i.e.,

$$f(x) = F_i(L_i^{-1}(x), f(L_i^{-1}(x)))$$

(3)

where $i = 1, 2, 3$ and $x \in L_i([0, 1])$.

From (1), (2) and the interpolation points $\{(0, 0), (\frac{1}{3}, h), (\frac{2}{3}, h), (1, 0)\}$, we get

$$L_1(x) = \frac{1}{3} x, \hspace{1cm} L_2(x) = \frac{1}{3} x + \frac{1}{3}, \hspace{1cm} L_3(x) = \frac{1}{3} x + \frac{2}{3}$$

$$F_1(x, y) = hx + s_1y, \hspace{1cm} F_2(x, y) = h + s_2y, \hspace{1cm} F_3(x, y) = h(1 - x) + s_3y.$$  

(4)

It follows from (3) that the AFIF $f_{(h,s_1,s_2,s_3)}(x)$ satisfies the following formula:

$$f_{(h,s_1,s_2,s_3)}(x) = \begin{cases} 3hx + s_1f_{(h,s_1,s_2,s_3)}(3x), & \text{if } 0 \leq x \leq 1/3, \\ h + s_2f_{(h,s_1,s_2,s_3)}(3x - 1), & \text{if } 1/3 \leq x \leq 2/3, \\ 3h(1 - x) + s_3f_{(h,s_1,s_2,s_3)}(3x - 2), & \text{if } 2/3 \leq x \leq 1. \end{cases}$$

(5)

**Remark 1** The continuous function satisfying (5) and $f(0) = f(1) = 0, f(1/3) = f(2/3) = h$ is unique.

We use the notation $f_h(x)$ to replace $f_{(h,s_1,s_2,s_3)}(x)$ which is determined by $\{s_i\}_{i=1}^3$ and

$\{(x_i, y_i)\}_{i=0}^3 = \{(0, 0), (1/3, h), (2/3, h), (1, 0)\}$.

**Lemma 1** $f_h(x) = h f_1(x)$ for any $h$ and $s_1, s_2, s_3$.

**Proof.** When $h = 1$, by (5), there holds

$$f_1(x) = \begin{cases} 3x + s_1f_1(3x), & \text{if } 0 \leq x \leq 1/3, \\ 1 + s_2f_1(3x - 1), & \text{if } 1/3 \leq x \leq 2/3, \\ 3(1 - x) + s_3f_1(3x - 2), & \text{if } 2/3 \leq x \leq 1. \end{cases}$$

(6)

Multiplying (6) by $h$ and compare with (5), furthermore, by Remark 1, we obtain $f_h(x) = h f_1(x)$.

If Theorem 1 is true for $h = 1$, then Theorem 1 follows from Lemma 1 for any $h$.

**Lemma 2** $f_{(h,s_1,s_2,s_3)}(x) = f_{(h,s_1,s_2,s_1)}(1 - x)$ for any $s_1, s_2, s_3$ and $x \in [0, 1]$.

**Proof.** Let $g(x) = f_{(1,s_3,s_2,s_1)}(1 - x)$. We shall distinguish three cases:

(i) When $0 \leq x \leq \frac{1}{3}$, i.e., $\frac{2}{3} \leq 1 - x \leq 1$, by (6), we have

$$g(x) = f_{(1,s_3,s_2,s_1)}(1 - x) = 3(1 - (1 - x)) + s_1f_{(1,s_3,s_2,s_1)}(1 - 3x) = 3x + s_1g(3x).$$

(ii) When $\frac{1}{3} \leq x \leq \frac{2}{3}$, i.e., $\frac{1}{3} \leq 1 - x \leq \frac{2}{3}$, by (6), we have

$$g(x) = f_{(1,s_3,s_2,s_1)}(1 - x) = 1 + s_2f_{(1,s_3,s_2,s_1)}(2 - 3x) = 1 + s_2g(3x - 1).$$

(iii) When $\frac{2}{3} \leq x \leq 1$, i.e., $0 \leq 1 - x \leq \frac{1}{3}$, by (6), we have

$$g(x) = f_{(1,s_3,s_2,s_1)}(1 - x) = 3(1 - x) + s_3f_{(1,s_3,s_2,s_1)}(3 - 3x) = 3(1 - x) + s_3g(3x - 2).$$

By Remark 1, we obtain that $f_{(1,s_1,s_2,s_3)}(x) = g(x) = f_{(1,s_3,s_2,s_1)}(1 - x)$.

**Remark 2** The symmetry of $s_1, s_3$ is given in Lemma 2.

**Lemma 3** Let $G(x) = f_{(1,\frac{1}{3},0,0)}(x)$. Then $G(\frac{1}{3m}, \frac{2}{3m})$ and $G(\frac{2m}{3}, \frac{1}{3m})$ are linear functions for any positive integer $n$. Furthermore, $0 \leq G(x) \leq 1$ for any $x \in [0, 1]$.
Proof. Firstly, by induction on \( n \), we will prove that \( G|_{\left[ \frac{1}{3^n}, \frac{2}{3^n} \right]} \) and \( G|_{\left[ \frac{2}{3^n}, \frac{1}{3^n} \right]} \) are linear functions for any positive integer \( n \).

When \( n = 1 \), by formula (6), we have \( G|_{\left[ \frac{1}{3}, \frac{2}{3} \right]}(x) \equiv 1 \) and \( G|_{\left[ \frac{2}{3}, \frac{1}{3} \right]}(x) = 3(1 - x) \). It is easy to see that \( G|_{\left[ \frac{1}{3}, \frac{2}{3} \right]} \) and \( G|_{\left[ \frac{2}{3}, \frac{1}{3} \right]} \) are linear functions.

Now assume \( G|_{\left[ \frac{1}{3^n}, \frac{2}{3^n} \right]} \) and \( G|_{\left[ \frac{2}{3^n}, \frac{1}{3^n} \right]} \) are linear, we will show \( G|_{\left[ \frac{1}{3^n+1}, \frac{2}{3^n+1} \right]} \) and \( G|_{\left[ \frac{2}{3^n+1}, \frac{1}{3^n+1} \right]} \) are also linear. In fact, suppose \( x \in \left[ \frac{1}{3^n+1}, \frac{2}{3^n+1} \right] \) or \( x \in \left[ \frac{2}{3^n+1}, \frac{1}{3^n+1} \right] \), then \( 3x \in \left[ \frac{1}{3^n}, \frac{2}{3^n} \right] \) or \( 3x \in \left[ \frac{2}{3^n}, \frac{1}{3^n} \right] \). Note that \( G(x) = 3x + \frac{1}{2}G(3x) \), which imply that \( G|_{\left[ \frac{1}{3^n+1}, \frac{2}{3^n+1} \right]} \) and \( G|_{\left[ \frac{2}{3^n+1}, \frac{1}{3^n+1} \right]} \) are also linear.

To prove that \( 0 \leq G(x) \leq 1 \) for any \( x \in [0, 1] \), we note that \( G|_{\left[ \frac{1}{3^n}, \frac{2}{3^n} \right]} \) and \( G|_{\left[ \frac{2}{3^n}, \frac{1}{3^n} \right]} \) are linear, that means we need only to prove \( 0 \leq G(\frac{1}{3^n+1}) \leq 1 \) and \( 0 \leq G(\frac{2}{3^n+1}) \leq 1 \) for any \( n \in \mathbb{N} \). In fact, by formula (6), we have \( G(\frac{1}{3^n+1}) = \frac{1}{3^n} + \frac{1}{3}G(\frac{1}{3^n}) = \cdots = \frac{n+1}{3^n} \) and \( G(\frac{2}{3^n+1}) = \frac{2}{3^n} + \frac{1}{3}G(\frac{2}{3^n}) = \cdots = \frac{2n+1}{3^n} \) for \( n \in \mathbb{N} \), which implies prove

\[
0 \leq G(\frac{1}{3^n+1}), G(\frac{2}{3^n+1}) \leq 1.
\]

Lemma 4 Let \( s_1, s_3 > 0 \) and \( s_1 + s_3 \leq \frac{1}{3} \), then \( 0 \leq f_{(1, s_1, 0, s_3)}(x) \leq 1 \) for any \( x \in [0, 1] \).

Proof. Let \( f(x) = f_{(1, s_1, 0, s_3)}(x) \). It follows from the expansion \( f(\psi(\omega)) = \sum s_{\omega(\omega)}h_{f}(\psi(\sigma^{j}\omega)) \) (5) that \( f(x) \geq 0 \). It suffices to show that \( f(x) \leq 1 \) for all \( x \in [0, 1] \).

Case (i): Assume \( s_1 + s_3 = \frac{1}{3} \). Let \( q_1(x) = x, q_2(x) = 1 \) and \( q_3(x) = 1 - x \). From (4) and \( s_2 = 0 \), we have \( F_1(x, y) = s_1y + q_1(x), F_2(x, y) = q_2(x) \) and \( F_3(x, y) = s_3y + q_3(x) \). Let \( F_{i, 1}(x, y) = \frac{q_i(x) + q_i(y)}{q_i(x)} \), \( F_{i, 1}(x, y) = \frac{q_i(x)}{q_i(x)} \) and \( y_{i, 1} = \frac{1}{s_1} - \frac{1}{s_3} \). We show that \( F_{i, 1}(x_3, y_{i, 1}) = F_{i, 1}(x_3, y_{i, 1}) = F_{i, 1}(1, -\frac{1}{3}) = 0, F_{2, 1}(0, y_{i, 1}) = F_{2, 1}(x_3, y_{i, 1}) = 0 \) and \( F_{3, 1}(x_3, y_{i, 1}) = F_{3, 1}(0, \frac{1}{3}) = 0 \). Furthermore, we have \( F_{1, 1}(x_3, y_{i, 1}) = F_{2, 1}(x_3, y_{i, 1}) \) and \( F_{3, 1}(x_3, y_{i, 1}) = F_{3, 1}(x_3, y_{i, 1}) \). Hence, there exists an AFIF \( g(x) \) that be determined by \( \{L_i(x), F_{i, 1}(x, y)\}_{i=1}^{3} \). Since \( y_{i, 1} = \frac{1}{s_1} > 0 \) and \( y_{i, 1} = -\frac{1}{s_3} < 0 \), we have

\[
f'(x) = g(x) \text{ and } g(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{3}], \\ 0, & \text{if } x \in [\frac{1}{3}, \frac{2}{3}], \\ < 0, & \text{if } x \in (\frac{2}{3}, 1]. \end{cases}
\]

Therefore, \( 0 \leq f(x) \leq 1 \) for all \( x \in [0, 1] \).

Case (ii): Assume \( s_1 + s_3 < \frac{1}{3} \). Let \( t_1 = s_1, t_2 = 0 \) and \( t_3 = \frac{1}{3} - s_1 \), then \( s_3 < t_3 \). By case (i), we have \( f_{(1, t_1, 0, t_3)}(x) \leq 1 \). It follows from the expansion of AFIF that \( f_{(1, s_1, 0, s_3)}(x) \leq f_{(1, t_1, 0, t_3)}(x) \). Hence \( f_{(1, s_1, 0, s_3)}(x) \leq 1 \).
Lemma 5 Let $0 < s_1, s_3 \leq \frac{1}{3}$ and $s_1 + s_3 > \frac{1}{3}$, then $\max_{x \in [0,1]} f_{(1,s_1,0,s_3)}(x) > 1$.

Proof. Let $\Omega$ be the symbolic space $\{\omega = (i_0, i_1, \ldots, i_t, \ldots) \mid i_t \in \{0, 1, 2\}$ for all $t$. Suppose $\omega_m = (i_0, i_1, \ldots, i_t, \ldots) \in \Omega$, where $i_t = \begin{cases} 1, & \text{if } 1 \leq t \leq m, \\ 0, & \text{if } t = 0 \text{ or } t \geq m + 1. \end{cases}$ Suppose $\omega^*_m = (j_0, j_1, \ldots, j_t, \ldots) \in \Omega$, where $j_t = \begin{cases} 1, & \text{if } t \leq m - 1, \\ 0, & \text{if } t \geq m. \end{cases}$ Then we have

$$f(\psi(\omega_m)) = \sum_{j=0}^{\infty} s_{\omega(j)} h_f(\psi(\sigma^j \omega_m)) = h_f(\psi(\omega_m)) + s_1 h_f(\psi(\omega^*_m)) + s_1 s_2 h_f(\psi(\omega^*_{m-1})) + \cdots + s_1 s_3^{m-1} h_f(\psi(\omega^*_1))$$

$$= (1 - \frac{1}{3})^m + (1 - \frac{1}{3})^{m-1} s_1 + (1 - \frac{1}{3})^{m-2} s_1 s_2 + \cdots + (1 - \frac{1}{3})^{m-1} s_1 s_2 s_3 - (\frac{1}{3})^{m-1} (1 + 3s_3 + \cdots + (3s_3)^{m-2}) s_1.$$

Since $0 < s_3 \leq \frac{1}{3}$, we obtain that $\lim_{m \to +\infty} f(\psi(\omega_m)) = 1 + \frac{s_3}{1 - s_3} > 1$. Hence there exists $x \in [0,1]$ such that $f(x) > 1$, due to the continuity of $f(x)$.

3 Proof Theorem 1

In order to prove Theorem 1, we need the following four propositions.

Proposition 1 If $0 \leq f_{(1,s_1,0,s_3)}(x) \leq 1$ for any $x \in [0,1]$, then $-\frac{1}{3} \leq s_1, s_3 \leq \frac{1}{3}$, $-1 \leq s_2 \leq 0$.

Proposition 2 Let $-\frac{1}{3} \leq s_1, s_3 \leq 0$ and $-1 \leq s_2 \leq 0$. Then $0 \leq f_{(1,s_1,0,s_3)}(x) \leq f_{(1,0,0,0)}(x)$ for any $x \in [0,1]$.

Proposition 3 Let $0 \leq s_1 \leq \frac{1}{3}$, $-1 \leq s_2 \leq 0$ and $-\frac{1}{3} \leq s_3 \leq 0$. Then $0 \leq f_{(1,s_1,0,s_3)}(x) \leq f_{(1,0,0,0)}(x)$ for any $x \in [0,1]$.

Proposition 4 Let $s_1, s_3 > 0$ and $-1 \leq s_2 \leq 0$. Then $0 \leq f_{(1,s_1,0,s_3)}(x) \leq 1$ for any $x \in [0,1]$ if and only if $s_1 + s_3 \leq \frac{1}{3}$.

Proof of Proposition 1:

Let $f(x) = f_{(1,s_1,0,s_3)}(x)$. It follows from (6) that $f(1/9) = 1/3 + s_1$, $f(2/9) = 2/3 + s_1$, $f(4/9) = 1 + s_2$, $f(5/9) = 1 + s_2$, $f(7/9) = 2/3 + s_3$, $f(8/9) = 1/3 + s_3$. Since $0 \leq f_{(1,s_1,0,s_3)}(x) \leq 1$, we have

$$-\frac{1}{3} \leq s_1, s_3 \leq \frac{1}{3} \text{ and } -1 \leq s_2 \leq 0.$$

Proof of Proposition 2:

Let $f(x) = f_{(1,s_1,s_2,s_3)}(x)$ and $H(x) = f_{(1,0,0,0)}(x)$ satisfying

$$H(x) = \begin{cases} 3x, & \text{if } 0 \leq x \leq 1/3, \\ 1, & \text{if } 1/3 \leq x \leq 2/3, \\ 3(1 - x), & \text{if } 2/3 \leq x \leq 1. \end{cases}$$

Given $k \in \mathbb{N}$, let $\Lambda_k = \{i/3^k \in [0,1] : i \in \mathbb{Z}\}$. It suffices to show that $0 \leq f(x) \leq H(x)$ for all $x \in \Lambda_k$ by induction on $k$. 

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For \( k = 1 \), it is obvious that \( 0 \leq f(x) = H(x) \) for \( x \in \Lambda_1 = \{0, \frac{1}{3}, \frac{2}{3}, 1\} \), due to the property of AFIF. Now we assume that for a given integer \( k \), \( 0 \leq f(x) \leq H(x) \) for any \( x \in \Lambda_k \). We shall show \( 0 \leq f(x) \leq H(x) \) for all \( x \in \Lambda_{k+1} \). We can distinguish three cases as follows.

(i) When \( x \in \Lambda_{k+1} \cap [0, \frac{2}{3}] \), i.e., \( 3x \in \Lambda_k \), by formulas (5) and (6), we have \( f(x) = 3x + s_1 f(3x) \leq 3x = H(x) \). On the other hand, we can show \( f(x) \geq 0 \) as follows: If \( 3x \in [0, \frac{1}{3}] \), then \( f(x) = 3x + s_1 f(3x) \geq 3x - |s_1| H(3x) = 3x(1 - 3|s_1|) \geq 0 \); If \( 3x \in [\frac{1}{3}, \frac{1}{2}] \), then \( f(x) \geq 3x - |s_1| \geq 0 \).

(ii) When \( x \in \Lambda_{k+1} \cap [\frac{1}{3}, \frac{2}{3}] \), i.e., \( 3x - 1 \in \Lambda_k \), by formulas (5) and (6), we have \( f(x) = 1 + s_2 f(3x - 1) \). Since \(-1 \leq s_2 f(3x - 1) \leq 0\),

\[
0 \leq f(x) \leq 1 = H(x).
\]

(iii) When \( x \in \Lambda_{k+1} \cap [\frac{2}{3}, 1] \), i.e., \( 3x - 2 \in \Lambda_k \), by formulas (5) and (6), we have \( f(x) = 3(1 - x) + s_3 f(3x - 2) \leq 3(1 - x) = H(x) \). On the other hand, we can show \( f(x) \geq 0 \) as follows: If \( 3x - 2 \in [0, \frac{2}{3}] \), then \( f(x) = 3(1 - x) + s_3 f(3x - 2) \geq 3(1 - x) - |s_3| \geq 0 \); If \( 3x - 2 \in [\frac{2}{3}, 1] \), then \( f(x) = 3(1 - x) + s_3 f(3x - 2) \geq 3(1 - x) - |s_3| H(3x - 2) = 3(1 - x) - 9 |s_3| (1 - x) = 3(1 - x)(1 - 3|s_3|) \geq 0 \).

Because of the continuity of \( f(x) \) and \( H(x) \), we obtain \( 0 \leq f(x) \leq H(x) \) for any \( x \in [0, 1] \).

Figure 3: Graphs of \( f_{(1,0,0,0)}(x) \) and \( f_{(1,-0.2,-0.7,-0.1)}(x) \).

Proof of Proposition 3:

Let \( f(x) = f_{(1, s_1, s_2, s_3)}(x) \) and \( G(x) = f_{(1, \frac{1}{3}, 0, 0)}(x) \) satisfying

\[
G(x) = \begin{cases} 
3x + G(3x)/3, & \text{if } 0 \leq x \leq 1/3, \\
1, & \text{if } 1/3 \leq x \leq 2/3, \\
3(1 - x), & \text{if } 2/3 \leq x \leq 1.
\end{cases}
\]

Given \( k \in \mathbb{N} \), let \( \Lambda_k = \{i/3^k \in [0, 1] : i \in \mathbb{Z}\} \). It suffices to show that \( 0 \leq f(x) \leq G(x) \) with \( x \in \Lambda_k \) by induction on \( k \).

For \( k = 1 \), it is obvious that \( 0 \leq f(x) = G(x) \) for \( x \in \Lambda_1 = \{0, \frac{1}{3}, \frac{2}{3}, 1\} \), due to the property of AFIF. Now we assume that for a given integer \( k \), \( 0 \leq f(x) \leq G(x) \) for any \( x \in \Lambda_k \). We shall show \( 0 \leq f(x) \leq G(x) \) for all \( x \in \Lambda_{k+1} \). We can distinguish three cases as follows.

(i) When \( x \in \Lambda_{k+1} \cap [0, \frac{1}{3}] \), i.e., \( 3x \in \Lambda_k \), by formulas (5) and (6), we have \( f(x) = 3x + s_1 f(3x) \leq 3x + \frac{1}{3} G(3x) = G(x) \). On the other hand, we have \( f(x) = 3x + s_1 f(3x) \geq 0 \) since \( s_1 \), \( f(3x) \geq 0 \).

(ii) When \( x \in \Lambda_{k+1} \cap [\frac{1}{3}, \frac{2}{3}] \), i.e., \( 3x - 1 \in \Lambda_k \), by formulas (5) and (6), we have \( f(x) = 1 + s_2 f(3x - 1) \). It follows from \(-1 \leq s_2 f(3x - 1) \leq 0 \) that \( 0 \leq f(x) \leq 1 = G(x) \).

(iii) When \( x \in \Lambda_{k+1} \cap [\frac{2}{3}, 1] \), i.e., \( 3x - 2 \in \Lambda_k \), by formulas (5) and (6), we have \( f(x) = 3(1 - x) + s_3 f(3x - 2) \leq 3(1 - x) = G(x) \). On the other hand we can show have \( f(x) \geq 0 \) as follows: If \( 3x - 2 \in [0, \frac{2}{3}] \), then \( f(x) = 3(1 - x) + s_3 f(3x - 2) \geq 3(1 - x) - |s_3| \geq 0 \); If \( 3x - 2 \in [\frac{2}{3}, 1] \), then \( f(x) = 3(1 - x) + s_3 f(3x - 2) \geq 3(1 - x) - |s_3| G(3x - 2) = 3(1 - x) - 9 |s_3| (1 - x) = 3(1 - x)(1 - 3|s_3|) \geq 0 \).

Because of the continuity of \( f(x) \) and \( G(x) \), we obtain \( 0 \leq f(x) \leq G(x) \) for any \( x \in [0, 1] \).

Proof of Proposition 4:

(1) Sufficient condition: Given \( k \in \mathbb{N} \), let \( \Lambda_k = \{i/3^k \in [0, 1] : i \in \mathbb{Z}\} \). It is easy to prove that \( 0 \leq f_{(1, s_1, s_2, s_3)}(x) \leq f_{(1, s_1, 0, s_3)}(x) \) for all \( x \in \Lambda_k \) by induction on \( k \). Due to the continuity of \( f_{(1, s_1, s_2, s_3)}(x) \) and \( f_{(1, s_1, 0, s_3)}(x) \), we have \( 0 \leq f_{(1, s_1, s_2, s_3)}(x) \leq f_{(1, s_1, 0, s_3)}(x) \) for any \( x \in [0, 1] \). Therefore, the sufficient condition follows from Lemma 4.

(2) Necessary condition: We obtain the necessary condition from Lemma 5.

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We complete the proof of Theorem 1 by using Proposition 1-4 and Lemma 1-5.

Figure 4: Graphs of $f_{(1,\frac{1}{2},0,0)}(x)$ and $f_{(1,0.2,-0.7,-0.1)}(x)$.

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References