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The Range of an Affine Fractal Interpolation Function

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Abstract:This paper discusses the range of a class of affine interpolation functions in terms of vertical scaling factors.

Key words:fractal; affine interpolation function; vertical scaling factor

1 Introduction

The research of fractal geometry and chaotic phenomena([7]-[11]) are very interesting. In particular, the affine fractal interpolation function provides a new approach to fit experimental data. It has been applied to model discrete sequence and signal ([3],[6]). In order to determine effectively vertical scaling factors when modeling experimental data, Dalla and Drakopoulos [2] gave some conditions that a vertical scaling factor must obey.

Let $\{(x_i, y_i)\}_{i=0}^N$ be a given data set, where $x_0 < x_1 < \cdots < x_N$. For $i = 1, 2, \cdots, N$, affine map ω_i is defined as

$$\omega_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & s_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_i \\ e_i \end{pmatrix}, \tag{1}$$

where real numbers a_i, c_i, s_i, d_i and e_i are chosen such that $|s_i| < 1$ and

$$\omega_i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix}, \omega_i \begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}. \tag{2}$$

Then there exists a unique attractor $E = \cup_{i=1}^N \omega_i(E)$ ([1]). We call such $\{s_i\}_{i=1}^N$ vertical scaling factors. Furthermore, E is the graph of a continuous function $f : [x_0, x_N] \to \mathbb{R}$ satisfying $f(x_i) = y_i$ $(i = 0, 1, \cdots, N)$. Such a function is said to be an *affine fractal interpolation function* or AFIF for short. By the definition of AFIF, an AFIF is determined by $\{(x_i, y_i)\}_{i=0}^N$ and vertical scaling factors $\{s_i\}_{i=1}^N$.

In particular, we let $f_{(h,s_1,s_2,s_3)}(x)$ denote the AFIF which is determined by vertical scaling factors $\{s_i\}_{i=1}^3$ and interpolation points $\{(x_i, y_i)\}_{i=0}^3 = \{(0,0), (\frac{1}{3}, h), (\frac{2}{3}, h), (1,0)\}$. In this paper, we consider the range of $f_{(h,s_1,s_2,s_3)}(x)$, and obtain the following main Theorem.

Theorem 1 *Let* $h > 0$. *Then* $0 \le f_{(h,s_1,s_2,s_3)}(x) \le h$ *for any* $x \in [0,1]$ *if and only if the vertical scaling factors* $(s_1, s_2, s_3) \in \Gamma$ *where* $\Gamma = [-\frac{1}{3}, \frac{1}{3}] \times [-1, 0] \times [-\frac{1}{3}, \frac{1}{3}] \backslash \{(x,y,z) : x + z > \frac{1}{3}\}$.

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2 Some Lemmas

Let $L_i(x) = a_i x + d_i$ and $F_i(x,y) = c_i x + s_i y + e_i$ for $i = 1,2,3$. Let $f(x) = f_{(h,s_1,s_2,s_3)}(x)$. It follows from the property of AFIF [1,5] that $f(L_i(x)) = F_i(x, f(x))$, where $i = 1,2,3$ and $x \in [0,1]$, i.e.,

$$f(x) = F_i(L_i^{-1}(x), f(L_i^{-1}(x))) \tag{3}$$

where $i = 1,2,3$ and $x \in L_i([0,1])$.

From (1), (2) and the interpolation points $\{(0,0),(\frac{1}{3},h),(\frac{2}{3},h),(1,0)\}$, we get

$$\begin{array}{lll} L_1(x) = \frac{1}{3}x, & L_2(x) = \frac{1}{3}x + \frac{1}{3}, & L_3(x) = \frac{1}{3}x + \frac{2}{3} \\ F_1(x,y) = hx + s_1 y, & F_2(x,y) = h + s_2 y, & F_3(x,y) = h(1-x) + s_3 y. \end{array} \tag{4}$$

It follows from (3) that the AFIF $f_{(h,s_1,s_2,s_3)}(x)$ satisfies the following formula:

$$f_{(h,s_1,s_2,s_3)}(x) = \begin{cases} 3hx + s_1 f_{(h,s_1,s_2,s_3)}(3x), & \text{if } 0 \le x \le 1/3, \\ h + s_2 f_{(h,s_1,s_2,s_3)}(3x-1), & \text{if } 1/3 \le x \le 2/3, \\ 3h(1-x) + s_3 f_{(h,s_1,s_2,s_3)}(3x-2), & \text{if } 2/3 \le x \le 1. \end{cases} \tag{5}$$

Remark 1 *The continuous function satisfying (5) and $f(0) = f(1) = 0, f(1/3) = f(2/3) = h$ is unique.*

We use the notation $f_h(x)$ to replace $f_{(h,s_1,s_2,s_3)}(x)$ which is determined by $\{s_i\}_{i=1}^3$ and

$$\{(x_i,y_i)\}_{i=0}^3 = \{(0,0),(1/3,h),(2/3,h),(1,0)\}.$$

Lemma 1 *$f_h(x) = hf_1(x)$ for any h and (s_1,s_2,s_3).*

Proof. When $h = 1$, by (5), there holds

$$f_1(x) = \begin{cases} 3x + s_1 f_1(3x), & \text{if } 0 \le x \le 1/3, \\ 1 + s_2 f_1(3x-1), & \text{if } 1/3 \le x \le 2/3, \\ 3(1-x) + s_3 f_1(3x-2), & \text{if } 2/3 \le x \le 1. \end{cases} \tag{6}$$

Multiplying (6) by h and compare with (5), furthermore, by Remark 1, we obtain $f_h(x) = hf_1(x)$.

If Theorem 1 is true for $h = 1$, then Theorem 1 follows from Lemma 1 for any h.

Lemma 2 *$f_{(h,s_1,s_2,s_3)}(x) = f_{(h,s_3,s_2,s_1)}(1-x)$ for any (s_1,s_2,s_3) and $x \in [0,1]$.*

Proof. Let $g(x) = f_{(1,s_3,s_2,s_1)}(1-x)$. We shall distinguish three cases:

(i) When $0 \le x \le \frac{1}{3}$, i.e., $\frac{2}{3} \le 1-x \le 1$, by (6), we have

$$g(x) = f_{(1,s_3,s_2,s_1)}(1-x) = 3(1-(1-x)) + s_1 f_{(1,s_3,s_2,s_1)}(1-3x) = 3x + s_1 g(3x).$$

(ii) When $\frac{1}{3} \le x \le \frac{2}{3}$, i.e., $\frac{1}{3} \le 1-x \le \frac{2}{3}$, by (6), we have

$$g(x) = f_{(1,s_3,s_2,s_1)}(1-x) = 1 + s_2 f_{(1,s_3,s_2,s_1)}(2-3x) = 1 + s_2 g(3x-1).$$

(iii) When $\frac{2}{3} \le x \le 1$, i.e., $0 \le 1-x \le \frac{1}{3}$, by (6), we have

$$g(x) = f_{(1,s_3,s_2,s_1)}(1-x) = 3(1-x) + s_3 f_{(1,s_3,s_2,s_1)}(3-3x) = 3(1-x) + s_3 g(3x-2).$$

By Remark 1, we obtain that $f_{(1,s_1,s_2,s_3)}(x) = g(x) = f_{(1,s_3,s_2,s_1)}(1-x)$.

Remark 2 *The symmetry of s_1, s_3 is given in Lemma 2.*

Lemma 3 *Let $G(x) = f_{(1,\frac{1}{3},0,0)}(x)$. Then $G|_{[\frac{1}{3^n},\frac{2}{3^n}]}$ and $G|_{[\frac{2}{3^n},\frac{1}{3^{n-1}}]}$ are linear functions for any positive integer n. Furthermore, $0 \le G(x) \le 1$ for any $x \in [0,1]$.*

Proof. Firstly, by induction on n, we will prove that $G|_{[\frac{1}{3^n},\frac{2}{3^n}]}$ and $G|_{[\frac{2}{3^n},\frac{1}{3^{n-1}}]}$ are linear functions for any positive integer n.

When $n = 1$, by formula (6), we have $G|_{[\frac{1}{3},\frac{2}{3}]}(x) \equiv 1$ and $G|_{[\frac{2}{3},1]}(x) = 3(1-x)$. It is easy to see that $G|_{[\frac{1}{3},\frac{2}{3}]}$ and $G|_{[\frac{2}{3},1]}$ are linear functions.

Now assume $G|_{[\frac{1}{3^n},\frac{2}{3^n}]}$ and $G|_{[\frac{2}{3^n},\frac{1}{3^{n-1}}]}$ are linear, we will show $G|_{[\frac{1}{3^{n+1}},\frac{2}{3^{n+1}}]}$ and $G|_{[\frac{2}{3^{n+1}},\frac{1}{3^n}]}$ are also linear. In fact, suppose $x \in [\frac{1}{3^{n+1}}, \frac{2}{3^{n+1}}]$ or $x \in [\frac{2}{3^{n+1}}, \frac{1}{3^n}]$, then $3x \in [\frac{1}{3^n}, \frac{2}{3^n}]$ or $3x \in [\frac{2}{3^n}, \frac{1}{3^{n-1}}]$. Note that $G(x) = 3x + \frac{1}{3}G(3x)$, which imply that $G|_{[\frac{1}{3^{n+1}},\frac{2}{3^{n+1}}]}$ and $G|_{[\frac{2}{3^{n+1}},\frac{1}{3^n}]}$ are also linear.

To prove that $0 \le G(x) \le 1$ for any $x \in [0,1]$, we note that $G|_{[\frac{1}{3^n},\frac{2}{3^n}]}$ and $G|_{[\frac{2}{3^n},\frac{1}{3^{n-1}}]}$ are linear, that means we need only to prove $0 \le G(\frac{1}{3^{n+1}}) \le 1$ and $0 \le G(\frac{2}{3^{n+1}}) \le 1$ for any $n \in \mathbb{N}$. In fact, by formula (6), we have $G(\frac{1}{3^{n+1}}) = \frac{1}{3^n} + \frac{1}{3}G(\frac{1}{3^n}) = \cdots = \frac{n+1}{3^n}$ and $G(\frac{2}{3^{n+1}}) = \frac{2}{3^n} + \frac{1}{3}G(\frac{2}{3^n}) = \cdots = \frac{2n+1}{3^n}$ for $n \in \mathbb{N}$, which implies prove

$$0 \le G(\frac{1}{3^{n+1}}), G(\frac{2}{3^{n+1}}) \le 1.$$

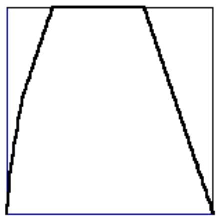

Figure 1: Graph of $f_{(1,\frac{1}{3},0,0)}(x)$.

Lemma 4 *Let $s_1, s_3 > 0$ and $s_1 + s_3 \le \frac{1}{3}$, then $0 \le f_{(1,s_1,0,s_3)}(x) \le 1$ for any $x \in [0,1]$.*

Proof. Let $f(x) = f_{(1,s_1,0,s_3)}(x)$. It follows from the expansion $f(\psi(\omega)) = \sum s_{\omega(j)} h_f(\psi(\sigma^j\omega))$ ([5]) that $f(x) \ge 0$. It suffices to show that $f(x) \le 1$ for all $x \in [0,1]$.

Case (i): Assume $s_1 + s_3 = \frac{1}{3}$. Let $q_1(x) = x$, $q_2(x) = 1$ and $q_3(x) = 1-x$. From (4) and $s_2 = 0$, we have $F_1(x,y) = s_1y + q_1(x)$, $F_2(x,y) = q_2(x)$ and $F_3(x,y) = s_3y + q_3(x)$. Let $F_{i,1}(x,y) = \frac{s_iy+q_i'(x)}{a_i} (i = 1,2,3)$, $y_{0,1} = \frac{q_1'(x_0)}{a_1-s_1}$ and $y_{3,1} = \frac{q_3'(x_3)}{a_3-s_3}$, that is, $F_{1,1}(x,y) = 3(s_1y+1)$, $F_{2,1}(x,y) = 0$, $F_{3,1}(x,y) = 3(s_3y-1)$, $y_{0,1} = \frac{1}{\frac{1}{3}-s_1}$ and $y_{3,1} = -\frac{1}{\frac{1}{3}-s_3}$. We show that $F_{1,1}(x_3,y_{3,1}) = F_{1,1}(1, -\frac{1}{\frac{1}{3}-s_3}) = 0$, $F_{2,1}(x_0,y_{0,1}) = 0$, $F_{2,1}(x_3,y_{3,1}) = 0$ and $F_{3,1}(x_0,y_{0,1}) = F_{3,1}(0, \frac{1}{\frac{1}{3}-s_1}) = 0$. Furthermore, we have $F_{1,1}(x_3,y_{3,1}) = F_{2,1}(x_0,y_{0,1})$ and $F_{2,1}(x_3,y_{3,1}) = F_{3,1}(x_0,y_{0,1})$. Hence, there exists an AFIF $g(x)$ that be determined by $\{L_i(x), F_{i,1}(x,y)\}_{i=1}^3$. Since $y_{0,1} = \frac{1}{\frac{1}{3}-s_1} > 0$ and $y_{3,1} = -\frac{1}{\frac{1}{3}-s_3} < 0$, we have $f'(x) = g(x)$ and $g(x) = \begin{cases} > 0, & \text{if } x \in [0,\frac{1}{3}), \\ = 0, & \text{if } x \in [\frac{1}{3},\frac{2}{3}], \\ < 0, & \text{if } x \in (\frac{2}{3},1]. \end{cases}$ Therefore, $0 \le f(x) \le 1$ for all $x \in [0,1]$.

Case (ii): Assume $s_1 + s_3 < \frac{1}{3}$. Let $t_1 = s_1$, $t_2 = 0$ and $t_3 = \frac{1}{3} - s_1$, then $s_3 < t_3$. By case (i), we have $f_{(1,t_1,0,t_3)}(x) \le 1$. It follows from the expansion of AFIF that $f_{(1,s_1,0,s_3)}(x) \le f_{(1,t_1,0,t_3)}(x)$. Hence $f_{(1,s_1,0,s_3)}(x) \le 1$.

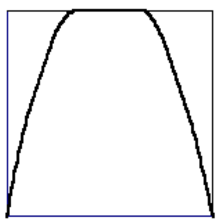

Figure 2: Graph of $f_{(1,0.19,0,0.1433...)}(x)$.

Lemma 5 *Let* $0 < s_1, s_3 \leq \frac{1}{3}$ *and* $s_1 + s_3 > \frac{1}{3}$, *then* $\max_{x\in[0,1]} f_{(1,s_1,0,s_3)}(x) > 1$.

Proof. Let Ω be the symbolic space $\{\omega = (i_0, i_1, \cdots, i_t, \cdots) \mid i_t \in \{0,1,2\} \text{ for all } t\}$.

Suppose $\omega_m = (i_0, i_1, \cdots, i_t, \cdots) \in \Omega$, where $i_t = \begin{cases} 1, & \text{if } 1 \leq t \leq m, \\ 0, & \text{if } t = 0 \text{ or } t \geq m+1. \end{cases}$

Suppose $\omega_m^* = (j_0, j_1, \cdots, j_t, \cdots) \in \Omega$, where $j_t = \begin{cases} 1, & \text{if } t \leq m-1, \\ 0, & \text{if } t \geq m. \end{cases}$

Then we have

$$\begin{aligned} & f(\psi(\omega_m)) = \sum_{j=0}^{\infty} s_{\omega(j)} h_f(\psi(\sigma^j \omega_m)) \\ = \; & h_f(\psi(\omega_m)) + s_1 h_f(\psi(\omega_m^*)) + s_1 s_3 h_f(\psi(\omega_{m-1}^*)) + \cdots + s_1 s_3^{m-1} h_f(\psi(\omega_1^*)) \\ = \; & (1 - (\frac{1}{3})^m) + (1 - (\frac{1}{3})^{m-1}) s_1 + (1 - (\frac{1}{3})^{m-2}) s_1 s_3 + \cdots + (1 - \frac{1}{3}) s_1 s_3^{m-2} + s_1 s_3^{m-1} \\ = \; & (1 - (\frac{1}{3})^m) + (1 + s_3 + \cdots + s_3^{m-1}) s_1 - (\frac{1}{3})^{m-1} (1 + 3s_3 + \cdots + (3s_3)^{m-2}) s_1. \end{aligned}$$

Since $0 < s_3 \leq \frac{1}{3}$, we obtain that $\lim_{m\to+\infty} f(\psi(\omega_m)) = 1 + \frac{s_1}{1-s_3} > 1$. Hence there exists $x \in [0,1]$ such that $f(x) > 1$, due to the continuity of $f(x)$.

3 Proof Theorem 1

In order to prove Theorem 1, we need the following four propositions.

Proposition 1 *If* $0 \leq f_{(1,s_1,s_2,s_3)}(x) \leq 1$ *for any* $x \in [0,1]$, *then* $-\frac{1}{3} \leq s_1, s_3 \leq \frac{1}{3}$, $-1 \leq s_2 \leq 0$.

Proposition 2 *Let* $-\frac{1}{3} \leq s_1, s_3 \leq 0$ *and* $-1 \leq s_2 \leq 0$. *Then* $0 \leq f_{(1,s_1,s_2,s_3)}(x) \leq f_{(1,0,0,0)}(x)$ *for any* $x \in [0,1]$.

Proposition 3 *Let* $0 \leq s_1 \leq \frac{1}{3}$,$-1 \leq s_2 \leq 0$ *and* $-\frac{1}{3} \leq s_3 \leq 0$. *Then* $0 \leq f_{(1,s_1,s_2,s_3)}(x) \leq f_{(1,\frac{1}{3},0,0)}(x)$ *for any* $x \in [0,1]$.

Proposition 4 *Let* $s_1, s_3 > 0$ *and* $-1 \leq s_2 \leq 0$. *Then* $0 \leq f_{(1,s_1,s_2,s_3)}(x) \leq 1$ *for any* $x \in [0,1]$ *if and only if* $s_1 + s_3 \leq \frac{1}{3}$.

Proof of Proposition 1:

Let $f(x) = f_{(1,s_1,s_2,s_3)}(x)$. It follows from (6) that $f(1/9) = 1/3 + s_1$, $f(2/9) = 2/3 + s_1$, $f(4/9) = 1 + s_2$, $f(5/9) = 1 + s_2$, $f(7/9) = 2/3 + s_3$, $f(8/9) = 1/3 + s_3$. since $0 \leq f_{(1,s_1,s_2,s_3)}(x) \leq 1$, we have

$$-\frac{1}{3} \leq s_1, s_3 \leq \frac{1}{3} \text{ and } -1 \leq s_2 \leq 0.$$

Proof of Proposition 2:

Let $f(x) = f_{(1,s_1,s_2,s_3)}(x)$ and $H(x) = f_{(1,0,0,0)}(x)$ satisfying

$$H(x) = \begin{cases} 3x, & \text{if } 0 \leq x \leq 1/3, \\ 1, & \text{if } 1/3 \leq x \leq 2/3, \\ 3(1-x), & \text{if } 2/3 \leq x \leq 1. \end{cases}$$

Given $k \in \mathbb{N}$, let $\Lambda_k = \{i/3^k \in [0,1] : i \in \mathbb{Z}\}$. It suffices to show that $0 \leq f(x) \leq H(x)$ for all $x \in \Lambda_k$ by induction on k.

For $k = 1$, it is obvious that $0 \le f(x) = H(x)$ for $x \in \Lambda_1 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, due to the property of AFIF. Now we assume that for a given integer k, $0 \le f(x) \le H(x)$ for any $x \in \Lambda_k$. We shall show $0 \le f(x) \le H(x)$ for all $x \in \Lambda_{k+1}$. We can distinguish three cases as follows.

(i) When $x \in \Lambda_{k+1} \cap [0, \frac{1}{3}]$, i.e., $3x \in \Lambda_k$,by formulas (5) and (6), we have $f(x) = 3x + s_1 f(3x) \le 3x = H(x)$. On the other hand, we can show $f(x) \ge 0$ as follows: If $3x \in [0, \frac{1}{3}]$, then $f(x) = 3x + s_1 f(3x) \ge 3x - |s_1| H(3x) = 3x(1 - 3|s_1|) \ge 0$; If $3x \in [\frac{1}{3}, 1]$, then $f(x) \ge 3x - |s_1| \ge 0$.

(ii) When $x \in \Lambda_{k+1} \cap [\frac{1}{3}, \frac{2}{3}]$, i.e., $3x - 1 \in \Lambda_k$,by formulas (5) and (6), we have $f(x) = 1 + s_2 f(3x - 1)$. Since $-1 \le s_2 f(3x - 1) \le 0$,

$$0 \le f(x) \le 1 = H(x).$$

(iii) When $x \in \Lambda_{k+1} \cap [\frac{2}{3}, 1]$, i.e., $3x - 2 \in \Lambda_k$,by formulas (5) and (6), we have $f(x) = 3(1 - x) + s_3 f(3x - 2) \le 3(1 - x) = H(x)$. On the other hand, we can show $f(x) \ge 0$ as follows: If $3x - 2 \in [0, \frac{2}{3}]$, then $f(x) = 3(1-x) + s_3 f(3x-2) \ge 3(1-x) - |s_3| \ge 0$; If $3x - 2 \in [\frac{2}{3}, 1]$,then $f(x) = 3(1-x) + s_3 f(3x-2) \ge 3(1-x) - |s_3| H(3x-2) = 3(1-x) - 9|s_3|(1-x) = 3(1-x)(1-3|s_3|) \ge 0$.

Because of the continuity of $f(x)$ and $H(x)$, we obtain $0 \le f(x) \le H(x)$ for any $x \in [0, 1]$.

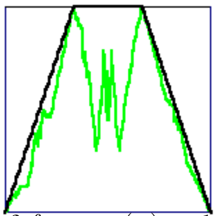

Figure 3: Graphs of $f_{(1,0,0,0)}(x)$ and $f_{(1,-0.2,-0.7,-0.1)}(x)$.

Proof of Proposition 3:

Let $f(x) = f_{(1,s_1,s_2,s_3)}(x)$ and $G(x) = f_{(1,\frac{1}{3},0,0)}(x)$ satisfying

$$G(x) = \begin{cases} 3x + G(3x)/3, & \text{if } 0 \le x \le 1/3, \\ 1, & \text{if } 1/3 \le x \le 2/3, \\ 3(1-x), & \text{if } 2/3 \le x \le 1. \end{cases}$$

Given $k \in \mathbb{N}$, let $\Lambda_k = \{i/3^k \in [0,1] : i \in \mathbb{Z}\}$. It suffices to show that $0 \le f(x) \le G(x)$ with $x \in \Lambda_k$ by induction on k.

For $k = 1$, it is obvious that $0 \le f(x) = G(x)$ for $x \in \Lambda_1 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, due to the property of AFIF. Now we assume that for a given integer k, $0 \le f(x) \le G(x)$ for any $x \in \Lambda_k$. We shall show $0 \le f(x) \le G(x)$ for all $x \in \Lambda_{k+1}$. We can distinguish three cases as follows.

(i) When $x \in \Lambda_{k+1} \cap [0, \frac{1}{3}]$, i.e., $3x \in \Lambda_k$, by formulas (5) and (6), we have $f(x) = 3x + s_1 f(3x) \le 3x + \frac{1}{3} G(3x) = G(x)$. On the other hand, we have $f(x) = 3x + s_1 f(3x) \ge 0$ since $s_1, f(3x) \ge 0$.

(ii) When $x \in \Lambda_{k+1} \cap [\frac{1}{3}, \frac{2}{3}]$, i.e., $3x - 1 \in \Lambda_k$, by formulas (5) and (6), we have $f(x) = 1 + s_2 f(3x - 1)$. It follows from $-1 \le s_2 f(3x - 1) \le 0$ that $0 \le f(x) \le 1 = G(x)$.

(iii) When $x \in \Lambda_{k+1} \cap [\frac{2}{3}, 1]$, i.e., $3x - 2 \in \Lambda_k$, by formulas (5) and (6), we have $f(x) = 3(1 - x) + s_3 f(3x - 2) \le 3(1 - x) = G(x)$. On the other hand we can show have $f(x) \ge 0$ as follows: If $3x - 2 \in [0, \frac{2}{3}]$, then $f(x) = 3(1-x) + s_3 f(3x-2) \ge 3(1-x) - |s_3| \ge 0$; If $3x - 2 \in [\frac{2}{3}, 1]$, then $f(x) = 3(1-x) + s_3 f(3x-2) \ge 3(1-x) - |s_3| G(3x-2) = 3(1-x) - 9|s_3|(1-x) = 3(1-x)(1-3|s_3|) \ge 0$.

Because of the continuity of $f(x)$ and $G(x)$, we obtain $0 \le f(x) \le G(x)$ for any $x \in [0, 1]$.

Proof of Proposition 4:

(1) Sufficient condition: Given $k \in \mathbb{N}$, let $\Lambda_k = \{i/3^k \in [0,1] : i \in \mathbb{Z}\}$. It is easy to prove that $0 \le f_{(1,s_1,s_2,s_3)}(x) \le f_{(1,s_1,0,s_3)}(x)$ for all $x \in \Lambda_k$ by induction on k. Due to the continuity of $f_{(1,s_1,s_2,s_3)}(x)$ and $f_{(1,s_1,0,s_3)}(x)$, we have $0 \le f_{(1,s_1,s_2,s_3)}(x) \le f_{(1,s_1,0,s_3)}(x)$ for any $x \in [0, 1]$. Therefore, the sufficient condition follows from Lemma 4.

(2) Necessary condition: We obtain the necessary condition from Lemma 5.

We complete the proof of Theorem 1 by using Proposition 1-4 and Lemma 1-5.

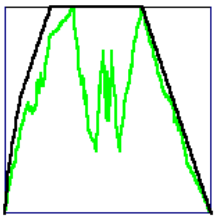

Figure 4: Graphs of $f_{(1,\frac{1}{3},0,0)}(x)$ and $f_{(1,0.2,-0.7,-0.1)}(x)$.

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